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GENERALIZED INTERPOLATION AND SOME APPLICATIONS IN ORDINARY DIFFERENTIAL EQUATIONS

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We consider some classes of generalized interpolation. In every class we distinguish generalized Lagrange, Newton and Hermite formulas. Our main interest is the interpolation in H^2 spaces. The aim of the present paper is to give a survey of generalized interpolations and to show how they can be used for the numerical solution of ordinary differential equations.

We obtain the following results:

The use of generalized interpolation formulas in H^2 spaces enables us to construct explicit linear multistep methods for approximate numerical computation of the solution of a system of ordinary differential equations of the first order

$$\dot{y} = f(t, y), \quad y(0) = y_0 \quad (y \in \boldsymbol{R}^m,\ t \geqslant 0),$$

which are A- or L-stable and are of arbitrarily high order. These methods involve at least one free parameter, which can be used to make the approximate numerical solution fit certain stiff problems. The coefficients in the methods depend on the stepsize h and on the points chosen in the interpolation.

I. ON GENERALIZED INTERPOLATION

Let us remind the classical Lagrange interpolation polynomial

$$L_n(f; z) = \sum_{r=0}^{n} f_r \prod_{\substack{j \neq r \\ j=0}}^{n} \frac{z-x_j}{x_r-x_j}.$$

If we replace $z-x_j$ by functions $h_j(z)$ with $h_j(x_j) = 0$ and $h_j(x_k) \neq 0$, for $j \neq k$, we get a generalized Lagrange interpolation function

$$L_n(f; z) = \sum_{r=0}^{n} f_r \prod_{\substack{j \neq r \\ j=0}}^{n} \frac{h_j(z)}{h_j(x_r)}.$$

It was introduced already by Thiele [1] in 1909. In the recent years generalized interpolation methods were widely considered. H. Engels [2] dealt with the following generalized interpolation problems: Lagrange, Newton and Aitken–Neville interpolations, the cubic spline and Hermite interpolations, the applications of generalized interpolations as quadrature methods and Richardson extrapolation. In another paper [3] he developed the Runge–Kutta methods on the basis of generalized interpolation. H. Engels has concentrated his investigations on constructive aspects.

In [9] a further class of functions $\{g_l(z),\ l = 0, 1, \ldots\}$ is introduced into generalized interpolation representations. The investigation of interpolations with pairs of functions $(h_j(z),\ g_j(z))$, $j = 0, 1, \ldots,$ leads to a better theoretical understanding of generalized interpolation and to some new results. In the following we give the main ideas of [9].

In Section I.1 we define some generalized interpolation representations and write down their properties. Then we are able to classify different generalized interpolation representations into types.

Each of these types is identified by a choice of a set of pairs of functions $\{(h_j(z),\ g_j(z)),\ j = 0, 1, \ldots\}$, and is characterized by some essential properties.

In Section I.2 we show that two types of interpolation are strongly connected with orthonormal series in $H^2(\mathfrak{B})$ spaces. The concept of interpolation types seems to be very advantageous in applications. And so, in Chapter II we consider the approximate numerical solution of initial value problems for ordinary differential equations, by means of certain types of generalized interpolation.

In the sequel we use, along with a set of points $\{x_j,\ j = 0, 1, \ldots\}$, two sets of functions

$$\{h_j(z),\ j = 0, 1, \ldots\}, \qquad \{g_j(z),\ j = 0, 1, \ldots\}.$$

We assume that the variable z and the points x_j, $j = 0, 1, \ldots,$ are elements of $\mathfrak{B}$, where $\mathfrak{B}$ is a finite or infinite connected domain of the complex z-plane. We require that

$$(1) \qquad h_j(x_j) = 0, \quad h_j(x_i) \neq 0, \quad \text{for } j \neq i,$$

$$(2) \qquad g_j(z) \neq 0, \quad \forall z \in \mathfrak{B}.$$

1. Definition and properties of some generalized interpolation representations

1.1. Two interpolation representations

Generalized Lagrange interpolation. Assume $x_j \neq x_k$, $j \neq k$, $j, k = 0, 1, \ldots$ Let us consider the following representation of $f(z)$:

(3) $$f(z) = L_n(f; z) + B_{n+1}(f; x_0, \ldots, x_n, z) W_n(z), \qquad n = 0, 1, 2, \ldots,$$

with

(4) $$L_n(f; z) = \sum_{r=0}^{n} f_r \prod_{\substack{j \neq r \\ j=0}}^{n} \frac{h_j(z)}{h_j(x_r)} \prod_{j=0}^{n} \frac{g_j(x_r)}{g_j(z)} \ [1]$$

and

(5) $$W_n(z) = \prod_{j=0}^{n} \frac{h_j(z)}{g_j(z)}.$$

Because of the requirements (1) and (2) imposed on h_j, g_j, the functions $L_n(f; z)$ and $W_n(z)$ fulfil the conditions

(6) $$L_n(f; x_r) = f_r, \qquad r = 0, 1, \ldots, n$$

and

(7) $$W_n(x_r) = 0, \qquad r = 0, 1, \ldots, n.$$

In (3) we have represented $f(z)$ by generalized Lagrange interpolation with a remainder term. Here L_n is called the generalized Lagrange operator and $L_n(f; z)$ the Lagrange function, both of order n.

In the classical case $(h_j(z) = z - x_j,\ g_j(z) \equiv 1,\ j = 0, 1, \ldots)$ the function $B_{n+1}(f; x_0, \ldots, x_n, z)$ is the divided difference of order $n+1$ of the function $f(z)$ with respect to $x_0, \ldots, x_n, z$. We call $B_{n+1}(\ldots)$ the generalized divided Lagrange difference of order $n+1$ of f with respect to $x_0, x_1, \ldots, x_n, z$.

Generalized Newton interpolation. In order to generalize Newton interpolation we choose the following representation of $f(z)$:

(8) $$f(z) = N_n(f; z) + A_{n+1}(f; x_0, \ldots, x_n, z) W_n(z)$$

with

(9) $$N_n(f; z) = \sum_{j=0}^{n} A_j \frac{g_j(x_j)}{g_j(z)} W_{j-1}(z),$$

$W_j(z)$ from (5) and $W_{-1}(z) = 1$.

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[1] We write $f_r = f(x_r)$.

The interpolation requirements are

$$N_n(f; x_r) = f_r, \quad r = 0, 1, \ldots, n. \tag{10}$$

If we assume that $x_j \neq x_k$ for $j \neq k$, $j, k = 0, 1, \ldots, n$, we can compute the coefficients A_j from

$$\sum_{j=0}^{r} A_j \frac{g_j(x_j)}{g_j(x_r)} W_{j-1}(x_r) = f_r, \quad r = 0, 1, \ldots, n. \tag{11}$$

In the case $h_j(z) = z - x_j$, $g_j(z) \equiv 1$, $j = 0, 1, \ldots, n$, we get the classical Newton formula.

In (8) we have represented $f(z)$ by generalized Newton interpolation with a remainder term. Here N_n is called the generalized Newton operator of order n and $N_n(f; z)$ the Newton function of order n.

We call the function $A_{n+1}(\ldots)$ the generalized divided Newton difference of order $n+1$ of f with respect to $x_0, \ldots, x_n, z$.

Remark. From a well-known theorem of complex function theory it follows that the Newton and the Lagrange functions converge to $f(z)$ as n tends to infinity, provided that the functions $f(z)$ and $h_j(z)$, $g_j(z)$, $j = 0, 1, \ldots, n$, are regular analytical in $\mathfrak{B}$ and the points x_j, $j = 0, 1, \ldots, n$ have an accumulation point in $\mathfrak{B}$.

1.2. Types of interpolation. We now briefly present some essential facts concerning generalized interpolation. We give no proofs here and refer to [9], but we illustrate the results by examples.

EXAMPLE 1. For $h_i(z) = z - x_i$, $g_i(z) \equiv 1$, $i = 0, 1, \ldots$, the generalized interpolation representations (3)–(7) and (8)–(11) are reduced to the classical Lagrange and Newton interpolation formulas. Their properties, in particular the identity

$$L_n(f; z) \equiv N_n(f; z), \quad n = 0, 1, \ldots,$$

are well known.

EXAMPLE 2. For rational interpolation we choose

$$h_i(z) = \frac{z - x_i}{1 - x_i z}, \quad g_i(z) \equiv 1, \quad |x_i| < 1, \quad i = 0, 1, \ldots, \quad |z| < 1.$$

The Lagrange and Newton functions for $n = 1$ are

$$L_1(f; z) = f_0 \frac{z - x_1}{1 - x_1 z} \cdot \frac{1 - x_1 x_0}{x_0 - x_1} + f_1 \frac{z - x_0}{1 - x_0 z} \cdot \frac{1 - x_0 x_1}{x_1 - x_0}$$

and

$$N_1(f;z) = A_0+A_1\frac{z-x_0}{1-x_0z}$$

with $A_0=f_0$ and $A_1=(f_1-f_0)(1-x_0x_1)/(x_1-x_0)$ from (11).

For the Newton interpolation we get

$$N_1(f;z) = f_0\frac{z-x_1}{1-x_0z}\cdot\frac{1-x_0^2}{x_0-x_1}+f_1\frac{z-x_0}{1-x_0z}\cdot\frac{1-x_0x_1}{x_1-x_0}.$$

We see that the interpolation conditions

$$L_1(f;x_r) = N_1(f;x_r) = f_r, \quad \text{for } r = 1, 2,$$

are fulfilled.

This example shows that the identity $L_1(f;z)\equiv N_1(f;z)$ is not naturally provided. Here it does not hold.

EXAMPLE 3. Another type of rational interpolation is obtained by choosing

$$h_i(z) = z-x_i, \quad g_i(z) = 1-x_iz, \quad i = 0, 1, \ldots$$

The Lagrange and Newton functions for $n=1$ are

$$L_1(f;z) = f_0\frac{z-x_1}{x_0-x_1}\cdot\frac{1-x_0^2}{1-x_0z}\cdot\frac{1-x_1x_0}{1-x_1z}+f_1\frac{z-x_0}{x_1-x_0}\cdot\frac{1-x_0x_1}{1-x_0z}\cdot\frac{1-x_1^2}{1-x_1z}$$

and

$$N_1(f;z) = A_0\frac{1-x_0^2}{1-x_0z}+A_1\frac{1-x_1^2}{1-x_1z}\cdot\frac{z-x_0}{1-x_0z}$$

with

$$A_0 = f_0, \quad A_1 = \big(f_1(1-x_0x_1)-f_0(1-x_0^2)\big)/(x_1-x_0).$$

If we expand the function $N_1(f;z)$ we see that in this case the identity $N_1(f;z)\equiv L_1(f;z)$ holds.

These examples lead to the questions:

For which kind of generalized interpolations does the identity between the Lagrange and Newton interpolation functions hold, and which of them have properties similar to those of the classical ones?

In [9] four types of interpolation are considered. Of these, the types I, II and III are the following:

I: $h_j(z) = u(z)-u(x_j)$, $g_j(z)\equiv 1$,

II: $h_j(z) = u(z)-u(x_j)$, $g_j(z) = 1-u(x_j)u(z)$,

III: $h_j(z) = u(z)v(x_j) - v(z)u(x_j)$, $g_j(z) = v(z)v(x_j) - u(z)u(x_j)$
(for each type $j = 0, 1, \ldots$).

The values of $u(x_j)$, $v(x_j)$ in $g_j(z)$ can also be replaced by their complex conjugates.

Let us remark that for $v(z) \equiv 1$ type III is identical with type II. If we use the notation

$$U(z) = \begin{cases} u(z) & \text{for type I and II,} \\ u(z)/v(z), & \text{for type III} \end{cases}$$

and require for all z:

(c1): $U(z)$ is single-valued,

(c2): $|U(z)| < 1$,

then the sets $\{h_j, g_j\}$ from types I, II, III fulfil the conditions (1) and (2). For type I the conditions (c2) is not necessary. We call $U(z)$ the basic function and $v^{-1}(z)$ the weight function of the interpolation.

THEOREM 1. *For types* I, II, III *the following identities hold*:

$$L_n(f; z) \equiv N_n(f; z),$$

$$A_{n+1}(f; \ldots) \equiv B_{n+1}(f; \ldots), \quad n = 0, 1, \ldots$$

We again turn attention to the examples.

Example 1 belongs to the interpolation type I and Example 3 to type II. Example 2 does not belong to any of these types. For interpolation of this kind we cannot give general theoretical assertions.

Therefore, in what follows we consider only interpolations of type I, II and III.

DEFINITION 1. $\Psi(z)$ is called a *fixed element of a linear interpolation operator* D_n if

$$D_n(\Psi; z) = \Psi(z), \quad \forall z \in \mathfrak{B}.$$

Also from the practical point of view it is desirable to characterize the fixed elements of interpolation operators.

In the classical case ($h_j(z) = z - x_j$, $g_i(z) \equiv 1$) the fixed elements of the operators L_n and N_n are $\sum_{j=0}^{n} c_j z^j$, with arbitrary constants c_j. For the generalized interpolation operators L_n and N_n of types I, II, III the fixed elements are:

I: $\sum_{j=0}^{n} c_j u^j(z)$,

II: $\sum_{j=0}^{n} c_j u^j(z) / \prod_{j=0}^{n} \big(1 - u_j u(z)\big)$,

$$\text{III: } \sum_{j=0}^{n} c_j u^j(z) v^{n-j}(z) \Big/ \prod_{j=0}^{n} \big(v(z)v_j - u(z)u_j\big)$$

with arbitrary complex or real c_j, $j = 0, 1, \ldots, n$, and $u_j = u(x_j)$, $v_j = v(x_j)$. We see that the fixed elements in type I are polynomials of the basic function $u(z)$. Further, we can see that the fixed elements in type II are rational functions of the basic function $u(z)$.

The degree of the numerator is n, and if $u_j \neq 0$ for $j = 0, 1, \ldots, n$, the degree of the denominator is $n+1$. Using the above defined basic function $U(z)$ we rewrite the fixed elements in type III in the form

$$\text{III: } \frac{1}{v(z)} \Big(\sum_{j=0}^{n} d_j U^j(z) \Big/ \prod_{j=0}^{n} \big(1 - U_j U(z)\big) \Big).$$

This means that the fixed elements in type III are rational functions in $U(z) = u(z)/v(z)$ multiplied by the weight function $v^{-1}(z)$.

1.3. Taylor series and Hermite interpolation. For the Lagrange and Newton representations we have assumed the points x_j, $j = 0, 1, \ldots$, to be distinct. For the proofs of some properties of interpolation operators (for example, the identities from Theorem 1 and the fixed elements) the condition $x_j \neq x_k$, for $j \neq k$ is not necessary.

If now x_l tends to x_k for some l and k, then the interpolation operator changes its shape, but its essential properties remain inaffected.

It is convenient to use the following concept for interpolation of types I, II and III. We define Taylor operators for type I. The interrelations between the types I through III allow us to use this definition for all the three types.

With the help of Taylor operators we can construct generalized Hermite operators for a general choice of arguments x_j.

To indicate that an interpolation function depends on the points x_j, we write

$$D_n(f; z) = D_n(f; z; x_0, \ldots, x_n).$$

DEFINITION 2. The operator T_n defined by

$$T_n(f; z; x_0) = N_n(f; z; x_0, \ldots, x_0)$$

is called the *Taylor operator of type* I *with respect to* x_0.

Because of the relations between the types I through III we set

$$T_n^{\mathrm{II}}(f; z; x_0) = g_0^{-(n+1)}(z) T_n^{\mathrm{I}}(f g_0^{n+1};\ z; x_0),$$
$$T_n^{\mathrm{III}}(f; z; x_0) = v^{-1}(z) g_0^{-(n+1)}(z) T_n^{\mathrm{I}}(f \cdot v g_0^{n+1}; z; x_0).$$

For type III we have assumed that T_n is viewed with respect to the basic function $U(z) = u(z)/v(z)$. For $h_0(z) = u(z) - u_0$ the operator T_n possesses the fixed elements $\sum_{j=0}^{n} c_j u^j(z)$. Also for the operators T_n of type II and III we can see that the fixed elements are the same as in the case of the operators L_n and N_n of the same type. Only the roots of the denominator are different.

DEFINITION 3. A Lagrange operator L_n is called a *Hermite operator* if at least two points of the set $\{x_j,\ j = 0, 1, \ldots, n\}$ are identical.

The Hermite operator

$$H_n(f; z) = L_{2n-1}(f; z; x_1, x_1, x_2, x_2, \ldots, x_n, x_n)$$

is of special interest for the derivation of quadrature formulas of Gaussian type (see [9]).

For our interpolation types I and II we obtain, for instance,

$$\text{(12)}\qquad H_n(f; z) = \sum_{r=1}^{n} \prod_{\substack{i\neq r\\ i=1}}^{n} \left(\frac{h_i(z)}{h_i(x_r)}\right)^2 \prod_{i=1}^{n} \left(\frac{g_i(x_r)}{g_i(z)}\right)^2 \times$$
$$\times \left\{ f_r \left[1 - 2\frac{h_r(z)}{h_r'(x_r)} \left(\sum_{j=1}^{n} \frac{1}{g_j'(x_r)} - \sum_{\substack{j\neq r\\ j=1}}^{n} \frac{1}{h_j'(x_r)} \right) \right] + f_r' \frac{h_r(z)}{h_r'(x_r)} \right\}$$

with

$$\text{(13)}\qquad H_n(f; x_r) = f_r, \quad H_n'(f; x_r) = f_r', \quad r = 1, 2, \ldots, n,$$

and the fixed elements

$$\text{(14)}\qquad \Psi_{2n-1}(z) = \prod_{i=1}^{n} \frac{1}{g_i^2(z)} \sum_{j=0}^{2n-1} c_j U^j(z).$$

If we multiply the right hand side of (12) by $v^{-1}(z)$ and replace f_r by $f_r v_r$ and f_r' by $f_r v_r' + f_r' v_r$ then (12) holds for type III, too. In this case we have assumed $h_j(z) = U(z) - U_j$, $g_j(z) = 1 - \overline{U}_j U(z)$. From this we get the fixed elements $\Psi_{2n-1}(z)$ from (14) divided by $v(z)$.

We obtain a rather general method for the construction of Hermite interpolations by using the following lemma:

LEMMA 1. *For $k \leqslant n$ the following relation holds:*

$$L_n(f; z; x_0, \ldots, x_n) = \prod_{i=k+1}^{n} \frac{h_i(z)}{g_i(z)} L_k\left(f \prod_{i=k+1}^{n} \frac{g_i}{h_i}; z; x_0, \ldots, x_k\right) +$$
$$+ \prod_{i=0}^{k} \frac{h_i(z)}{g_i(z)} L_{n-k-1}\left(f \prod_{i=0}^{k} \frac{g_i}{h_i}; z; x_{k+1}, \ldots, x_n\right).$$

If we want to obtain a Hermite interpolation with $x_0 = x_1 = \ldots = x_k$, we have to replace the L_k from Lemma 1 by the corresponding Taylor function and $h_i, g_i, i = 0, 1, \ldots, k$, by h_0 and g_0.

2. Interpolation and orthonormal systems in $H^2(\mathfrak{B})$ spaces

The aim of this section is to show that the Newton interpolations of type II and III are orthonormal series in suitable $H^2(\mathfrak{B})$ spaces. For the Newton series of type II we use a space with an inner product defined on the boundary $\mathfrak{C}$ of the domain $\mathfrak{B}$. In the case of type III we have an inner product with a distribution function $V(z) \not\equiv 1$. In both cases the basic function $U(z) = u(z)/v(z)$ is the generating element of an orthonormal system in $H^2(\mathfrak{B})$. Again we give results from [9].

2.1. The space $H^2(\mathfrak{B})$

ASSUMPTION 1. Let $\mathfrak{B}$ be a bounded convex simply connected domain in the complex z-plane with rectifiable boundary $\mathfrak{C}$ of length $\lambda = \lambda(\mathfrak{C})$.

By introducing the inner product

$$(f, g) = \frac{1}{\lambda} \int_{\mathfrak{C}} f(t)\overline{g(t)}\, ds \qquad (ds = |dt|), \tag{1}$$

we get the space $H^2(\mathfrak{B})$ with the norm

$$\|f\| = \frac{1}{\lambda} \Big(\int_{\mathfrak{C}} |f(t)|^2 ds \Big)^{1/2}. \tag{2}$$

In $H^2(\mathfrak{B})$ there exists an orthonormal system $\{\varphi_i\}$:

$$(\varphi_i, \varphi_j) = \delta_{ij}. \tag{3}$$

For $f \in H^2(\mathfrak{B})$ it yields the expansion

$$g(z) = \sum_{j=0}^{\infty} a_j \varphi_j(z), \qquad a_j = (g, \varphi_j),$$

with

$$\sum_{j=0}^{\infty} |a_j|^2 \leqslant \|f\|^2 \qquad \text{(Bessel inequality)}.$$

If $\{\varphi_j\}$ is a closed system, it follows that

$$g(z) \equiv f(z)$$

and the equality in the Bessel inequality holds. Then the kernel function of the system $\{\varphi_j\}$

$$K(z, t) = \sum_{j=0}^{\infty} \varphi_j(z)\overline{\varphi_j(t)}$$

has the reproducing property

$$f(z) = \big(f(t), K(z, t)\big), \qquad \text{for } f \in H^2(\mathfrak{B}). \tag{4}$$

2.2. Basic functions

ASSUMPTION 2. Let $u(z)$ be a single-valued function in $\overline{\mathfrak{B}} = \mathfrak{B} \cup \mathfrak{C}$ with $|u(z)| = 1$ on $\mathfrak{C}$ and with a single root in the interior of $\mathfrak{B}$.

Let us remark that Assumption 2 expresses a connection between the domain $\mathfrak{B}$ and a function $u(z)$, which we choose as our basic function. It is clear that $u(z)$ defines a conformal mapping of the domain $\mathfrak{B}$ onto the unit disc. We give some lemmas.

LEMMA 1. *The system $\{u^j(z),\ j = 0, 1, \ldots\}$ is orthonormal and closed in $H^2(\mathfrak{B})$.*

The orthonormality is easy to prove by using the definition of the inner product in $H^2(\mathfrak{B})$ and the fact that $|u(z)| = 1$ on $\mathfrak{C}$. Since $u(z)$ has a single root in the interior of $\mathfrak{B}$, the system is closed in $H^2(\mathfrak{B})$. So we can write the kernel function in the form

$$K(z, t) = \sum_{j=0}^{\infty} u^j(z)\overline{u^j(t)} = \frac{1}{1 - u(z)\overline{u(t)}}. \tag{5}$$

2.3. Kernel expansion and interpolation

LEMMA 2. *For $a \in \mathfrak{B}$ ($|u(a)| < 1$) it holds:*

$$\begin{aligned} K(z, t) &= \frac{1}{1 - u(z)\overline{u(t)}} \\ &= \frac{1 - |u(a)|^2}{\big(1 - u(z)\overline{u(a)}\big)\big(1 - \overline{u(t)}u(a)\big)} + \\ &\quad + \frac{\big(u(z) - u(a)\big)\big(\overline{u(t)} - \overline{u(a)}\big)}{\big(1 - u(z)\overline{u(a)}\big)\big(1 - \overline{u(t)}u(a)\big)} \, \frac{1}{1 - u(z)\overline{u(t)}}. \end{aligned}$$

We can prove this lemma by elementary operations. Here $\bar{d}$ denotes the complex conjugate of d.

The repeated application of this lemma for any constants $x_0, x_1, \ldots$ with $|u(x_j)| < 1$, $j = 0, 1, \ldots$, leads to the following theorem.

THEOREM 1. *For* $x_j \in \mathfrak{B}$, $j = 0, 1, \ldots$, *the expansion*

$$K(z,t) = \sum_{j=0}^{\infty} p_j(z)\overline{p_j(t)},$$

with

$$p_j(z) = \frac{\sqrt{1-|u_j|^2}}{1-\overline{u_j}u(z)} \prod_{i=0}^{j-1} \frac{u(z)-u_i}{1-\overline{u_i}u(z)}$$

and

$$(p_i, p_k) = \delta_{ik}$$

is valid.

The orthonormality $(p_j, p_k) = \delta_{jk}$ can be proved by using the relation $\left|\dfrac{u(z)-u(a)}{1-\overline{u(a)}u(z)}\right| = 1$ for $z \in \mathfrak{C}$.

Now we represent the function $f(z)$ by the system of orthonormal functions $\{p_i(z)\}$.

THEOREM 2. *Let* $f \in H^2(\mathfrak{B})$, $x_j \in \mathfrak{B}$, $j = 0, 1, \ldots$ *If there is an accumulation point for the sequence* $\{x_j\}$ *in* $\mathfrak{B}$, *then*

$$f(z) = \sum_{i=0}^{\infty} A_i(1-|u_i|^2)q_i(z),$$

with

$$q_i(z) = \frac{1}{\sqrt{1-|u_i|^2}}\, p_i(z) \quad \textit{and} \quad A_i = (f, q_i).$$

The series converges absolutly and uniformly in $\mathfrak{B}$.

The assumption about the accumulation point of $\{x_j\}$ is needed in order that the identity theorem of complex function theory be applicable.

From Theorem 2 we get

$$\|f\|^2 = \sum_{i=0}^{\infty} |A_i|^2(1-|u_i|^2). \tag{6}$$

Because of the property $p_j(x_k) = 0$, $k = 0, 1, \ldots, j-1$, we can compute the coefficients A_j from

$$f(x_r) = \sum_{j=0}^{r} A_j(1-|u_j|^2)q_j(x_r), \qquad r = 0, 1, \ldots, \tag{7}$$

$$f_0 = A_0,$$

$$f_1 = A_0\frac{1-|u_0|^2}{1-\overline{u}_0u_1}+A_1\frac{u_1-u_0}{1-\overline{u}_0u_1},$$

$$\dots\dots\dots\dots\dots\dots\dots$$

It is clear that Theorem 2 provides an interpolation representation of type II. This is a generalized Newton interpolation if we set

$$h_l(z) = u(z)-u(x_l), \qquad g_l(z) = 1-\overline{u(x_l)}u(z). \tag{8}$$

If we now write

$$F_n(z) = \sum_{l=0}^{n} A_l(1-|u_l|^2)q_l(z) \tag{9}$$

then from Theorem 1 from I.1 follows

THEOREM 3. *The equality*

$$F_n(z) = N_n(f;z) \equiv L_n(f;z)$$

holds with $h_l(z)$, $g_l(z)$ *and* $F_n(z)$ *defined by* (8) *and* (9).

Because of the orthonormality of the functions $\{p_l(z)\}$ we obtain

COROLLARY 1. *Let* $F^* \in H^2(\mathfrak{B})$ *and* $F^*(x_r) = f(x_r)$, $r = 0, 1, \dots, n$; *then* $F_n(z)$ *is the unique function such that*

$$\|F_n\| = \min_{F^*\in H^2(\mathfrak{B})} \|F^*\|.$$

In connection with Corollary 1 let us remark that the interpolation of type II is optimal in $H^2(\mathfrak{B})$ with respect to the chosen points $x_0, \dots, x_n$. This proposition holds also for quadratures based on interpolation of type II. In this case we obtain Wilf's quadratures by using $u(z) = z$ (see [9], Chapter III).

2.4. Examples. Now we give some examples and discuss the possibilities of the choice of $\mathfrak{B}$ or u.

First we restrict our investigation to holomorphic functions $u(z)$. The following two cases are possible.

(a) The domain $\mathfrak{B} = \{z \mid |u(z)| < 1, z \in C\}$ is bounded. This means that the boundary curve $|u(z)| = 1$ is closed. Here it is possible that an infinite number of domains exist for one basic function $u(z)$.

(b) The domain $\mathfrak{B}$ is unbounded, which means that the curve $|u(z)| = 1$ is not closed.

In this case it can happen that we do not need the whole boundary ($|u(z)| = 1$) of $\mathfrak{B}$ to define the inner product in $H^2(\mathfrak{B})$.

EXAMPLE 3′. We consider Example 3 from Section I.1 in more details. The function $u(z) = z$ gives the identical mapping of the unit disc $\mathfrak{E}$, onto itself; therefore $\mathfrak{B} = \mathfrak{E} = \{z \mid |z| < 1\}$, $\mathfrak{C} = \{z \mid |z| = 1\}$. Thus we consider the space $H^2(\mathfrak{B})$. Let us require $|x_l| < 1$, $l = 0, 1, \ldots$ There exists the orthogonal system $\{z^l, l = 0, 1, \ldots\}$ and the kernel function $K(z, t) = \frac{1}{1-z\bar{t}}$. Besides, from Theorem 1 we get another orthonormal system in $H^2(\mathfrak{E})$

$$p_j(z) = \frac{\sqrt{g_j(x_j)}}{g_j(z)} \prod_{i=0}^{j-1} \frac{h_i(z)}{g_i(z)} = \frac{\sqrt{1-|x_j|^2}}{1-\bar{x}_j z} \prod_{i=0}^{j-1} \frac{z-x_i}{1-\bar{x}_i z}$$

and the kernel expansion

$$K(z, t) = \frac{1}{1-z\bar{t}} = \sum_{j=0}^{\infty} p_j(z)\overline{p_j(t)}.$$

According to Theorem 2, we write the orthonormal series for a function $f(z)$ in the form

$$f(z) = F_n(z) + A_{n+1}(f; x_0, \ldots, x_n, z) \prod_{i=0}^{n} \frac{h_i(z)}{g_i(z)}$$

with

$$F_n(z) = \sum_{j=0}^{n} A_j \frac{g_j(x_j)}{g_j(z)} \prod_{i=0}^{j-1} \frac{h_i(z)}{g_i(z)} = \sum_{j=0}^{n} A_j \frac{1-|x_j|^2}{1-\bar{x}_j z} \prod_{i=0}^{j-1} \frac{z-x_i}{1-\bar{x}_i z},$$

$n = 0, 1, \ldots$, where the coefficients A_j can be computed from (7).

This orthogonal series F_n was found by Takenaka [4] in 1925. The remainder term is called the Walsh remainder term [5].

We know that

$$F_n(z) = N_n(f; z) \equiv L_n(f; z)$$

with

$$L_n(f; z) = \sum_{r=0}^{n} f_r \prod_{\substack{i \neq r \\ i=0}}^{n} \frac{z-x_i}{x_r-x_i} \prod_{i=0}^{n} \frac{1-\bar{x}_i x_r}{1-\bar{x}_i z}.$$

The fixed elements of L_n and N_n are

$$\sum_{j=0}^{n} c_j z^j \Big/ \prod_{j=0}^{n} (1-\bar{x}_j z).$$

This interpolation of the type II is a rational one. The degree of the numerator is n. The degree of the denominator is $n+1-k$, where the integer k is the number of the points x_l, $l = 0, 1, \ldots, n$, which are equal to zero. Because of the remark after Corollary 1, this rational interpolation is optimal in $H^2(\mathfrak{C})$ with respect to the points x_j, $j = 0, 1, \ldots, n$. Let us write down the norms in $H^2(\mathfrak{C})$:

$$\|f\|^2 = \sum_{j=0}^{\infty} |A_j|^2(1-|x_j|^2)$$

and

$$\|N_n(f;z)\|^2 = \|L_n(f;z)\|^2 = \sum_{j=0}^{n} |A_j|^2(1-|x_j|^2).$$

EXAMPLE 4. We choose $u(z) = e^{-z}$. The domain $\mathfrak{B}$ with $|u(z)| < 1$ is given by $\mathrm{Re}(z) > 0$. This means that the function $u(z)$ defines a conformal mapping of the right half-plane onto the unit disc. If we choose $\mathfrak{C}$: $0 \leqslant \mathrm{Im}(z) < 2\pi$, then the set $\{e^{-lz}, l = 0, 1, \ldots\}$ is a closed orthogonal system in $H^2(\mathfrak{B})$ with the inner product

$$(f, g) = \frac{1}{2\pi} \int_{\mathfrak{C}} f(t)\overline{g(t)}\, ds, \qquad ds = |dt|.$$

The kernel function and its interpolatorical expansion are

$$K(z, t) = \frac{1}{1-e^{-z}\overline{e^{-t}}} = \sum_{j=0}^{\infty} p_j(z)\overline{p_j(t)},$$

$$p_j(z) = \frac{(1-|e^{-2x_j})^{1/2}}{1-\overline{e^{-x_j}}e^{-z}} \prod_{i=0}^{j-1} \frac{e^{-z}-e^{-x_i}}{1-\overline{e^{-x_i}}e^{-z}}.$$

The Newton interpolation is again an orthogonal series. We write down the Newton and the Lagrange interpolation functions:

$$N_n(f;z) = \sum_{j=0}^{n} A_j \frac{1-|e^{-x_j}|^2}{1-\overline{e^{-x_j}}e^{-z}} \prod_{i=0}^{j-1} \frac{e^{-z}-e^{-x_i}}{1-\overline{e^{-x_i}}e^{-z}},$$

$$L_n(f;z) = \sum_{r=0}^{n} f_r \prod_{\substack{i\neq r\\ i=0}}^{n} \frac{e^{-z}-e^{-x_i}}{e^{-x_r}-e^{-x_i}} \prod_{i=0}^{n} \frac{1-\overline{e^{-x_i}}e^{-x_r}}{1-\overline{e^{-x_i}}e^{-z}}.$$

Of course, they are identical. The fixed elements are

$$\sum_{j=0}^{n} c_j e^{-jz} \Big/ \prod_{i=0}^{n} (1-\overline{e^{-x_i}}e^{-z}).$$

These interpolations are optimal in $H^2(\mathfrak{B})$ with respect to the points x_i, $i = 0, 1, \dots, n$.

We have in $H^2(\mathfrak{B})$

$$\|f\|^2 = \prod_{j=0}^{\infty} |A_j|^2(1-|e^{-x_j}|^2)$$

and

$$\|N_n(f; z)\|^2 = \|L_n(f; z)\|^2 = \sum_{j=0}^{n} |A_j|^2(1-|e^{-x_j}|^2).$$

EXAMPLE 5. We now give an example for type III. Let $u(z) = zv(z)$ where $v(z)$ is arbitrary such that $v(z) \neq 0$ for $|z| \leqslant 1$. With $U(z) = u(z)/v(z) = z$ we obtain the interpolation functions

$$\begin{aligned} L_n(f; z) &= \sum_{r=0}^{n} f_r \prod_{\substack{i\neq r\\ i=0}}^{n} \frac{u(z)v_i - v(z)u_i}{u_r v_i - v_r u_i} \prod_{i=0}^{n} \frac{v_r\bar{v}_i - u_r\bar{u}_i}{v(z)\bar{v}_i - u(z)\bar{u}_i} \\ &= \frac{1}{v(z)} \sum_{r=0}^{n} f_r v_r \prod_{\substack{i\neq r\\ i=0}}^{n} \frac{U(z)-U_i}{U_r - U_i} \prod_{i=0}^{n} \frac{1-\bar{U}_i U_r}{1-\bar{U}_i U(z)} \end{aligned}$$

and

$$\begin{aligned} N_n(f; z) &= \sum_{j=0}^{n} A_j \frac{|v_j|^2 - |u_j|^2}{v(z)\bar{v}_j - u(z)\bar{u}_j} \prod_{i=0}^{j-1} \frac{u(z)v_i - v(z)u_i}{v(z)\bar{v}_i - u(z)\bar{u}_i} \\ &= \frac{1}{v(z)} \sum_{j=0}^{n} A_j^* v_j \frac{1-|U_j|^2}{1-\bar{U}_j U(z)} \prod_{i=0}^{j-1} \frac{U(z)-U_i}{1-\bar{U}_i U(z)}. \end{aligned}$$

We know that the fixed elements are

$$\frac{1}{v(z)}\Big(\sum_{i=0}^{n} d_i U^i(z) \Big/ \prod_{i=0}^{n} \big(1-U_i U(z)\big)\Big).$$

The connection with the formulas of type II can easily be seen. If we use the notation $L_n^{\mathrm{II}}, N_n^{\mathrm{II}}, A_n^{\mathrm{II}}$ for type II and $L_n^{\mathrm{III}}, N_n^{\mathrm{III}}, A_n^{\mathrm{III}}$ for type III, then under the condition $U(z) = u(z)/v(z)$, with $v(z) \neq 0$ for $z \in \mathfrak{B}$, we can write

$$\begin{aligned} v(z)L_n^{\mathrm{III}}(f; z) &= L_n^{\mathrm{II}}(fv; z), \\ v(z)N_n^{\mathrm{III}}(f; z) &= N_n^{\mathrm{II}}(fv; z), \\ v_i A_i^{\mathrm{III}}(f) &= A_i^{\mathrm{II}}(fv). \end{aligned}$$

The orthogonality in the space $H^2(\mathfrak{B})$ is transferred to the space $H^2(\mathfrak{B}, V)$, where V is a distribution function. The space $H^2(\mathfrak{B}, V)$ is generated by the inner product

$$(f, g)_v = \frac{1}{\lambda} \int_{\mathfrak{C}} f(t)\overline{g(t)}\,|dV(t)| = \frac{1}{\lambda} \int_{\mathfrak{C}} f(t)\overline{g(t)}\overline{v^2(t)}\,ds,$$

with

$$\frac{dV(t)}{ds} = v^2(t).$$

The kernel is

$$K(z, t) = \frac{1}{v(z)\overline{v(t)} - u(z)\overline{u(t)}} = \frac{1}{v(z)\overline{v(t)}\left(1 - U(z)\,\overline{U(t)}\right)}.$$

In $H^2(\mathfrak{B}, V)$ the orthonormal series provides the norm

$$\begin{aligned}\|f\|^2 &= \sum_{j=0}^{\infty} |A_j^{\mathrm{III}}(f)|^2(|v_j|^2 - |u_j|^2) = \sum_{j=0}^{\infty} |A_j^{\mathrm{III}}(f)v_j|^2(1 - |U_j|^2)\\ &= \sum_{j=0}^{\infty} |A_j^{\mathrm{II}}(fv)|^2(1 - |U_j|^2).\end{aligned}$$

The equality holds between the norms of $H^2(\mathfrak{B}, V)$ and $H^2(\mathfrak{B})$:

$$\|f\|_V = \|fv\|.$$

II. LINEAR k-STEP METHODS FOR ORDINARY DIFFERENTIAL EQUATIONS

Let us define the general linear k-step method for the approximate numerical computation of the solution $y = y(t)$ of a system of ordinary differential equations of first order

$$\dot{y} = f(t, y), \quad y(0) = y_0 \quad (y \in \boldsymbol{R}^m,\ t \geqslant 0), \tag{1}$$

by the formula

$$y_{s+k} = \alpha_{k-1}y_{s+k-1} + \ldots + \alpha_0 y_s + \beta_k f_{s+k} + \beta_{k-1} f_{s+k-1} + \ldots + \beta_0 f_s. \tag{2}$$

Here it is assumed that the coefficients α_i, β_i, $i = 0, 1, \ldots, k-1$, β_k are real numbers, t_j are real numbers with $t_j < t_{j+1}$, and $f_j = f(t_j, y_j)$. If the vectors $y_0, y_1, \ldots, y_{k-1}$ are given then $y_k, y_{k+1}, \ldots$, can be computed from (2). This offers no difficulties, if the method is explicit ($\beta_k = 0$).

If $\beta_k \neq 0$, i.e. if the method is implicit, some conditions on t_j, $j = 0, 1, \ldots$, and on f are required in order to guarantee the existence and uniqueness of the vectors $y_k, y_{k+1}, \ldots$

Let us remark that we have not introduced the stepsize h in (2).

It is possible to choose the points t_j arbitrary real. The coefficients α_i, β_i depend on the t_j's.

1. Derivation of the methods

We assume that $t_0, \ldots, t_{k-1}, y_0, \ldots, y_{k-1}, f_0, \ldots, f_{k-1}$ are given. We want to compute y_k.

This is an extrapolation problem. We can solve this task formally if we use a generalized Lagrange interpolation formula $L_r(y, t)$, $r \leqslant 2k-1$, over some or all points $t_0, t_0, t_1, t_1, \ldots, t_{k-1}, t_{k-1}$, and compute $y_k = L_r(y; t_k)$. Here we always choose the special case of $r = 2k-1$.

We use the Lagrange function $L_{2k-1}(y; t)$ with

$$L_{2k-1}(y; t_r) = y_r, \quad L'_{2k-1}(y; t_r) = f_r, \quad r = 0, 1, \ldots, k-1, \tag{3}$$

and obtain

$$y_k = L_{2k-1}(y; t_k) = L_{2k-1}(y; t_k; t_0, t_0, \ldots, t_{2k-1}, t_{2k-1}). \tag{4}$$

We can apply this formula using the special Hermite form (I.1(12)) of the Lagrange formula. It is claer that (4) provides an explicit ($\beta_k = 0$) form of (2).

Now we give an implicit form of (2). For this we need an interpolation formula $L^*_{2k}(y; t)$ with

$$L^*_{2k}(y; t_r) = y_r, \quad r = 0, 1, \ldots, k-1,$$

and

$$L^{*\prime}_{2k}(y; t_r) = y'_r = f_r, \quad r = 0, 1, \ldots, k.$$

Using the interpolation formula L_{2k-1} from (3) we express $L^*_{2k}(y; t)$ by

$$L^*_{2k}(y; t) = L_{2k-1}(y; t) + \frac{y'_k - L'_{2k-1}(y; t_k)}{\varphi'_{2k}(t_k)} \varphi_{2k}(t), \tag{5}$$

with

$$\varphi_{2k}(t) = \prod_{i=0}^{k-1} \big(h_i(t)/g_i(t)\big)^2 \frac{g_k(x_k)}{g_k(t)}. \tag{6}$$

Then an implicit method is given by

$$y_k = L^*_{2k}(y; t_k). \tag{7}$$

Choosing $u(t) = t$ we get an example for the interpolation type I (classical interpolation) and $k = 1$. By (3) we get

$$L_1(y; t; t_0, t_0) = y_0 + (t - t_0) y'_0.$$

Taking $h = t_1 - t_0$ we obtain from (4) the explicit Euler method

$$y_1 = y_0 + hy_0' = L_1(y; t_1; t_0, t_0).$$

Let us now show an implicit method for $k = 1$. Setting $\varphi_2(t) = (t-t_0)^2$ we get

$$L_2^*(y; t) = y_0 + (t-t_0)y_0' + \frac{y_1' - y_0'}{2(t_1 - t_0)}(t-t_0)^2.$$

So (7) becomes

$$y_1 = y_0 + \frac{h}{2}(y_0' + y_1') = L_2^*(y; t_1).$$

This is the trapezoidal rule.

2. Properties of the methods

We restrict our investigations of the interpolation representations to the basic function $U(z) = z$. For the type I we meet the classical case of polynomial interpolation. Let us first consider this case.

If we choose $t_j - t_{j-1} = h$, $\forall j$, we have $\beta_j = h\beta_j^*$, $j = 0, 1, \ldots, k$, in (2) and α_j, β_j^* are real constants. Observe that we can derive all linear k-step methods (2) with constant coefficients α_j, β_j^* from the classical interpolation formulas. For this case the following two theorems of G. Dahlquist [6] are well known.

THEOREM 1. *An explicit k-step method cannot be A-stable.*

THEOREM 2. *The order p, of an A-stable linear multistep method cannot exceed* 2.

2.1. Explicit methods based on interpolation of type II. We are going to consider methods (2) employing interpolation of type II (methods of type II).

Now we use the formula I.1(12),

$$H_k(\varphi; z) = \sum_{r=0}^{k-1} \prod_{\substack{j \neq r \\ j=0}}^{k-1} \left(\frac{z - x_j}{x_r - x_j}\right)^2 \prod_{j=0}^{k-1} \left(\frac{1 - x_j x_r}{1 - x_j z}\right)^2 \times \tag{8}$$
$$\times \left\{\varphi_r \left[1 - 2(z - x_r)\left(\sum_{j=0}^{k-1} \frac{x_j}{1 - x_j x_r} + \sum_{\substack{j \neq r \\ j=0}}^{k-1} \frac{1}{x_r - x_j}\right)\right] + \varphi_r'(z - x_r)\right\}.$$

Let us set $\varphi_r = y_{s+r}$, $\varphi_r' = \varrho \cdot f_{s+r} = \varrho \cdot f(t_{s+r}, y_{s+r})$, $r = 0, 1, \ldots, k-1$ and let the real points x_j, $|x_j| < 1$, $j = 0, 1, \ldots, k$, be given by the linear transformation

$$x_j = \varrho(t_{s+j} - t_s) + x_0, \quad j = 0, 1, \ldots, k, \quad \varrho = \frac{x_1 - x_0}{t_{s+1} - t_s}. \tag{9}$$

Then we get an explicit linear k-step method (2) by carrying out the extrapolation

$$y_{k+s} = H_k(y; x_k), \quad s = 0, 1, \ldots \tag{10}$$

From (8) we get the coefficients a_r, β_r of the k-step method in the form

$$a_r = R_r\left[1 - 2(x_k - x_r)\left(\sum_{j=0}^{k-1} \frac{x_j}{1 - x_j x_r} + \sum_{\substack{j \neq r \\ j=0}}^{k-1} \frac{1}{x_r - x_j}\right)\right] \tag{11}$$

$$\beta_r = R_r(x_k - x_r), \tag{12}$$

with

$$R_r = \prod_{\substack{j \neq r \\ j=0}}^{k-1} \left(\frac{x_k - x_j}{x_r - x_j}\right)^2 \prod_{j=0}^{k-1} \left(\frac{1 - x_j x_r}{1 - x_j x_k}\right)^2, \tag{13}$$

for $r = 0, 1, \ldots, k-1$ and $\beta_k = 0$.

The order of consistency of this method is $2k-1$.

2.2. L-acceptability. Now we are going to investigate the properties of the explicit linear methods just described. We can consider the rational function (8) or, more generally, a function of interpolation type II, with $u(z) = z$ as a k-point Padé-type approximation [8].

If we want to show the A- or L-stability of a method (2) based on the rational function (8), we have to prove the A- or L-acceptability of (8).

DEFINITION 1 ([7], [8]). A rational approximation $R(z)$ to e^{-z} is said to be (i) *A-acceptable*, if $|R(z)| < 1$ whenever $\operatorname{Re}(z) > 0$, (ii) *$A(0)$-acceptable*, if $|R(z)| < 1$ whenever z is real and positive, and (iii) *L-acceptable*, if it is A-acceptable and, in addition, satisfies $|R(z)| \to 0$ as $\operatorname{Re}(z) \to +\infty$.

Let us consider the general interpolation function

$$L_n(\varphi; z) = L_n(\varphi; z; x_0, \ldots, x_n)$$

of type II with real x_j, $j = 0, 1, \ldots, n$, and $-1 < x_0 \leqslant x_1 \leqslant \ldots \leqslant x_n < 1$. According to Definition 1, we want to show that under certain conditions

$$|L_n(e^{-(z-x_0)}; z)| \leqslant 1, \quad \text{for all } z \text{ with } \operatorname{Re}(z) \geqslant x_0. \tag{14}$$

That is the A-acceptability of $L_n(e^{-(z-x_0)}; z)$.

In order to prove (14) we have to verify the conditions (see [8], p. 28):

1. $|L_n(e^{-(z-x_0)};\ x_0+it)| \leqslant 1,\ \forall t \in \boldsymbol{R}$,
2. $\lim\limits_{|z|\to\infty} |L_n(e^{-(z-x_0)};\ z)| = c \leqslant 1$,
3. $L_n(e^{-(z-x_0)};\ z)$ is analytic for $\operatorname{Re}(z) \geqslant x_0$.

The function $L_n(e^{-(z-x_0)}; z)$ is L-acceptable if $c = 0$ can be taken in condition 2. If conditions 2 and 3 hold only, then the approximation is $A(0)$-acceptable, and with $c = 0$ in condition 2 the approximation is called $L(0)$-acceptable.

We are interested in L- and $L(0)$-acceptability. The function $L_n(e^{-(z-x_0)}; z)$ is of the form

$$L_n(e^{-(z-x_0)}; z) = \sum_{j=0}^{n} c_j z^j \Big/ \prod_{j=0}^{n} (1-x_j z).$$

It is immediately clear that condition 2 with $c = 0$ and condition 3 are fulfilled, if we choose

$$-1 < x_j < 0, \quad j = 0, 1, \ldots, n. \tag{15}$$

Condition 2 with $c = 0$ is fulfilled, since the degree of the denominator is greater than the degree of the numerator. In order to verify condition 1 we consider the relation

$$|L_n(e^{-(z-x_0)};\ x_0+it)|^2 = \frac{1+a_1t^2+\ldots+a_nt^{2n}}{1+b_1t^2+\ldots+b_nt^{2n}+b_{n+1}t^{2n+2}} \leqslant 1 \tag{16}$$

for $t \in \boldsymbol{R}$. Since the rational function in (16) approximates the term $e^{-it}e^{+it} = 1$ with order $O(t^{n+1})$, it is sufficient to show that

$$b_r \geqslant a_r, \quad \text{for } r = \left[\frac{n+2}{2}\right], \ldots, n. \tag{17}$$

($[d]$ denotes the integer part of d.)

It is to be expected that we get a condition for the points x_j which restrict their position more than (15).

We are going to express the coefficients a_r, b_r in (16) by x_j's in order to prove (17). We can do this in the following way. We have

$$L_n^{\mathrm{II}}(e^{-(z-x_0)};\ x_0+\eta) = \frac{L_n^{\mathrm{I}}\left(e^{-(z-x_0)} \prod\limits_{j=0}^{n} \dfrac{1-x_jz}{1-x_jx_0};\ x_0+\eta\right)}{\prod\limits_{j=0}^{n} \dfrac{1-x_j(x_0+\eta)}{1-x_jx_0}} \tag{18}$$

where II and I denote interpolation types. Writing

$$c_j = x_j/(1-x_jx_0) \tag{19}$$

we get

$$L_n^{\mathrm{II}}(e^{-(z-x_0)};\ x_0+\eta) = \frac{1+d_1\eta+\ldots+d_n\eta^n}{\prod_{j=0}^{n}(1-c_j\eta)}. \tag{20}$$

We calculate the coefficients d_j from the identity

$$\left(1-\eta+\frac{1}{2}\eta^2-\frac{1}{6}\eta^3+\ldots+(-1)^n\frac{1}{n!}\eta^n\right)\prod_{j=0}^{n}(1-c_j\eta) = 1+d_1\eta+\ldots+d_n\eta^n+O(\eta^{n+1}), \tag{21}$$

finding that

$$\begin{aligned} d_1 &= -1-\sum_{j=0}^{n}c_j,\\ d_2 &= \tfrac{1}{2}+\sum_{j=0}^{n}c_j+\sum_{j=0}^{n}\sum_{r>j}^{n}c_jc_r,\\ d_3 &= -\tfrac{1}{6}-\tfrac{1}{2}\sum_{j=0}^{n}c_j-\sum_{j=0}^{n}\sum_{r>j}^{n}c_jc_r-\sum_{j=0}^{n}\sum_{r>j}^{n}\sum_{s>r}^{n}c_jc_rc_s, \ \ldots \end{aligned} \tag{22}$$

Using (20) we obtain the function occurring in (16) in the form

$$|L_n^{\mathrm{II}}(e^{-(z-x_0)};\ x_0+it)|^2 = \frac{1+a_1t^2+\ldots+a_nt^{2n}}{\prod_{j=0}^{n}(1+c_j^2t^2)} \tag{23}$$

where

$$a_k = d_k^2+2\sum_{j=1}^{n}(-1)^j d_{k-j}d_{k+j}, \tag{24}$$

with $d_0=1$ and $d_{n+j}=0$, $d_{-j}=0$, for $j\geqslant 1$.

Now we write the coefficients b_k in (16) in the form

$$\begin{aligned} b_{n+1} &= \prod_{j=0}^{n}c_j^2,\\ b_n &= \sum_{r=0}^{n}\prod_{j\neq r}^{n}c_j^2,\\ b_{n-1} &= \sum_{r=0}^{n}\sum_{s>r}^{n}\prod_{j\neq r,s}^{n}c_j^2, \ \ldots, \end{aligned} \tag{25}$$

and from (24) and (22) we obtain the coefficients a_k:

$$a_n = \left\{\frac{1}{n!} + \frac{1}{(n-1)!}\sum_{j=0}^{n} c_j + \frac{1}{(n-2)!}\sum_{r=0}^{n}\sum_{s>r}^{n} c_r c_s + \ldots + \sum_{r=0}^{n}\prod_{j\neq r}^{n} c_j\right\}^2,$$

$$a_{n-1} = \left\{\frac{1}{(n-1)!} + \frac{1}{(n-2)!}\sum_{j=0}^{n-1} c_j + \frac{1}{(n-3)!}\sum_{j=0}^{n-1}\sum_{r>j}^{n-1} c_j c_r + \right.$$
$$\left. + \ldots + \sum_{r=0}^{n-1}\prod_{j\neq r}^{n-1} c_j\right\}^2 -$$

(26) $$-2\left\{\frac{1}{(n-2)!} + \frac{1}{(n-3)!}\sum_{j=0}^{n-2} c_j + \ldots + \sum_{r=0}^{n-2}\prod_{j\neq r}^{n-2} c_j\right\}\times$$
$$\times\left\{\frac{1}{n!} + \frac{1}{(n-1)!}\sum_{j=0}^{n} c_j + \ldots + \sum_{r=0}^{n}\prod_{j\neq r}^{n} c_j\right\}, \ldots$$

So we have represented the coefficients a_k, b_k in the form

$$a_k(c_0, \ldots, c_n) \quad \text{and} \quad b_k(c_0, \ldots, c_n), \quad \text{with } c_j\text{'s given by (19).}$$

From (25) and (26) we are able to derive condition (17) for given $c_j < 0$, $j = 0, 1, \ldots, n$.

The existence of a vector $(c_0, \ldots, c_n)$ which fulfils (17) is immediately clear because there are always c_j, $j = 0, 1, \ldots, n$ with

$$a_r(c_0, \ldots, c_n) \leqslant 0, \quad r = \left[\frac{n+2}{2}\right], \ldots, n.$$

For the coefficients in (25) the inequality $b_k \geqslant 0$, $\forall k$, holds. Further, it is clear that for $c_j = 0$ and for $c_j \to -\infty$, $j = 0, 1, \ldots, n$, the condition (17) is not valid. But there exists a range $[\Gamma_n^*, \gamma_n^*]$ such that for $c_j \in [\Gamma_n^*, \gamma_n^*]$, $j = 0, 1, \ldots, n$, condition (17) is fulfilled.

For $n = 1, \ldots, 10$ the condition

(27) $$c_j \in [-1, -\tfrac{1}{3}], \quad j = 0, 1, \ldots, n,$$

is sufficient for the inequality (17) to hold. This means that constants Γ_n, γ_n, $n = 1, 2, \ldots, 10$, exist, with

(28) $$-1 < \Gamma_n \leqslant -0.618034, \qquad -\frac{1}{3}\Big/\left(1 + \frac{0.618034}{3}\right) \leqslant \gamma_n < 0,$$

such that

(29) $$x_j \in [\Gamma_n, \gamma_n]$$

is sufficient for (17).

For $n = 1, 2, 3$ we can give the exact constants Γ_n, γ_n:

$$\Gamma_1 = -0.7491, \quad \Gamma_2 = -0.879, \quad \Gamma_3 = -0.82,$$
$$\gamma_1 = -0.2714, \quad \gamma_2 = -0.176, \quad \gamma_3 = -0.277.$$

Summarizing, we formulate

THEOREM 3. *Given $n \geqslant 1$. If $x_j \in (-1, 0)$, $j = 0, 1, \ldots, n$, then $L_n(e^{-(z-x_0)}; z)$ is $L(0)$-acceptable and there are two constants Γ_n, γ_n with $-1 < \Gamma_n < \gamma_n < 0$, such that $x_j \in [\Gamma_n, \gamma_n]$, $j = 0, 1, \ldots, n$ is sufficient for the L-acceptability of $L_n(e^{-(z-x_0)}; z)$.*

2.3. Stability. Now we state certain stability properties of the explicit method (2) constructed on the basis of interpolation of type II.

We restrict our attention to the special Hermite form (8) of $L_n(g, z)$ with $n = 2k-1$ and with the points $x_0, x_0, x_1, x_1, \ldots, x_{k-1}, x_{k-1}$.

THEOREM 4. *The explicit ($\beta_k = 0$) method* (2) *with coefficients given by* (11), (12), (13) *and*

$$x_j = \varrho(t_{s+j} - t_s) + x_0, \quad \varrho = \frac{x_1 - x_0}{t_{s+1} - t_s}$$

is:

(i) *$L(0)$-stable if $-1 < x_j < 0$, $j = 0, 1, \ldots, k-1$,*

(ii) *L-stable if $x_j \in [\Gamma_n, \gamma_n]$, $j = 0, 1, \ldots, k-1$ for suitable Γ_n, γ_n with $-1 < \Gamma_n < \gamma_n < 0$, and suitable ϱ,*

(iii) *stiff-stable if* (ii) *and $|x_k| < 1$.*

The order of consistency is $2k-1$.

The assertions (i) and (ii) are true in view of Theorem 3. The numerical solution of the differential equation (1) is accurate for $|x_k| < 1$, because the function (8) or, more generally, $L_n(g; z)$, converges absolutely and uniformly for $|z| < 1$ as $n \to \infty$. For $x_k \geqslant 1$ the L-stability is secured if $x_j \in [\Gamma_n, \gamma_n]$. So the method is stiff-stable (see [7]) for $|x_k| < 1$.

Remarks. By formula (9) we have transformed the initial value problem from the t-plane into the unit circle of the z-plane. Of course, we can express the coefficients α_r, β_r, $r = 0, 1, \ldots, k-1$, of the linear explicit method (2) by the original points t_j if we use (9).

Theorem 4 can be interpreted in the following way. Let the points $t_s, t_{s+1}, \ldots, t_{s+k}$ be given. By means of the transformation (9) a circle with radius $R = (t_{s+1} - t_s)/(x_1 - x_0)$ and centre $t_M = t_s - x_0 R$ is defined in the t-plane. Only the positions of the points t_{s+j} in this circle, $j = 0, 1, \ldots, k$, and the radius R determine the stability properties of explicit methods (2) based on interpolation of type II.

In this way the coefficients α_r, β_r from (11) and (12) depend on the points t_{s+j-1}. If we work with a constant stepsize $h = t_{s+j} - t_{s+j-1}$, $j = 0, 1, \ldots,$ the coefficients $\alpha_r = \alpha_r(h)$, $\beta_r = \beta_r(h)$ are the same constants for all computations of the vectors y_{s+k}, $s = 0, 1, \ldots$ But if it is necessary to change the stepsize, we have to provide new coefficients.

2.4. Introduction of parameters. Linear explicit methods (2) of type II are optimal in H^2-spaces with respect to the points x_j used. In general, it is possible to optimize over these points (see [9]). Besides, we can extend the methods of Theorem 4 to the spaces $H^2(\mathfrak{E}, V)$. This means that we can consider methods which are constructed on the basis of the interpolation type III. We can fix a weight function $v(z)$ in the linear methods of Theorem 4. This can be easily done because of the relation between the types II and III:

$$L_n^{\mathrm{III}}(y; z) = \frac{1}{v(z)} L_n^{\mathrm{II}}(yv; z) \quad \text{(see also Example 5 from I.2.4).}$$

Introducing a parameter p in the function v, $v(z) = v(z; p)$, we find the optimal H^2-spaces over a set $v^2(z; p)$ of distribution functions, if the equation

$$f(t_k, \tilde{y}) = \tilde{y}'(t_k), \quad \text{where} \quad \tilde{y}(z) = \frac{1}{v(z; p)} L_n\big(yv(p; z); z\big),$$

can be solved.

By the special choice of

$$v(z; p) = e^{pz} \tag{30}$$

and of suitable p_0 the iterative computation of p_{j+1},

$$\begin{aligned} y_{kj} &= e^{-p_j z_k} L_n(ye^{p_j z}; z_k), \\ f(t_k, y_{kj}) &= e^{-p_j z_k} L_n'(ye^{p_j z}; z_k) - p_{j+1} y_{kj}, \end{aligned} \tag{31}$$

for $j = 0, 1, \ldots,$ produces a sequence convergent to a fixed point p^* or y_k^*, respectively. It is clear that p must be a vector of same dimension as y. If we work with the method of Theorem 4 and if we want to iterate according to (31), we need, besides the coefficients α_r, β_r defined by means of the special function $H_k(y; z)$, also the coefficients α_r^*, β_r^*, $r = 0, 1, \ldots, k-1$, which are computed from $H_k'(y; z)$.

3. Remarks

The present paper should have shown that methods based on generalized interpolation provide some new aspects in the matter of linear k-step methods and that their further investigations may be of interest.

Let us remark that the treatment of implicit methods constructed according to (5) to (7) is more difficult than the explicit case. Working out the implicit methods of type II we obtain

$$y_k = L_{2k-1}(y\,;x_k) - L_{2k-1}(y\,;x_k)D + y_k' D,$$

with

$$D = \varphi_{2k}(x_k)/\varphi_{2k}'(x_k) = 2\sum_{j=0}^{k} \frac{x_j}{1-x_j x_k} + 2\sum_{j=0}^{k-1} \frac{1}{x_k - x_j}.$$

By using the scalar test equation $y' = qy$ we get

$$y_k = \{L_{2k-1}(y\,;x_k) - L_{2k-1}'(y\,;x_k)D\}/\{D-q\}.$$

Because of the term $D-q$ in the denominator we have a more complicated situation with regard to stability than that occurring in the explicit case. Therefore we cannot expect as far reaching assertions as those stated in Theorem 4.

A further possibility of constructing methods with special properties of stability and approximation consists in using basic functions other than $U(z) = z$. For example, we get the right half-plane for the domain of the Hardy-space if we use $U(z) = e^{-z}$. In this case we have to define the consistency with regard to $\sum_{j=0}^{n} d_j e^{-jz}$ (see also Example 4 from I.2.4).

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