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APPROXIMATION OF INITIAL AND BOUNDARY VALUE PROBLEMS FOR QUASILINEAR FIRST ORDER EQUATIONS

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1. Introduction

We consider for $x \in]0,1[$ and $t \geqslant 0$ the equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0, \tag{1}$$

with $f \in C^1(\boldsymbol{R})$, and the initial and boundary conditions

(2) $u(\cdot, 0) = u_0 \in \mathrm{BV}(]0,1[)$ (space of Bounded Variation Functions),

$$\left.\begin{aligned} &u(0,\cdot) = a_0 \in \boldsymbol{R} &(3)\\ &u(1,\cdot) = a_1 \in \boldsymbol{R} &(4)\end{aligned}\right\} \quad \text{(if needed)}.$$

We shall see later that the right way of writing the boundary conditions is, instead of (3), (4), the following for $i = 0$ or 1,

$$\min_{k \in I(a_i, \gamma_i u)} \{n_i \operatorname{sg}(a_i - k)(f(\gamma_i u) - f(k))\} = 0 \qquad \text{(a.e. on }]0, +\infty[), \tag{5}$$

where $n_0 = -1$, $n_1 = 1$, $\gamma_i u$ is the trace of the solution u on the boundary $x = i$ and $I(\alpha, \beta)$ is a notation for the real closed interval, the bounds of which are α and β.

The plane of these lecture is the following. A few examples are given in Section 2, to show that (3) and (4) are not always satisfied. Some notations are introduced, and some results are recalled in Section 3. Section 4 deals with the vanishing viscosity method; some estimates are proved, which are used in Section 5 to state Existence and Uniqueness of a weak solution (satisfying an entropy condition) of (1), (2), (5). The convergence of the approximation by the Godunov scheme is proved in Section 6, and by the Glimm scheme or the Lax Friedrichs scheme in Section 7. Then conclusions and references are given.

2. Examples

We give three examples; the first and the second one are linear.

1. For $f(u) = u$, the solution to be found is

$$u(x, t) = \begin{cases} a_0 & \text{if } x < t, \\ u_0(x-t) & \text{if } x > t, \end{cases}$$

which shows that (4) is not useful.

2. For $f(u) = -u$, we get as the solution

$$u(x, t) = \begin{cases} a_1 & \text{if } x > t, \\ u_0(x+t) & \text{if } x < t, \end{cases}$$

and now (3) is not used.

3. For $f(u) = u^2/2$, (1) is known as the Burgers equation, and for $a_0 = -1$, $a_1 = 1$, $u_0 = 0$, the solution is zero and no boundary conditions are used. For $a_0 = 1$, $a_1 = -1$, $u_0 = 0$, the solution is given by

$$u(x, t) = \begin{cases} 0 & \text{if } t < x < 1-t, \\ -1 & \text{if } x > \max(\frac{1}{2}, 1-t), \\ 1 & \text{if } x < \min(\frac{1}{2}, t), \end{cases}$$

and both (3) and (4) are used. Note that two shocks meet, to give only one. For $a_0 = -1$, $a_1 = u_0 = 1$, the solution is

$$u(x, t) = \min(x/t, 1),$$

which seems as if a_0 is put equal to zero. Since f is a decreasing function on $[-1, 0[$, all these values are sent outside by (1) and since f is an increasing function on $]0, 1]$, these values are taken in account. The condition (5) improved this argument.

3. Some notations and results of functional analysis

We shall use the sign function sg and its approximation sg_η, defined by

$$(6)\qquad sg(x) = \begin{cases} 0 & \text{if } x = 0, \\ x/|x| & \text{if } x \neq 0, \end{cases} \qquad sg_\eta(x) = \begin{cases} sg(x) & \text{if } n \leqslant |x|, \\ x/\eta & \text{if } n > |x|, \end{cases} \quad (\eta > 0).$$

Let $T > 0$ and denote by Ω the set $]0, 1[\times]0, T[$. The space $BV(]0, 1[)$ is the space of Bounded Variation functions defined on $]0, 1[$, with real values; it is a Banach space for the norm

$$\|v\| = |v(0)| + \lim_{\delta\to 0} \int_0^{1-\delta} \frac{1}{\delta} |v(x+\delta) - v(x)|\, dx. \tag{7}$$

We introduce, as Tonelli Cesari, its generalization to the bidimensional case

$$\mathrm{BV}(\Omega) = \Big\{w \in L^\infty(\Omega),\ \int_0^T \|w(\cdot, t)\|\, dt + \int_0^1 \|w(x, \cdot)\|\, dx < +\infty\Big\}. \tag{8}$$

Since a function of $\mathrm{BV}(]0,1[)$ has a left and a right limit at any point, we can define a trace $\gamma_0 u(\cdot)$ at $x = 0$ and a trace $\gamma_1 u(\cdot)$ at $x = 1$ almost everywhere on $]0, T[$, for any function u in $\mathrm{BV}(\Omega)$. Moreover, from $u \in L^\infty(\Omega)$, by the Lebesgue theorem, these traces are reached through a strong convergence in $L^1(]0, T[)$ as x tends to 0 or to 1. By the same arguments, a trace $\gamma u(x, 0)$ may be defined for almost all x in $]0,1[$ which is also reached by a $L^1(]0,1[)$-convergence on u as t tends to 0.

We also use the Sobolev space

$$W^{1,1}(\Omega) = \Big\{w \in L^1(\Omega),\ \iint_\Omega \Big(\Big|\frac{\partial w}{\partial x}\Big| + \Big|\frac{\partial w}{\partial t}\Big|\Big)\, dx dt < +\infty\Big\}, \tag{9}$$

whose inclusion in $L^1(\Omega)$ is relatively compact. Moreover, $W^{1,1}(\Omega)$ is included in $\mathrm{BV}(\Omega)$, and by passing to the limit W on a sequence w_m which is bounded in $W^{1,1}(\Omega)$, we get from (7) and (8) that W also belongs to $\mathrm{BV}(\Omega)$.

The following lemma, which is due to Saks, will be used:

$$\forall\, w \in C^1(]0,1[), \quad \lim_{\eta \downarrow 0} \int_{\{x \mid |w(x)| < \eta\}} \Big|\frac{dw}{dx}\Big|\, dx = 0. \tag{10}$$

We also introduce the real function I_η defined on $\boldsymbol{R}$ by

$$I_\eta(y) = \int_0^y \mathrm{sg}_\eta(x)\, dx. \tag{11}$$

4. Vanishing viscosity method

Some estimates in $W^{1,1}(\Omega)$ will be derived for the solution u_ε, which is admitted to exist in $C^2(\overline{\Omega})$ for $\varepsilon > 0$, of the parabolic equation

$$\frac{\partial u_\varepsilon}{\partial t} + \frac{\partial}{\partial x} f(u_\varepsilon) - \varepsilon \frac{\partial^2 u_\varepsilon}{\partial x^2} = 0, \tag{12}$$

and satisfying to the boundary conditions (3) and (4) for all $t > 0$, and to a regularized initial condition

$$u_\varepsilon(\cdot, 0) = v_0 \in C^2([0, 1]) \tag{13}$$

such that

$$|v_0'|_{L^1(]0,1[)} \leqslant C(\|u_0\| + |a_0| + |a_1|)$$

and

$$v_0(0) = v_0'(0) = v_0''(0) = v_0(1) = v_0'(1) = v_0''(1) = 0.$$

The parameter ε physically corresponds to a coefficient of viscosity, and will tend to zero. In order to ensure Uniqueness for (1), (2), (5), we shall use in the estimates that ε was a positive number before passing to the limit.

Let us begin by an estimate of $\left|\dfrac{\partial u_\varepsilon}{\partial x}\right|_{L^1(\Omega)}$. We take the derivative of (12) with respect to x, multiply by $\operatorname{sg}_\eta(\partial u_\varepsilon/\partial x)$ and integrate by parts on $]0, 1[$, to get

$$\frac{d}{dt}\int_0^1 I_\eta\left(\frac{\partial u_\varepsilon}{\partial x}\right) dx + \left[\operatorname{sg}_\eta\left(\frac{\partial u_\varepsilon}{\partial x}\right)\frac{\partial}{\partial x}\left(f(u_\varepsilon) - \varepsilon\frac{\partial u_\varepsilon}{\partial x}\right)\right]_0^1 \tag{14}$$
$$= \int_0^1 \operatorname{sg}_\eta'\left(\frac{\partial u_\varepsilon}{\partial x}\right) f'(u_\varepsilon)\,\frac{\partial u_\varepsilon}{\partial x}\,\frac{\partial^2 u_\varepsilon}{\partial x^2}\,dx - \varepsilon\int_0^1 \operatorname{sg}_\eta'\left(\frac{\partial u_\varepsilon}{\partial x}\right)\left|\frac{\partial^2 u_\varepsilon}{\partial x^2}\right|^2 dx.$$

From (12), the second term is obviously zero, since the boundary condition are constant; the last term has a constant sign, we use it to transform (14) into an inequality. The third term tends to zero as η tends to zero, by (10). Now, for $t > s$, we get by integrating on $]s, t[$

$$\left|\frac{\partial u_\varepsilon}{\partial x}(\cdot, t)\right|_{L^1(]0,1[)} \leqslant \left|\frac{\partial u_\varepsilon}{\partial x}(\cdot, s)\right|_{L^1(]0,1[)},$$

and for s tending to zero, for all $t \in]0, T[$,

$$\int_0^1 \left|\frac{\partial u_\varepsilon}{\partial x}(x, t)\right| dx \leqslant C(\|u_0\| + |a_0| + |a_1|). \tag{15}$$

This gives also, from

$$|u_\varepsilon(x, t)| \leqslant |a_0| + \int_0^1 \left|\frac{\partial u_\varepsilon}{\partial x}(x, t)\right| dx, \tag{16}$$

that u_ε is uniformly bounded in $L^\infty(\Omega)$; this estimate may be also directly obtained from the principle of maximum. By the same arguments, we get an estimate of $|\partial u_\varepsilon/\partial t|_{L^1(]0,1[)}$ by deriving (12) with respect to t and multiplying by $\mathrm{sg}_\eta(\partial u_\varepsilon/\partial t)$. The analogue of (14) is,

$$\frac{d}{dt}\int_0^1 I_\eta\left(\frac{\partial u_\varepsilon}{\partial t}\right)dx+\left[\mathrm{sg}\left(\frac{\partial u_\varepsilon}{\partial t}\right)\frac{\partial}{\partial t}\left(f(u)-\varepsilon\frac{\partial u_\varepsilon}{\partial x}\right)\right]_0^1$$
$$=\int_0^1 \mathrm{sg}'_\eta\left(\frac{\partial u_\varepsilon}{\partial t}\right)f'(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial t}\,\frac{\partial^2 u_\varepsilon}{\partial x\partial t}\,dx-\varepsilon\int_0^1 \mathrm{sg}'_\eta\left(\frac{\partial u_\varepsilon}{\partial t}\right)\left|\frac{\partial^2 u_\varepsilon}{\partial x\partial t}\right|^2 dx,$$

where $\mathrm{sg}_\eta(\partial u_\varepsilon/\partial t)$ is zero on the boundaries. We finally get that for $\varepsilon>0$, u_ε belongs to a bounded set of $W^{1,1}(\Omega)$, and then a sequence u_{ε_m} can be extracted, with ε_m tending to zero as m tends to infinity, which converges in $L^1(\Omega)$ towards a function u lying in $\mathrm{BV}(\Omega)$.

5. Existence and uniqueness

We now prove that the function u, we have just exhibit, is the solution in a weak sense of (1), (2), (5). Our first goal is to give a definition which characterizes this solution, recalling that it was obtained from (12). We introduce the set

$$\Phi(\Omega)=\{\varphi\in C^2([0,1]\times[0,T]),\varphi\geqslant 0,\ \varphi(\cdot,T)=0\},$$

and for $\varphi\in\Phi(\Omega)$, $k\in\boldsymbol{R}$, we multiply (12) by $\varphi\,\mathrm{sg}(u-k)$ and integrate by parts, to get (note that $\dfrac{\partial k}{\partial t}=\dfrac{\partial}{\partial x}f(k)=0$ are introduced)

$$(17.1)\qquad 0=\iint_\Omega I_\eta(u_\varepsilon-k)\frac{\partial\varphi}{\partial t}\,dx\,dt+\int_0^1 I_\eta(v_0-k)\varphi(x,0)\,dx+$$

$$(17.2)\qquad +\iint_\Omega \big(f(u_\varepsilon)-f(k)\big)\mathrm{sg}_\eta(u_\varepsilon-k)\frac{\partial\varphi}{\partial x}\,dx\,dt+$$

$$(17.3)\qquad +\iint_\Omega \big(f(u_\varepsilon)-f(k)\big)\mathrm{sg}'_\eta(u_\varepsilon-k)\frac{\partial}{\partial x}(u_\varepsilon-k)\varphi\,dx\,dt-$$

$$(17.4)\qquad -\varepsilon\iint_\Omega \mathrm{sg}'_\eta(u_\varepsilon-k)\varphi\left|\frac{\partial u_\varepsilon}{\partial x}\right|^2 dx\,dt-$$

$$(17.5)\qquad -\varepsilon\iint_\Omega \frac{\partial u_\varepsilon}{\partial x}\mathrm{sg}_\eta(u_\varepsilon-k)\frac{\partial\varphi}{\partial x}\,dx\,dt-$$

$$(17.6)\qquad -\sum_{i=0,1}\left(f(a_i)-f(k)-\varepsilon\frac{\partial u_\varepsilon}{\partial x}(i,t)\right)\varphi(i,t)\,\mathrm{sg}_\eta(a_i-k)n_i.$$

We first write that (17.4) is negative, and then let η tends to zero, to get that (17.3) is zero at the limit by (10). The limit on the other terms is obvious. Now we put $\varepsilon = \varepsilon_m$ and let m tend to infinity (ε_m tends to zero); we have only to replace u_ε by u on all term, but a part of (17.6), for which to known the weak limit of $\varepsilon \dfrac{\partial u_\varepsilon}{\partial x}(i, \cdot)$ is needed.

For a non negative $\psi \in C^1(]0, T[)$, with $\psi(0) = \psi(T) = 0$, and $1 > \delta > 0$, we multiply (12) by $\psi(t)\max(0, 1-x/\delta)$ and integrate on Ω, to get

$$0 = -\int_0^T\int_0^\delta \left(\psi'(t)\left(1-\frac{x}{\delta}\right)u_\varepsilon + \psi\frac{1}{\delta}f(u_\varepsilon) - \psi\frac{\varepsilon}{\delta}\frac{\partial u_\varepsilon}{\partial x}\right)dt dx + \\ + \int_0^T \varepsilon\frac{\partial u_\varepsilon}{\partial x}(0, t)\psi(t)\,dt - \int_0^T f(a_0)\psi(t)\,dt.$$

We make first ε $(=\varepsilon_m)$ and then δ tending to zero. Since $u \in \mathrm{BV}(\Omega)$ we get

$$\lim_{\substack{\varepsilon_m \to 0\\(m\to\infty)}} \int_0^T \varepsilon_m \frac{\partial u_{\varepsilon_m}}{\partial x}\psi(t)\,dt = \int_0^T \{f(a_0) - f(\gamma u(t))\}\psi(t)\,dt. \tag{18}$$

Now, from (17) we obtain, using (18) and a similar result for $x = 1$, the inequality (where u_0 is written instead of v_0 since the passage to this limit is obvious)

$$\iint_\Omega \left(|u-k|\frac{\partial\varphi}{\partial t} + \mathrm{sg}(u-k)\big(f(u)-f(k)\big)\frac{\partial\varphi}{\partial x}\right)dx dt + \\ + \int_0^1 |u_0 - k|\varphi(x, 0)\,dx \\ \geqslant \sum_{i=0,1}\int_0^T n_i \mathrm{sg}(a_i - k)\{f(\gamma_i u(t)) - f(k)\}\varphi(i, t)\,dt, \tag{19}$$

since $f(a_i)$ get away by (18) and (17.6). We have the following

THEOREM 1. *The problem* (1), (2), (5) *has a unique solution* $u \in \mathrm{BV}(\Omega)$, *which is characterized by* (19) *for all* $k \in \boldsymbol{R}$ *and all* φ *in* $\Phi(\Omega)$.

Proof. The existence is stated. By taking $\varphi(x, t) = \psi(t)\max(0, 1-x/\delta)$ in (19), we get, as δ tends to zero, the inequality

$$\{f(\gamma_i u(t) - f(k))\}\{\mathrm{sg}(a_i - k) - \mathrm{sg}\,\gamma_i u(t) - k\} \geqslant 0 \qquad (\text{a.e. }]0, T[) \tag{20}$$

which gives (5) for k in $I(a_i, \gamma_i u(t))$ and is obvious for k outside.

It remains to prove uniqueness; this is done by using the same arguments as Kruzkov in [3]. Let u and v be two solutions (i.e., satisfying

(19)), with the initial data u_0 and v_0, and the same boundary conditions a_0 and a_1. From (3) we get

$$\iint \left\{ |u-v| \frac{\partial\varphi}{\partial t} + \mathrm{sg}(u-v)\big(f(u)-f(v)\big) \frac{\partial\varphi}{\partial x} \right\} dx\,dt \geqslant 0 \tag{21}$$

for any test function φ in $\Phi(\Omega)$, equal to zero on the boundaries of Ω. Now we take the following form for φ, with ψ defined as above,

$$\varphi(x, t) = \psi(t)\{1 - \max(0, 1 - x/\delta) - \max\big(0, 1 - (1-x)/\delta\big)\},$$

in (21) and let δ tends to zero, to obtain

$$\iint |u-v|\,\psi'(t)\,dx\,dt \geqslant \sum_{i=0,1} \int_0^T n_i \mathrm{sg}(\gamma_i u - \gamma_i v)\big(f(\gamma_i u) - f(\gamma_i v)\big)\psi(t)\,dt. \tag{22}$$

The last term in (22) is nonnegative. To show it, we introduce two functions $k_i(t)$, for $i = 0, 1$, in $L^\infty(]0, T[)$ such that, almost everywhere on $]0, T[$, $k_i(t) \in I\big(\gamma_i u(t), a_i\big) \cap I\big(\gamma_i v(t), a_i\big)$, and use (5), already deduced from (20). Now we can write from (22), for any nonnegative ψ,

$$\int_0^T \big(|u(\cdot, t) - v(\cdot, t)|_{L^1(]0,1[)}\big)\psi'(t)\,dt \geqslant 0,$$

which states that the semigroup operator associated to (1) and (5) is a nonlinear contraction in $L^1(]0, 1[)$. Obviously, we have for almost all t in $]0, T[$,

$$|u(\cdot, t) - v(\cdot, t)|_{L^1(]0,1[)} \leqslant |u_0 - v_0|_{L^1(]0,1[)}, \tag{23}$$

and then uniqueness by taking $u_0 = v_0$.

6. Approximation by the Godunov scheme

Let $I \in \boldsymbol{N}$ and $h = 1/I$. We introduce the space

$$V_h = \{v \in L^2(]0, 1[),\ v \text{ is constant on each } I_{i+1/2} =]ih, (i+1)h[,\ i = 0, \ldots, I-1\} \tag{24}$$

and project the initial condition on V_h, using the L^2-norm (i.e. averaging). We get the values

$$u^0_{i+1/2} = \frac{1}{h} \int_{I_{i+1/2}} u_0(x)\,dx, \tag{25}$$

and, denoting by P the operator of projection, the discrete initial condition

$$u^0_h = P u_0; \quad u^0_h(x) = u^0_{i+1/2} \quad \text{if } x \in I_{i+1/2}. \tag{26}$$

Obviously, the operator P preserves the L^∞ norm, and the BV$(]0,1[)$ seminorm defined by $|v| = \|v\| - |v(0)|$ from (7). It is also a contraction for the L^1 norm.

We denote by q a positive parameter, which will control the stability, and take as time meshsize $\Delta t = qh$. We suppose that, at time $n\Delta t$, the approximate function $u_h^n \in V_h$ is known, and try to built $u_h^{n+1} \in V_h$. First we solve exactly the problem (1), (5), starting with the initial condition u_h^n at time $n\Delta t$, on the strip $J_n =]n\Delta t, (n+1)\Delta t[$. The solution is a constant on each straight segment $\{ih\} \times J_n$ when the following stability condition

$$q \sup_{|k| \leqslant |u_0|_{L^\infty(]0,1[)}} |f'(k)| \leqslant 1, \tag{27}$$

known as the Courant–Friedrichs–Lewy stability condition is satisfied. Then, at time $(n+1)\Delta t$, we use the operator P to project this solution on V_h. Since, this projection needs only the values of the solution, on each $\{ih\} \times J_n$, given by $u_i^n \in I(u_{i-1/2}^n, u_{i+1/2}^n)$ such that

$$\mathrm{sg}(u_{i+1/2}^n - u_{i-1/2}^n) f(u_i^n) = \min_{k \in I(u_{i-1/2}^n, u_{i+1/2}^n)} \big(\mathrm{sg}(u_{i+1/2}^n - u_{i-1/2}^n) f(k)\big), \tag{28}$$

with $u_{-1/2}^n = a_0$, $u_{I+1/2}^n = a_1$, we get the projected values, for $i = 0, \dots$ $\dots, I-1$,

$$u_{i+1/2}^{n+1} = u_{i+1/2}^n - q\big(f(u_{i+1}^n) - f(u_i^n)\big). \tag{29}$$

The Godunov scheme is (28), (29), and we have the following

THEOREM 2. *Provided that* (27) *is verified, then the approximate solution built by the Godunov scheme converges in $L^1(\Omega)$, as h tends to zero, to the unique solution of* (1), (2), (5), *which was characterized by* (19).

Proof. We define the approximate solution as the solution on each strip J_n, and note it $u_h(x,t)$. From (19), in each strip, we can write, for any $k \in \boldsymbol{R}$ and $\varphi \in \Phi(\Omega)$,

$$\iint_\Omega \Big\{|u_h - k| \frac{\partial \varphi}{\partial t} + \mathrm{sg}(u_h - k)\big(f(u_k) - f(k)\big)\frac{\partial \varphi}{\partial x}\Big\} dx dt + \int_0^1 |Pu_0 - k| \varphi(x, 0) dx \tag{30}$$

$$\geqslant \sum_{i=0,1} \int_0^1 n_i \mathrm{sg}(a_i - k)\big(f(\gamma_i u_k) - f(k)\big) \varphi(i, t) dt - \sum_n \int_0^1 \big(|Pu_h(x, n\Delta t) - k| - |u_h(x, n\Delta t) - k|\big) \varphi(x, n\Delta t) dx.$$

For any function $v \in L^\infty(\Omega)$, we have $Pv-k = P(v-k)$, and since P is a contraction in $L^1(]0,1[)$, the last term is nonnegative. Moreover, P preserves some estimates on u_h from a strip to another, such as the L^∞ norm and the variation (i.e. seminorm $\|v\|-|v(0)|$ on $\mathrm{BV}(]0,1[)$, the L^1 convergence of a sequence u_{h_m}, as h_m tends to zero, is implied, by using the same arguments as in Theorem 1. Denoting by $u \in \mathrm{BV}(\Omega)$ this limit, we get that the first member in (30) converges to the first member in (19). We look now at the time integrals on the boundaries; denoting by $\lambda_i \in L^\infty(]0,T[)$ the weak star limit in $L^\infty(]0,T[)$ of a subsequence of $f(\gamma_i u_{h_m})$, we get from (30) at the limit

$$\iint_\Omega |u-k|\frac{\partial\varphi}{\partial t} + \mathrm{sg}(u-k)\big(f(u)-f(k)\big)\frac{\partial\varphi}{\partial x}\,dx\,dt + \int_0^1 |u_0-k|\varphi(x,0)\,dx \geqslant \sum_i \int_0^T n_i\,\mathrm{sg}(a_i-k)\big(\lambda_i(t)-f(k)\big)\varphi(i,t)\,dt. \tag{31}$$

By taking $\varphi(x,t) = \psi(t)\max(0,1-x/\delta)$, as above, we get at the limit, that

$$\mathrm{sg}(a_0-k)\big(\lambda_0(t)-f(k)\big) - \mathrm{sg}\big(\gamma_0 u(t)-k\big)\Big(f\big(\gamma_0 u(t)\big)-f(k)\Big) \geqslant 0 \qquad (\text{a.e. }]0,T[)$$

which gives, for $k \notin I\big(\gamma_0 u(t), a_0\big)$, that $\lambda_0(t)$ equals $f\big(\gamma_0 u(t)\big)$ almost everywhere. We have the analogue at $x = 1$, and Theorem 2 is proved.

7. Approximation by the Lax Friedrichs scheme and by the Glimm scheme

For any even $I \in N$ and $h = 1/I$, we define two spaces V_h^l, for $l = 0,1$ by

$$V_h^l = \{v \in L^2(\Omega),\ v \text{ constant on each } I_i =](i-1)h,(i+1)h[,\ i+l \text{ even}\}$$

and denote by P_l $(l = 0,1)$ the operator of L^2 projection (averaging) on V_h^l. The Lax Friedrichs scheme consists to project the initial condition on V_h^1, to solve (1), (5) on a strip $]0,\Delta t[$, with $\Delta t = qh$, to project the obtained solution on V_h^0, to solve (1), (5) on the strip $]\Delta t, 2\Delta t[$ and then to project on V_h^1, etc... If the stability condition (27) is fullfilled then we are ensured that the solution is a constant on the triangles

$$\Big\{(x,t),\ |x-ih| + \frac{1}{q}|t-nqh| \leqslant h\Big\},$$

and thus the scheme is easy to write, at the time level $n\Delta t$,

$$u_i^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - \frac{q}{2}\big(f(u_{i+1}^n) - f(u_{i-1}^n)\big) \qquad (i+n \text{ odd}). \tag{32}$$

For n odd only we use the boundary conditions, by writing for example,

$$u_0^n = a_0; \quad u_I^n = a_1; \tag{33}$$

and taking in account that, for the projections, the intervals near the boundaries have a length equal to h, instead of $2h$ for the other intervals. Note that (33) is not exactly the good way to introduce (5), since an analogue of (28) really appears.

The same spaces V_h^l are used for the Glimm scheme, which differs from the Lax Friedrichs scheme by interpolating at a random point on each I_i instead of a projection using the L^2 norm, at each time level $n\Delta t$. We also need the stability conditions (27), and following Glimm [2], we have to integrate an analogue of (30) with respect to a_n, if $a_n \in]0,1[$ is the uniform random value such that the interpolation at time level $n\Delta t$ is performed at the points $(i+2a_n-1)h$, in such a way that the arguments used for the operator P above are suitable here.

Theorem 2 is true for the Lax Friedrichs scheme (32), and for the Glimm scheme, but for $(a_1, \ldots, a_n, \ldots)$ belonging to a negligible set of $\prod_{k=1}^{\infty} [0,1]$. This scheme may be written

$$u_i^{n+1} = u_h^n\big((i+2a_n-1)h, (n+1)\Delta t\big),$$

if u_h^n is the exact solution on the strip $]0,1[\times J_n$, from the initial data in V_h^l at time level $n\Delta t$.

All these results are suitable, even when the boundary conditions are non constant data or when (1) has a second member; see [4]. Theorem 1 is true for a multidimensional problem (see [1]), but the approximation by the Godunov scheme is proved to be convergent only with an hypothesis of monotony on the components of f (see [6]). Quasi order two schemes may also be applied, and give numerically, good shape for shocks, see [5]. The profiles are rather good for the Godunov scheme, but not for the Lax Friedrichs one.

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