Extensions of invariant measures on Euclidean spaces

by

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Abstract. Sierpiński (see Szpilrajn [8]) asked if there exists a maximal extension of the Lebesgue measure on the Euclidean space E^n, invariant with respect to all isometries of this space. Our result implies a negative answer to this problem. We also show that for semiregular measures the existence of proper invariant extensions depends on the size of 2^{ω}.

0. Terminology. We use the standard set theoretic notation. For any set X, $P(X)$ denotes the family of all subsets of X and $|X|$ the cardinality of X. Ordinals are identified with sets of their predecessors and cardinals with initial ordinals. If $f\colon X \to Y$ is a function and $A \subset X$ then $f[A]$ denotes the image of A. R denotes the set of reals, Q the set of rationals and ω the set of natural numbers.

A *measure* on a set X is a non-negative, extended real-valued function m defined on a σ-algebra $\mathfrak{M}$ of subsets of X containing all singletons such that:

$m(\{x\}) = 0$ for any $x \in X$,

$m(X) > 0$,

$m(\bigcup_{n\in\omega} A_n) = \sum_{n\in\omega} m(A_n)$ for pairwise disjoint sets A_n from $\mathfrak{M}$.

Elements of the σ-algebra $\mathfrak{M}$ are called *measurable sets*.

A measure on X is:

— *complete* iff all subsets of sets of measure zero are measurable;

— *universal* iff it is defined on $P(X)$;

— *uniform* iff for any $A \subset X$, $m(A) = 0$ whenever $|A| < |X|$;

— *σ-finite* iff X is a countable union of sets of finite measure;

— *semiregular* iff every set of positive measure contains a set of positive finite measure.

If G is any group of bijections of a set X, then a measure m defined on a σ-algebra $\mathfrak{M}$ of subsets of X is *G-invariant* iff $g[A] \in \mathfrak{M}$ and $m(g[A]) = m(A)$ for any $g \in G$ and $A \in \mathfrak{M}$.

If m_i is a measure defined on a σ-algebra $\mathfrak{M}_i$ of subsets of X ($i = 1, 2$) then m_1 is an extension of m_2 iff $\mathfrak{M}_1 \supset \mathfrak{M}_2$ and $m_1(A) = m_2(A)$ for any $A \in \mathfrak{M}_2$.

Looking for extensions of G-invariant measures we shall always assume without loss of generality that the extended measure is complete. (The measure completion of a G-invariant measure is also G-invariant).

If G is any group of bijections of a set X then a subset $A \subset X$ is G-absolutely negligible iff for any σ-finite G-invariant measure m on X there exists a G-invariant extension of m defined on a σ-algebra $\mathfrak{M}$ containing A and for any such extension $\tilde{m}$, $\tilde{m}(A) = 0$.

A cardinal $\varkappa$ is large iff there exists a universal semiregular measure on $\varkappa$. Otherwise it is small. It is clear that $\varkappa$ is large iff it is greater or equal to a real-valued measurable cardinal and hence the existence of large cardinals cannot be proved in ZFC.

1. Preliminaries. Sierpiński (quoted in Szpilrajn [8]) asked if there exists a maximal G-invariant extension of the Lebesgue measure on the Euclidean space E^n where G is the group of all isometries of this space. A few partial solutions of this problem have been obtained. Hulanicki [3] proved the following

LEMMA 1.1. *Let G be any group of bijections of a set X such that $|G| \leqslant |X|$ and $|X|$ is small. If m is any uniform measure on X, then there exists a non-measurable set $Z \subset X$ such that*

$$m(g[Z] \triangle Z) = 0 \quad \textit{for every } g \in G.$$

From the above lemma he inferred the following consequence.

THEOREM 1.2. *Let G be any group of bijections of a set X such that $|G| \leqslant |X|$ and $|X|$ is small. Then every uniform G-invariant measure on X assuming positive finite values has a proper G-invariant extension.*

As a corollary he got a negative answer to the problem of Sierpiński, assuming that 2^ω is a small cardinal. (This result was obtained earlier by Pkhakadze [7] using similar methods.) In order to see this it suffices to remark that if G is the group of all isometries of the space E^n then every σ-finite G-invariant measure on E^n has a uniform G-invariant extension and hence, in view of the above theorem, also a proper G-invariant extension.

In the proof of Theorem 1.2. from Lemma 1.1., Hulanicki used the extension theorem of Łoś and Marczewski [4]. Applying it to semiregular measures we would risk to loose semiregularity of the extension because the technique of Łoś and Marczewski does not guarantee it. However, a different argument allows us to omit that difficulty. We shall use the following easy fact essentially due to Szpilrajn [8].

PROPOSITION 1.3. *Let G be any group of bijections of a set X and m any semiregular G-invariant measure on X. If there exists a non-measurable subset $A \subset X$ such that for any countable set $\{g_n: n \in \omega\} \subset G$ the set $\bigcup_{n \in \omega} g_n[A]$ has inner measure zero then m has a proper semiregular G-invariant extension.*

The next proposition is the counterpart of Theorem 1.2 for semiregular measures mentioned above (cf. Pelc [6])

PROPOSITION 1.4. *Let G be any group of bijections of a set X such that $|G| \leqslant |X|$ and $|X|$ is small. Then every semiregular uniform G-invariant measure m on X has a proper semiregular G-invariant extension.*

Proof. Let Z be the set from Lemma 1.1. and $Y \subset Z$ be measurable and such that the set $Z_1 = Z \setminus Y$ has inner measure zero. It is easy to see that Y has also the following property:

$$m(g[Y] \triangle Y) = 0 \quad \text{for every } g \in G.$$

Hence the set Z_1 satisfies the following conditions (cf. Pkhakadze [7], Theorem 3.22):
(a) Z_1 is non-measurable,
(b) Z_1 has inner measure zero,
(c) $m(g[Z_1] \triangle Z_1) = 0$ for every $g \in G$.
It follows from (b) and (c) that the set $\bigcup_{n \in \omega} g_n[Z_1]$ has inner measure zero for any countable subset $\{g_n: n \in \omega\}$ of G. Hence we get our conclusion by Proposition 1.3.

We do not know if the assumption of uniformity can be removed from the general formulation of Proposition 1.4. It will be done in a special case in Theorem 3.1. Clearly the assumption about the cardinality of X is necessary. Namely, for any set X of large cardinality we can find a group G of bijections of X of the same cardinality, for which the proposition fails: take as G the group of those bijections of X which move only finitely many elements. Then every measure on X is G-invariant and hence any universal semiregular measure on X provides a counterexample.

The next step towards the solution of Sierpiński's problem is due to Harazišvili (see [1], [2]). The tool he used were G-absolutely negligible sets. Notice that his definition (cf. also Pkhakadze [7]), though different from ours, turns out to be equivalent for any group G of isometries of E^n containing all translations.

Harazišvili proved the following facts:

THEOREM 1.5. *Let G be the group of translations of the Euclidean space E^n. Then E^n is a countable union of G-absolutely negligible sets.*

THEOREM 1.6. *Let G be the group of all isometries of the real line E^1. Then E^1 is a countable union of G-absolutely negligible sets.*

THEOREM 1.7. *Assume the continuum hypothesis. Let G be the group of all isometries of the Euclidean space E^n. Then E^n is a countable union of G-absolutely negligible sets.*

Clearly each of those results implies a negative answer to Sierpiński's problem in the respective special case because for any σ-finite G-invariant measure m one of the countably many G-absolutely negligible sets must be non measurable and m can be extended G-invariantly over it.

Harazišvili (cf. [1], [2]) stated the following problem: Let G be the group of all isometries of the Euclidean space E^n. Do there exist countably many G-absolutely negligible subsets of E^n whose union is E^n?

The positive answer to this problem implies a negative answer to Sierpiński's question and generalizes Theorems 1.6 and 1.7. In Section 2 we prove an even stronger result, thus solving Sierpiński's and Harazišvili's problems and generalizing Theorems 1.5, 1.6, 1.7. Since isometrically invariant σ-finite measures always turn out to have proper invariant extensions, we are also going to investigate semiregular measures as their natural generalization. In this case, discussed in Section 3, the existence of invariant extensions turns out to depend on the size of 2^ω, and hence becomes a set theoretical problem.

The following auxiliary fact is an important tool in the theory of G-absolutely negligible sets. We state it in the general setting and omit the easy proof.

PROPOSITION 1.9. *Let G be any group of bijections of a set X. If a subset $A\subset X$ satisfies the following property:*
for any countable set $\{g_n: n\in\omega\}\subset G$ there exists an uncountable subset $H\subset G$ such that $h_1[\bigcup_{n\in\omega} g_n[A]]\cap h_2[\bigcup_{n\in\omega} g_n[A]] = \emptyset$, for distinct $h_1, h_2\in H$
then A is G-absolutely negligible.

2. σ-finite measures.

THEOREM 2.1. *Let G be any group of isometries of the Euclidean space E^n which contains all translations. There exists a countable family $\{N_m: m\in\omega\}$ of G-absolutely negligible sets such that $\bigcup_{m\in\omega} N_m = E^n$.*

Proof. Let $\{K_\zeta: \zeta\leqslant 2^\omega\}$ be a family of subfields of the field of reals R such that: $K_0 = Q$, $K_{2^\omega} = R$, $K_\zeta\subsetneq K_\eta$ for $\zeta<\eta\leqslant 2^\omega$ and $K_\lambda = \bigcup_{\zeta<\lambda} K_\zeta$ for limit ordinals λ. For any $\zeta<2^\omega$ let $B_{\zeta+1}\subset K_{\zeta+1}\setminus K_\zeta$ be such that $B_{\zeta+1}\cup\{1\}$ is a linear basis of the space $K_{\zeta+1}$ over the field K_ζ.

We put $B^* = \{b_1\dots b_k: 0<k<\omega\ \&\ (\exists\eta_1<\dots<\eta_k<2^\omega)$ (η_i are successor ordinals and $b_i\in B_{\eta_i})\}$.

LEMMA 2.2. *The family $B^*\cup\{1\}$ is a linear basis of R over the field K_0.*

Proof. It is enough to prove by induction on $0<\zeta\leqslant 2^\omega$ that

(*) if $B^*_\zeta = \{b_1\dots b_k: 0<k<\omega\ \&\ (\exists\varrho_1<\dots<\eta_k\leqslant\zeta)$ (η_i are successor ordinals and $b_i\in B_{\eta_i})\}$; then $B^*_\zeta\cup\{1\}$ is a linear basis of K_ζ over K_0.

Fix any $0<\vartheta\leqslant 2^\omega$ and suppose that (*) is true for every $0<\zeta<\vartheta$. If ϑ is a limit ordinal then the set B^*_ϑ is linearly independent over K_0 because $B^*_\vartheta = \bigcup_{\zeta<\vartheta} B^*_\zeta$ and the system of sets B^*_ζ are increasing. The space K_ϑ is spanned by $B^*_\vartheta\cup\{1\}$, because $K_\vartheta = \bigcup_{\zeta<\vartheta} K_\zeta$.

If $\vartheta = \eta+1$ then

$$\{1\}\cup B^*_\vartheta = \{c\cdot b: c\in B^*_\eta\cup\{1\}, b\in B_{\eta+1}\cup\{1\}\}.$$

Since by definition $B_{\eta+1}\cup\{1\}$ is a linear basis of $K_{\eta+1}$ over K_η and by the inductive hypothesis $B^*_\eta\cup\{1\}$ is a linear basis of K_η over K_0, it easily follows that $B^*_\vartheta\cup\{1\}$ is a linear basis of $K_{\eta+1}$ over K_0. This completes the proof of the lemma.

Let S be the family of finite subsets of the cardinal 2^ω. We define a function $r: R\to S$ as follows:

(a) $r(0) = \emptyset$, $r(1) = \{0\}$,
(b) $r(b) = \{0, \zeta+1\}$ for any $b\in B_{\zeta+1}$, $\zeta<2^\omega$,
(c) $r(b) = \bigcup_{1\leqslant i\leqslant k} r(b_i)$ for $b\in B^*$, where $b = b_1\dots b_k$ is the unique representation from the definition of B^*,
(d) $r(x) = \bigcup_{1\leqslant i\leqslant m} r(x_i)$, where $x = k_1x_1+\dots+k_mx_m$ ($k_i\in K_0\setminus\{0\}$, $x_i\in B^*\cup\{1\}$) is the unique representation of x in the basis $B^*\cup\{1\}$.

The following lemma states some simple properties of the function r. We leave it without proof.

LEMMA 2.3. *Let η be a limit ordinal.*
1. *If $x\in K_\eta$ then $r(x)\subset\eta$;*
2. *$r(x)\,\triangle\, r(y)\subset r(x+y)\subset r(x)\cup r(y)$, for any reals x, y;*
3. *$r(x\cdot y)\subset r(y)\cup\eta$, for any $x\in K_\eta$, $y\in R$;*
4. *$r(y)\setminus\eta\subset r(x\cdot y)$, for any $x\in K_\eta\setminus\{0\}$, $y\in R$.*

For any natural number m, let

$$\mathscr{A}_m = \{\lambda+k: \lambda<2^\omega,\ \lambda \text{ is a limit ordinal},\ k<m\},$$
$$X_m = \{x\in R: r(x)\subset\mathscr{A}_m\} \quad\text{and}\quad N_m = (X_m)^n$$

where n is the fixed dimension of our Euclidean space.

Clearly, $E^n = \bigcup_{m\in\omega} N_m$ because $\bigcup_{m\in\omega}\mathscr{A}_m = 2^\omega$ and $r(x)$ is always a finite set. It suffices to show that every set N_m is G-absolutely negligible. We use Proposition 1.9. Fix a natural number m and let $\{g_k: k\in\omega\}$ be any countable set of isometries from G. Every isometry g_k can be represented as a superposition $w_k\circ A_k$ where w_k is a translation $\langle w_1^k, \dots, w_n^k\rangle$ and A_k is an isometry fixing the origin of coordinates, given by the matrix

$$A_k = \begin{bmatrix} a_{11}^k & \dots & a_{1n}^k \\ \dots & \dots & \dots \\ a_{n1}^k & \dots & a_{nn}^k \end{bmatrix}, \qquad \det(A_k) = \pm 1.$$

By abuse of notation we identify vectors with translations and matrices with respective isometries.

Let $W_k = \{w_1^k, \dots, w_n^k\}\cup\{a_{ij}^k: 1\leqslant i, j\leqslant n\}$ and take a limit ordinal $\lambda_0<2^\omega$ such that $\bigcup_{k\in\omega} W_k\subset K_{\lambda_0}$. For any limit ordinal ζ such that $\lambda_0<\zeta<2^\omega$, let $b_\zeta\in B_{\zeta+m}$ and put $w_\zeta = \langle b_\zeta, 0, \dots, 0\rangle\in E^n$.

By Proposition 1.9 it is enough to prove

$$w_\zeta[\bigcup_{k\in\omega} g_k[N_m]]\cap w_\eta[\bigcup_{k\in\omega} g_k[N_m]] = \emptyset,$$

for any limit ordinals ζ, η such that $\lambda_0<\zeta<\eta<2^\omega$.

Hence if suffices to show that for any pair (s, t) of natural numbers:

$$w_\zeta[g_s[N_m]] \cap w_\eta[g_t[N_m]] = \emptyset$$

or, in other words, that if $Z = (g_t^{-1}) \circ (-w_\eta) \circ (w_\zeta) \circ (g_s)[N_m]$ then $Z \cap N_m = \emptyset$. Let $x = \langle x_1, \ldots, x_n \rangle \in N_m$. Hence if $y = \langle y_1, \ldots, y_n \rangle$ denotes the image $g_s(x)$, we have $y_i = a_{i1}^s x_1 + \ldots + a_{in}^s x_n + w_i^s$, for $i = 1, \ldots, n$ and $r(y_i) \subset \mathscr{A}_m \cup \lambda_0$ by Lemma 2.3. Let furthermore $v = \langle v_1, \ldots, v_n \rangle = (-w_\eta) \circ (w_\zeta)(y)$. Hence $v_i = y_i$ for $2 \leqslant i \leqslant n$ and $v_1 = y_1 + b_\zeta - b_\eta$. By Lemma 2.3 we get $\zeta + m \in r(v_1)$.

By definition we have $g_t^{-1} = (A_t^{-1}) \circ (-w_t)$. Let

$$A_t^{-1} = \begin{bmatrix} d_{11} & \ldots & d_{1n} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ d_{n1} & \ldots & d_{nn} \end{bmatrix}, \quad \det(A_t^{-1}) = \pm 1.$$

Clearly, $\{d_{ij}\colon 1 \leqslant i, j \leqslant n\} \subset K_{\lambda_0}$. Let finally $z = \langle z_1, \ldots, z_n \rangle = g_t^{-1}(v)$. Hence for any $i \leqslant n$, $z_i = d_{i1}(v_1 - w_1^t) + r_i$ where $r_i = d_{i2}(v_2 - w_2^t) + \ldots + d_{in}(v_n - w_n^t)$.

By Lemma 2.3. we get $r(r_i) \subset \mathscr{A}_m \cup \lambda_0$, $\zeta + m \in r(v_1 - w_1^t)$ and $r(d_{i1}) \subset \lambda_0$. However, since $\det(A_t^{-1}) \neq 0$, it follows that $d_{i_0 1} \neq 0$ for some $i_0 \leqslant n$. Hence by Lemma 2.3., $\zeta + m \in r\big(d_{i_0 1}(v_1 - w_1^t)\big)$ which implies $\zeta + m \in r(z_{i_0})$.

Hence, if $z \in Z$ then $z \notin N_m$ because $\zeta + m \notin \mathscr{A}_m$. This gives $Z \cap N_m = \emptyset$ and completes the proof of our theorem.

The next corollary follows immediately from Theorem 2.1. and gives a negative answer to the problem of Sierpiński.

COROLLARY 2.4. *Let G be any group of isometries of the Euclidean space E^n, which contains all translations. Then every σ-finite G-invariant measure on E^n has a proper G-invariant extension.*

It is easy to see that some assumptions on the group G of isometries have to be imposed in the above results. If 2^ω is large and G is e.g. any countable group of translations, then there exists a universal σ-finite G-invariant measure on E^n, hence Corollary 2.4. fails. It would be interesting to find an exact characterization of those groups G of isometries of E^n for which Corollary 2.4. is true.

3. Semiregular measures. Since in the σ-finite case every measure on E^n invariant with respect to all isometries can be properly extended with preservation of this property, it seems natural to investigate the extension problem in a more general setting. The only place where σ-finiteness was used in the proof of Theorem 2.1. was the application of Proposition 1.9. In fact, instead of the uncountable set H of isometries required in this proposition, we have shown a set of cardinality 2^ω thus proving Theorem 2.1. in a slightly more general situation: for measures which do not admit pairwise disjoint families of cardinality 2^ω of sets of positive measure.

Semiregular measures can be equivalently defined as those for which every set of positive measure has a partition into subsets of positive finite measure (without any specific restriction on the size of the partition). Hence semiregularity seems the reasonable assumption for which there may be some hope of a different answer

to the extension problem. On the one hand no argument of the type used before can work, on the other hand we avoid e.g. the trivial case of measure giving value 0 to countable and value ∞ to uncountable sets. Let us recall that there exist natural examples of semiregular and not σ-finite measures in Euclidean spaces, e.g. the one-dimensional Hausdorff measure on the Euclidean plane. It is moreover a measure invariant with respect to all isometries.

The result of this section shows that in the semiregular case the solution of the extension problem depends on the size of 2^ω. In particular Corollary 2.4. fails for semiregular measures when 2^ω is a large cardinal. Notice that our theorem remains true (with the same proof) for measures assuming at least one positive finite value (cf. Theorem 1.2.).

THEOREM 3.1. *Let G be any group of isometries of the Euclidean space E^n. The following are equivalent*:

(a) 2^ω *is large*,
(b) *there exists a universal semiregular G-invariant measure on E^n*,
(c) *there exists a maximal semiregular G-invariant measure on E^n*.

Proof.
(a)$\Rightarrow$ (b) Denote by $\mathscr{S}$ the family of all lines in the space E^n. Fix on each $S \in \mathscr{S}$ two points: 0_S and 1_S with Euclidean distance between them equal to 1. Thus for any $S \in \mathscr{S}$ there exists a bijection $\varphi_S\colon S \to R$ such that for any isometry T of E^n, if $T[S_1] = S_2$ then the transformation $\varphi_{S_2} \circ T \circ \varphi_{S_1}^{-1}\colon R \to R$ is an isometry of the reals.

We shall use the following result of Pelc [5]: on every abelian group of large cardinality there exists a semiregular universal invariant measure which is moreover invariant with respect to the "inverse element" operation.

This theorem applied to the additive group of reals gives a semiregular universal measure m invariant with respect to all isometries of the reals. Let m_S be the respective measure on S obtained via the bijection φ_S.

We define for any $A \subset E^n$:

$$\mu(A) = \sum_{S \in \mathscr{S}} m_S(A \cap S).$$

The above infinite sum is defined as the supremum of sums over finite subsets. μ is clearly a semiregular universal measure on E^n. It is enough to show that μ is invariant with respect to all isometries. Let T be any isometry of E^n. Since T induces a permutation of the set $\mathscr{S}$ we get:

$$\mu(T[A]) = \sum_{S \in \mathscr{S}} m_S(T[A] \cap S) = \sum_{S \in \mathscr{S}} m_{T[S]}(T[A] \cap T[S])$$

It is enough to show that for any $S \in \mathscr{S}$

$$m_S(A \cap S) = m_{T[S]}(T[A] \cap T[S]).$$

Indeed, since m is a measure invariant with respect to all isometries of the reals, we get:

$$\begin{aligned} m_S(A \cap S) &= m(\varphi_S[A \cap S]) \\ &= m(\varphi_{T[S]} \circ T \circ \varphi_S^{-1}[\varphi_S[A \cap S]]) \\ &= m(\varphi_{T[S]}[T[A] \cap T[S]]) \\ &= m_{T[S]}(T[A] \cap T[S]). \end{aligned}$$

(b) $\Rightarrow$ (c) This is obvious, since universal measures are maximal.

(c) $\Rightarrow$ (a) Assume that 2^ω is small and let μ be any semiregular G-invariant measure defined on a σ-algebra $\mathfrak{M}$ of subsets of E^n. Denote by λ the smallest cardinality of a subset of positive outer measure. If every set of cardinality λ has inner measure zero, we are done by Proposition 1.3. Hence we may assume that there exist sets of cardinality λ and positive inner measure and thus also measurable sets of cardinality λ and positive measure. Let k be the smallest integer for which a set of cardinality λ and positive measure is contained in a k-dimensional hyperplane. Fix such a set A contained in a hyperplane E.

We define by induction a non-decreasing family $\{G_m: m \in \omega\}$ of subgroups of G. As G_0 take the trivial group. If G_m is already constructed let $B'_m = \bigcup_{h \in G_m} h[A]$ and B_m be such a measurable subset of B_m that $B'_m \setminus B_m$ has inner measure zero. It is easy to show that $\mu(h[B_m] \triangle B_m) = 0$ for any $h \in G_m$. Denote by G'_{m+1} the set

$$\{g \in G\colon \mu(g[B_m] \cap B_m) > 0\}$$

and let G_{m+1} be the group generated by G'_{m+1}. It is easy to see that the groups G_m are actually non-decreasing. We may also assume that $B_m \subset B_{m+1}$ for any natural m.

The following statement will be proved by simultaneous induction on m: For every $m \in \omega$ and $g \in G_m$, $g[E] = E$ and $H_m = \{g \restriction E\colon g \in G_m\}$ is a group of isometries of E of cardinality $\leqslant \lambda$.

Suppose we are done for m and let $g \in G'_{m+1}$. If $g[E] \neq E$ then $g[E] \cap E$ is an l-dimensional hyperplane for $l < k$ and the set $g[B_m] \cap B_m$ has positive measure and cardinality λ. By the inductive hypothesis $B_m \subset E$ and hence

$$g[B_m] \cap B_m \subset g[E] \cap E$$

which contradicts the minimality of the dimension k. This proves $g[E] = E$ for any $g \in G'_{m+1}$ and hence also for any $g \in G_{m+1}$.

In order to show $|H_{m+1}| \leqslant \lambda$ take again $g \in G'_{m+1}$ and notice that the set $g[B_m] \cap B_m$ must contain $k+1$ points $x_1, \ldots, x_{k+1}$ which are not elements of the same $(k-1)$-dimensional hyperplane. Let us call such points independent. Hence the points $x'_1 = g^{-1}(x_1), \ldots, x'_{k+1} = g^{-1}(x_{k+1})$ also form an independent subset of B_m.

It follows that for any isometry $g \in G'_{m+1}$ there exist independent $(k+1)$-element sets $I_1, I_2 \subset B_m$ such that $g[I_1] = I_2$. For any such pair I_1, I_2 there are however

only finitely many isometries h of the hyperplane E for which $h[I_1] = I_2$. Since $|B_m| = \lambda$ by the inductive hypothesis, it follows that $[\{g \restriction E\colon g \in G'_{m+1}\}] \leqslant \lambda$ and hence $|H_{m+1}| \leqslant \lambda$ which finishes the proof of our statement by induction.

Put $H = \bigcup_{m \in \omega} H_m$. Since H_m were formed a non-decreasing system groups of cardinalities at most λ, H is also a group of such cardinality. Denote by C the set $\bigcup_{m \in \omega} B_m$. We show that any $g \in G$ such that $g \restriction E \notin H$ the following holds:

$$(*) \qquad \mu(g[C] \cap C) = 0.$$

Indeed

$$g[C] \cap C = g[\bigcup_{m \in \omega} B_m] \cap \bigcup_{m \in \omega} B_m \subset \bigcup_{l, m \in \omega} g[B_m] \cap B_l \subset \bigcup_{m \in \omega} g[B_m] \cap B_m,$$

because the sets B_m are non-decreasing. For any $m \in \omega$ we have $g \notin G_{m+1}$. Hence for any $m \in \omega$

$$\mu(g[B_m] \cap B_m) = 0$$

which implies $(*)$.

Next we show that for any $h \in G$ such that $h \restriction E \in H$ we have

$$(**) \qquad \mu(h[C] \triangle C) = 0.$$

Indeed, for some $m \in \omega$, $h \in G_m$ which implies $\mu(h[B_r] \triangle B_r) = 0$ for any $r \geqslant m$. Hence

$$\mu(h[\bigcup_{r \geqslant m} B_r] \triangle \bigcup_{r \geqslant m} B_r) = 0$$

which implies $(**)$ because $\bigcup_{r \geqslant m} B_r = C$.

By an argument of Pkhakadze [7] and Hulanicki [3] we get a non-measurable set $Z \subset C$ such that $\mu(h[Z] \triangle Z) = 0$ for any $h \in H$. Then, similarly as in the proof of Proposition 1.4, we find a subset $Z_1 \subset Z$ satisfying the following conditions:

(a) $Z_1 \notin \mathfrak{M}$,

(b) Z_1 has inner measure zero,

(c) $\mu(h[Z_1] \triangle Z_1) = 0$ for any $h \in H$.

In view of Proposition 1.3. it is enough to show that for any countable set $L \subset G$ the set $\bigcup_{l \in L} l[Z_1]$ has inner measure zero. Let $G^* = \{g \restriction E\colon g \in G\}$. We shall equivalently show the above property for any countable $L \subset G^*$.

Suppose that $T \subset \bigcup_{l \in L} l[Z_1]$, $\mu(T) > 0$ and let L' be a subset of L whose elements belong to distinct left cosets of H in G^* and such that for any $l \in L$ there is an isometry $l' \in L'$ belonging to the same coset. Since for distinct $l_1, l_2 \in L'$ the sets $l_1[C]$ and $l_2[C]$ are both measurable and $\mu(l_1[C] \cap l_2[C]) = 0$, we get that for some $l_0 \in L'$

$$\mu(T \cap l_0[C]) > 0.$$

Let $\{h_i: i \in \omega\}$ be such elements of H that every $l \in L$ belonging to the same coset that l_0 is of the form $l_0 \circ h_i$. Clearly

$$\mu((T \cap l_0[C]) \setminus \bigcup_{i \in \omega} l_0 \circ h_i[Z_1]) = 0\,.$$

However in view of property (c),

$$\mu((\bigcup_{i \in \omega} l_0 \circ h_i[Z_1]) \triangle l_0[Z_1]) = 0$$

which implies that the set $l_0[Z_1]$ contains a set of positive measure. This contradicts property (b).

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Über den Homotopietyp von Linsenraumprodukten

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Abstract. In this paper we derive a necessary criterion for products of lens spaces to have the same homotopy type. The criterion is a generalization of the Franz–Whitehead–criterion for a single factor.

I. Einführung und Ergebnis. Linsenräume wurden 1935 von W. Franz [2] und K. Reidemeister [7] kombinatorisch und 1941 von W. Franz [3] und J. H. C. Whitehead [9] bezüglich ihres Homotopietyps klassifiziert. Diese Arbeiten waren grundlegend für die Theorie des einfachen Homotopietyps, und Linsenräume waren die ersten Beispiele für die Stärke der damit zusammenhängenden Torsionsinvariante.

Für Mannigfaltigkeiten mit endlicher zyklischer Fundamentalgruppe sind Linsenräume besonders wichtige Beispiele. Dasselbe gilt für Produkte von Linsenräumen bezüglich endlicher abelscher Gruppen. Zwischen diesen Produkten können Diffeomorphismen existieren, die nicht von solchen der Faktoren herrühren: Einer der Sätze in [5] besagt z.B., daß der Diffeomorphietyp eines Produktes $\prod_{i=1}^{s} L_{m_i}(r_i)$ aus dreidimensionalen Linsenräumen $L_{m_i}(r_i)$ für $s \geqslant 2$ allein durch die Fundamentalgruppe bestimmt ist, falls mindestens zwei der Drehnenner m_i teilerfremd sind und mindestens eine Verdrillungszahl die Bedingung $r_i \equiv \pm 1 \bmod m_i$ erfüllt.

Im Anhang von [5] wurden jedoch bereits ohne Beweis Beispiele dafür angegeben, daß auch für $s \geqslant 2$ die Fundamentalgruppe nicht immer den Homotopietyp bestimmt. Die dieser Bemerkung zugrundeliegende Idee wird in der vorliegenden Arbeit ausgeführt, d.h. das Homotopietypkriterium von Franz und Whitehead auf Produkte $(2n-1)$-dimensionaler Linsenräume $(n>1)$ verallgemeinert. Wir beweisen die Verallgemeinerung als *notwendiges* Kriterium. Spezialfälle davon wurden bereits von R. Quetting [6] in Zusammenarbeit mit den Autoren erzielt. In einer weiteren Arbeit [4] wird gezeigt, daß für Produkte dreidimensionaler Linsenräume das Kriterium auch *hinreichend* ist.

Folgende Begriffe und Erläuterungen seien der Formulierung des Resultates vorausgeschickt:

Sei m eine natürliche Zahl und $(r_1, \ldots, r_n)$ ein n-tupel zu m teilerfremder ganzer