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Polyhedral-shape concordance implying homeomorphic complements

by

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Abstract. If two compacta X and Y satisfy the inessential loops condition in the interior of a piecewise linear m-manifold M, $m \geqslant 6$, and are shape concordant in M, and if X has the shape of a compact k-polyhedron, $k \leqslant m-3$, then $M-X$ is homeomorphic to $M-Y$.

A compact subset X of a manifold N satisfies the *inessential loops condition* (abbreviated ILC) if for each neighborhood U of X, there is a neighborhood V of X such that each loop in $V-X$ which is null homotopic in V is null homotopic in $U-X$. Throughout the paper, $I=[0,1]$.

Two compacta X_0 and X_1 (satisfying ILC) in the interior of a manifold M is said to be (ILC) *shape concordant* if there is a compactum Z (satisfying ILC) in $M\times I$ such that $X_i\times\{i\} = Z\cap(M\times\{i\})\hookrightarrow Z$ is a shape equivalence for each $i=0,1$. Similarly, if X_0, X_1 and Z are polyhedra in the corresponding PL manifolds, we can define the notion of *polyhedral concordance*.

Sher has proved in [S] that "*If X and Y are* ILC *compacta in a* PL *manifold M^m* ($\partial M=\emptyset$ and $m\geqslant 6$) *and* ILC *shape concordant in M by Z, and if X has the shape of a k-polyhedron* ($k\leqslant m-3$), *then $M-X\cong M-Y$.*" In this note, we will show that if X and Y satisfy ILC in M and are shape concordant, then $M-X$ is still homeomorphic to $M-Y$ without assuming that Z satisfies ILC in $M\times I$ (Theorem 3.4).

For standard notions and notations in piecewise linear (abbreviated PL) topology as: simplicial collapse, PL homeomorphism, ambient isotopy, singular set $S(f)$ of a PL map f, derived neighborhood, boundary ∂Q of a PL manifold Q, $\mathrm{Fr}_Q N$, $\mathrm{Int}_Q N, \ldots$, we refer to [Hd]. Given an open subset W of ∂Q, by an *open collar* of W in Q, we mean the image of a PL open embedding $h\colon W\times[0,1)\to Q$ such that $h(x,0)=x$ for all $x\in W$. If Q is a PL manifold, $\dot{Q}$ denotes $Q-\partial Q$.

We will suppress the base points from our notations of homotopy groups. If $(X,A)\subset(Y,B)$, the homomorphism $\pi_q(X,A)\to\pi_q(Y,B)$ will be the inclusion-induced one if it is not specified otherwise.

We assume that the reader is familiar with the fundamentals of shape [B] and ANR-systems [M–S].

In the following, we will use the same notation for a PL space and its underlying space if there is no danger of confusion. For a polyhedron X, by writing X^r, we mean $\dim X = r$.

§ 1. An embedding theorem. The central result of this section is Theorem 1.4 which is a more general form of Theorem 8.2 in [Hd] and is a tool for the proof of our main result, Theorem 3.3.

First, we will prove some technical lemmas needed for the proof of Theorem 1.4. Let us define some notations. Given an integer q, for each integer $k \leqslant q-3$, define by induction a sequence $\{\eta_0, \eta_1, \ldots, \eta_k\}$: $\eta_0 = \eta_1 = 1$, $\eta_j = 1+\eta_{j-1}$ if $1 < j \leqslant q-3$. Set $r_k = \eta_0+\eta_1+\ldots+\eta_k$. Define $\mu_0 \equiv \mu_0(k) = r_k$ and $\mu_{j+1} = \mu_j - \eta_j$, $j = 0, \ldots$ $\ldots, k-1$.

LEMMA 1.1 (Engulfing Lemma). *Let Q^q be a* PL *manifold and R^{q-1} a compact* PL *submanifold of ∂Q. Assume that $U_0 \supset U_1 \supset \ldots \supset U_{\mu_0(k)}$ is a sequence of open neighborhoods of R in Q such that for each $i = 0, 1, 2, \ldots, k$ and $j = 1, \ldots, \mu_0(k)$, the homomorphism $\pi_i(U_j, R) \to \pi_i(U_{j-1}, R)$ is trivial, and let V be an open collar of $\mathring{R}$ in $U_{\mu_0(k)}$. If X and Y are compact subpolyhedra of $\mathring{U}_{\mu_0(k)} \cup \mathring{R}$ and V, respectively, such that $\dim(X-R) \leqslant k$ and $\dim(Y-R) \leqslant l$ where $k, l \leqslant q-3$, then there is a* PL *homeomorphism $h\colon U_0 \to U_0$, fixed on $\partial U_0 \cup Y$ and outside a compact subset of U_0, such that $X \subset h(V)$.*

Proof. Let Z $(X \cap R \subset Z \subset V)$ be a compact subpolyhedron of X and write $K = (X \cup Y) - R$, $L = (Z \cup Y) - R$. Then, $\overline{K-L}$ is a compact subpolyhedron of $\mathring{U}_{\mu_0(k)} \subset \mathring{U}_0$ and $L \subset \mathring{V}$. We only need to show that there is an ambient isotopy h_t of $\mathring{U}_0$ $(0 \leqslant t \leqslant 1)$, keeping L and outside a compact set of $\mathring{U}_0$ fixed, such that $h_1(\mathring{V}) \supset K$.

The proof which is similar to that of Theorem 4.2.1 [R] will be by induction on $k = \dim(K-L)$. The following is an outline of the proof. Let (K, L) also denote a simplicial triangulation of (K, L) and let $K^{(i)} = L \cup \{\sigma |\ \sigma \in K, \dim\sigma \leqslant i\}$.

Here, we deform $\mathring{V}$ in $\mathring{U}_{\mu_1}$ to engulf $K^{(0)}$, then in $\mathring{U}_{\mu_2}$ to engulf $K^{(1)}$. Assume that the lemma has been prove for all $i < k$ and we already deform $\mathring{V}$ in $\mathring{U}_{\mu_k}$ to engulf $K^{(k-1)}$. For convenience, we can assume that $K^{(k-1)} \subset \mathring{V} \subset \mathring{U}_{\mu_k}$. Write $K_k = \overline{K^{(k)} - K^{(k-1)}}$ and $K_{k,k-1} = K_k \cap K^{(k-1)}$.

From the triviality of $\pi_k(U_{\mu_k}, R) \to \pi_k(U_{\mu_k-1}, R)$, it follows that $\pi_k(\mathring{U}_{\mu_k}, \mathring{V}) \to$ $\to \pi_k(\mathring{U}_{\mu_k-1}, \mathring{V})$ is trivial; therefore, we can define a map $\varphi\colon (K_k \times I) \cup (K \times \{1\}) \to$ $\to \mathring{U}_{\mu_k-1}$ such that $\varphi(K_{k,k-1} \times I) \subset \mathring{V}$, $\varphi(x, 1) = x$ for all $x \in K$ and $\varphi(x, 0) \in \mathring{V}$ for all $x \in K_k$. Then, by approximating φ if necessary, we can assume further that there is a simplicial triangulation (S, T) of $((K_k \times I) \cup (K \times \{1\}), (K_{k,k-1} \times I) \cup$ $\cup (K \times \{1\}))$ and Q_1 of Q such that $S \searrow T$, φ is a nondegenerate simplicial map, and $\dim(S(\varphi|\sigma \cup \tau)) \leqslant \dim\sigma + \dim\tau - q$ for all simplices σ and τ of S. Let $\Sigma = \bigcup \{S(\varphi|\sigma \cup \tau)) | \sigma, \tau \in S;\ \dim\sigma = k+1, \dim\tau = k\} \subset S$. Then, $\dim\Sigma \leqslant k-2$ (refer to p. 152 [R]). Now, since $S \searrow T$, there is from Lemma 7.3 [Hd] a subcomplex A of S with $A \supset \Sigma$ and $\dim A \leqslant k-1$ such that $S \searrow A \cup T \searrow T$. Then, by induction hypothesis, we can first deform $\mathring{V}$ in $\mathring{U}_{\mu_k-1-\eta_k} = \mathring{U}_0$ to engulf $\varphi(A \cup T)$, then $\varphi(S^{(k)})$, containing $\varphi(K \times \{1\}) = K$, by pushing out the deformed $\mathring{V}$ in $\mathring{U}_0$ along the image under φ of the collapse $S \searrow A \cup T$, where $S^{(k)}$ denotes the k-skeleton of S. The proof is complete.

The following is a relative version of Lemma 7.3 [Hd].

LEMMA 1.2. *Let L be a compact* PL *subspace of a* PL *manifold M and N a regular neighborhood of L in M meeting ∂M regularly. If X^x is a subpolyhedron of N such that $X \cap \partial M \subset L \cap \partial M$, then there is a subpolyhedron T of N such that $X \subset T$, $T \cap \partial M \subset L \cap \partial M$, $\dim T \leqslant x+1$ and $N \searrow T \cup L \searrow L$.*

Proof. Write $A = N \cap \partial M$ and C a regular neighborhood of A in N. Then, there is a PL homeomorphism $f\colon C \cong A \times [0, 2]$ with $f(x) = (x, 0)$ for $x \in A$. For each $i = 1, 2$, let A_i denote the simplicial triangulation of $A \times [0, i]$ whose 0-skeleton $A_i^{(0)}$ is the set $A^{(0)} \times \{0, i\}$ (refer to Lemma 1.4 [Hd]); Q_i the subspace of A_i with $Q_i^{(0)} = A_i^{(0)} - (L^{(0)} \times \{i\})$; and $P_i = f^{-1}(Q_i)$. Since $X \cap \partial M \subset L \cap \partial M$, without loss of generality, we can assume that $\overline{N-P_2} \supset X \cup L$. We will show that $N \searrow \overline{N-P_1}$ and $\overline{N-P_1} \cong N$ by a PL homeomorphism, say h, fixing $\overline{N-P_2}$ (in particular $X \cup L$). Then, the lemma will follow.

First, we show that $N \searrow \overline{N-P_1}$. Let $B_1, \ldots, B_r$ be the simplices of $A-L$ in order of decreasing dimension. By use the PL homeomorphism f above, we only need to prove that there is a composite of elementary collapses ([Hd], p. 42): $Q_1 \searrow^e W_1 \searrow^e$ $W_2 \searrow^e \ldots \searrow^e W_r = \mathrm{Fr}\, Q_1$ where $W_j = \mathrm{Fr}\, Q_1 \cup (Q_1 - (\mathring{B}_1 \cup \ldots \cup \mathring{B}_j) \times [0, 1])$. We will prove by induction on $j \leqslant r$. For $j = 1$, write $B_1 = \sigma * \tau$ the join of σ in L and τ in $A-L$ ($\sigma = \emptyset$ if $B_1 \cap L = \emptyset$). Observe that $Q_1 \cap (B_1 \times [0, 1])$ is a PL cell having $B_1 \times \{0\}$ as a free face; so, we have an elementary collapse $Q_1 \searrow^e W_1$. For the inductive step, observe that $B_{j+1} \times \{0\}$ is a free face of the PL cell $Q_1 \cap (B_{j+1} \times [0, 1])$ in W_j; hence, $W_j \searrow^e W_{j+1}$.

Second, write $P_{12} = \overline{P_2 - P_1}$. Define

$$h\colon P_2^{(0)} \to ((A^{(0)} - L^{(0)}) \times \{1, 2\}) \cup (L^{(0)} \times \{0\})$$

by (1) $h(v, 0) = (v, 1)$ if $v \in A^{(0)} - L^{(0)}$; (2) $h(v, 2) = (v, 2)$ if $v \in \overline{A^{(0)}} - L^{(0)}$; and (3) $h(v, 0) = (v, 0)$ if $v \in L^{(0)}$. Then, there is a simplicial triangulation R of P_{12} with $R^{(0)} = ((A^{(0)} - L^{(0)}) \times \{1, 2\}) \cup (L^{(0)} \times \{0\})$ such that we can extend h linearly over each simplex of P_2. Then, extend h via identity over $\overline{N-P_2}$.

Finally, from Lemma 7.3 [Hd], there is a subpolyhedron T_0 of N such that $\dim T_0 \leqslant x+1$, $X - L \subset T_0$ and $N \searrow T_0 \cup L \searrow L$. Define $T = h^{-1}(T_0)$, then $N \searrow$ $\searrow \overline{N-P_1} \searrow h^{-1}(T_0) \cup L = T \cup L \searrow L$. Since h keeps $X \cup L$ fixed, $X - L \subset T$. Therefore, the lemma is proved.

LEMMA 1.3. *Let Q^q be a* PL *manifold and R^{q-1} a compact* PL *submanifold of ∂Q. Assume that $U_0 \supset U_1 \supset \ldots \supset U_{\mu_0(k)}$, $k \leqslant q-3$, is a sequence of open neighborhoods of R in Q such that for each $i = 0, 1, \ldots, k$, $j = 1, 2, \ldots, \mu_0(k)$, the homomorphism $\pi_1(U_j, R) \to \pi_1(U_{j-1}, R)$ is trivial. Let X^k and C be compact subpolyhedra of $\mathring{U}_{\mu_0(k)} \cup \mathring{R}$ such that $C \searrow C \cap R \subset \mathring{R}$. Then, there is a compact subpolyhedron C'*

of U_0 *such that*

(1) $$C \cup X \subset C' \searrow C' \cap R \subset \mathring{R},$$

and

(2) $$\dim(C'-C) \leqslant k+1 .$$

Proof. Let $Z = \overline{X-C}$. Then, $\dim(Z \cap C) \leqslant k-1$ and $C \cup X = C \cup Z$. Now, since $Z \cap C$ and $C \cap R$ are subpolyhedra of C with $C \searrow C \cap R$, it follows from Lemma 7.3 [Hd] that there is a subpolyhedron T of C containing $(Z \cap C)-(C \cap R)$ such that $C \searrow (C \cap R) \cup T \searrow C \cap R$ and that $\dim T \leqslant k$. Consequently, $C \cup X = C \cup Z \searrow (C \cap R) \cup Y$, where $Y = T \cup Z \subset \mathring{U}_{\mu_0(k)} \cup \mathring{R}$. Observe that $\dim Y \leqslant \max\{\dim T, \dim X\} \leqslant k$.

From Engulfing Lemma 1.1, for a given open collar V of $\mathring{R}$ in $U_{\mu_0(k)}$, there is a PL homeomorphism h: $U_0 \to U_0$, fixing ∂U_0 and outside a compact subset of U_0, such that $Y \subset h(V)$. Consequently, $((C \cup X) \cap R) \cup h^{-1}(Y)$ is a compact subpolyhedron of V. Therefore, there is a compact PL submanifold P^{q-1} of $\mathring{R}$ such that $(C \cup X) \cap R \subset P$ and $[(C \cup X) \cap R] \cup h^{-1}(Y) \subset W \equiv \operatorname{Int}_Q N$, where N is a regular neighborhood of P in V that meets ∂V regularly. So, $[(C \cup X) \cap R] \cup \cup Y \subset h(W) \subset h(N)$.

By pushing out $h(W)$ along the collapse $C \cup X \searrow Y \cup (C \cap R)$, we can obtain a PL homeomorphism g of U_0, fixing $\partial U_0 \cup Y$, such that $C \cup X \subset gh(W) \subset gh(N)$. Let N_1 be a second derived neighborhood of $P \cup C$ in $\operatorname{Int}_{U_0}(gh(N))$. Then, $N_1 \searrow P \cup \cup C \searrow P$ since $C \searrow C \cap R \subset P$; hence, N_1 and $gh(N)$ are regular neighborhoods of P in U_0 with $N_1 \subset \operatorname{Int}_{U_0}(gh(N))$ and both meet ∂U_0 regularly ([Hd], p. 65). Therefore, by Corollary 2.16.2 [Hd], (p. 74), $\overline{gh(N)-N_1}$ is PL homeomorphic to $(\operatorname{Fr}_{U_0} N_1) \times I$. Consequently, $gh(N) \searrow N_1 \searrow P \cup C$.

From Lemma 1.2, there is a subpolyhedron S of $gh(N)$ such that

$$gh(N) \searrow P \cup C \cup S \searrow P \cup C \searrow P,\ X \subset S,\ S \cap \partial U_0 \subset P, \quad \text{and} \quad \dim S \leqslant k+1 .$$

Finally, define $C' = C \cup S$, then $C' \searrow (C \cup S) \cap P = C' \cap P = C' \cap R \subset \mathring{R}$ since the above collapse leaves $P \subset \mathring{R}$ fixed; and it is clear that $\dim(C'-C) \leqslant \dim S \leqslant k+1$. The proof is complete.

For convenience, let us define some notations. Let m, q be integers such that $m \leqslant q-3$. Define $s_0 = \mu_0(1)+\mu_0(2)+ \dots \mu_0(2m-q+1)$, and inductively $s_{i+1} = s_i - \mu_0(2m-q+1-i)$ for each $i = 1, \dots, 2m-q+1$. Observe that $s_{2m-q} = \mu_0(1)$ and $s_{2m-q+1} = 0$.

THEOREM 1.4. *Let* Q^q *be a* PL *manifold and* R^{q-1} *a compact* PL *submanifold of* ∂Q. *Assume that* R *has a sequence of open neighborhoods* $U_0 \supset U_1 \supset U_2 \supset \dots \supset U_{s_0}$ *in* Q *such that for each* $i \leqslant 2m-q+1$ *and for each* $j = 1, 2, \dots, s_0$, *the homomorphism* $\pi_i(U_j, R) \to \pi_i(U_{j-1}, R)$ *is trivial. Let* M^m *be a compact* PL *manifold* $(m \leqslant q-3)$ *such that the pair* $(M, \partial M)$ *is* $(2m-q)$-*connected, and* f: $(M, \partial M) \to (U_{s_0}, R)$ *a continuous map. Then,* f *is homotopic to a* PL *embedding* φ: $(M, \partial M) \to (\mathring{U}_0 \cup \mathring{R}, \mathring{R})$ *via a homotopy of pairs* $(M \times I, \partial M \times I) \to (U_0, R)$.

Proof. By using collars of ∂R and ∂Q in R and Q respectively, we can homotope f to a map g: $(M, \partial M) \to (Q, \mathring{R})$ such that $g(\mathring{M}) \subset \mathring{Q}$. Moreover, by general positioning, we can assume that f is in general position and nondegenerate. In particular,

$$\dim(S(f) \cap \mathring{M}) \leqslant 2m-q .$$

Let $X_0 = \overline{S(f)-\partial M}$. By Theorem 7.8 [Hd], there is a compact PL subspace C of M such that $X_0 \subset C \searrow C \cap \partial M$ and $\dim C \leqslant 2m-q+1$. So, $\dim f(C) \leqslant 2m-q+1 < m \leqslant q-3$. By Lemma 1.3, there is a compact PL subspace D of U_{s_1} such that $f(C) \subset D \searrow D \cap R \subset \mathring{R}$ and $\dim D \leqslant 2m-q+2$. By General Position Theorem 4.6 [Hd], there exists a PL homeomorphism h: $U_{s_1} \to U_{s_1}$ realized by an ambient isotopy such that

(a)$_1$ $h = \mathrm{id}$ on $f(C) \cup \partial U_{s_1}$ and outside a compact subset of U_{s_1}, and

(b)$_1$ $\dim\{[h(D)-(f(C) \cup \partial U_{s_1})] \cap f(M)\} \leqslant (2m-q+2)+m-q \leqslant 2m-q-1$

since $m-q \leqslant -3$.

Therefore, $f^{-1}(h(D)) = C \cup X \cup Y$ where $\dim X \leqslant 2m-q-1$ (because f is nondegenerate) and $Y \subset \partial M$.

Define $C_1 = C$, $D_1 = h(D)$, $X_1 = X$ and $Y_1 = Y$. Suppose by induction that we have found PL subspaces C_i, X_i, Y_i of M, and D_i of U_{s_i} such that

(1)$_i$ $X_0 \subset C_i \searrow C_i \cap \partial M$,

(2)$_i$ $D_i \searrow D_i \cap R \subset \mathring{R}$, and

(3)$_i$ $f^{-1}(D_i) = C_i \cup X_i \cup Y_i$ where $\dim X_i \leqslant 2m-q-i$ and $Y_i \subset \partial M$.

Then, by Lemma 7.8 [Hd], there is a compact PL subspace $C_{i+1} \subset M$ such that

(α) $C_i \cup X_i \subset C_{i+1} \searrow C_{i+1} \cap \partial M$, and

(β) $\dim(C_{i+1}-C_i) \leqslant 2m-q-i+1$.

By Lemma 1.3, there is a compact PL subspace $\tilde{D}$ of $U_{s_i-\mu_0(2m-q-i+1)} = U_{s_{i+1}}$ such that

(*) $D_i \cup f(C_{i+1}) \subset \tilde{D} \searrow \tilde{D} \cap R \subset \mathring{R}$, and

(**) $\dim(\tilde{D}-D_i) \leqslant 2m-q-i+2$.

By General Position Theorem 4.6 [Hd], there is a PL homeomorphism g: $U_{s_{i+1}} \to U_{s_{i+1}}$ realized by an ambient isotopy such that

(a)$_{i+1}$ $g = \mathrm{id}$ on $f(C_{i+1}) \cup \partial U_{s_{i+1}}$ and outside a compact subset of $U_{s_{i+1}}$, and

(b)$_{i+1}$ $\dim\{[g(\tilde{D})-(f(C_{i+1}) \cup \partial U_{s_{i+1}})] \cap f(M)\} \leqslant (2m-q-i+2)+m-q \leqslant 2m-q-i-1$ by (**). (Observe that $D_i \subset f(C_{i+1}) \cup \partial U_{s_{i+1}}$ by (3)$_i$ and (α) above.)

Define $D_{i+1} = g(\tilde{D})$. Then, $f^{-1}(D_{i+1})$ can be written as $C_{i+1} \cup X_{i+1} \cup Y_{i+1}$ where $Y_{i+1} \subset \partial M$ and $\dim X_{i+1} \leqslant 2m-q-i-1$ by use of (b)$_{i+1}$ and the nondegeneracy of f. Therefore, the proof of the inductive step is complete.

Observe that if we define $k = 2m-q+1$, then $X_k = \emptyset$ and $f^{-1}(D_k) = C_k \cup Y_k$ where $Y_k \subset \partial M$ and $D_k \subset U_0$. Therefore, $S(f) \subset C_k \cup \partial M$, $C_k \searrow C_k \cap \partial M$, $D_k \searrow D_k \cap \cap \partial U_0$ so that $f^{-1}(D_k \cup \partial Q) = C_k \cup \partial M$.

Now, the rest of the proof is the same as the corresponding part in the proof of Theorem 8.2 [Hd] (pp. 179–180). For the sake of the completeness, we include here an outline of the proof. Let P be a PL submanifold neighborhood of $f(\partial M)$ in ∂Q such that $D_k \cap \partial Q \subset P$; hence, $D_k \searrow D_k \cap P$. Let K and L be triangulations of M and Q so that $f\colon K \to L$ is simplicial and C_k, D_k are triangulated as subcomplexes. Let $N_1 = N(\partial M \cup C_k, K'')$ and $N_2 = N(P \cup D_k - \partial P, L'')$ (the simplicial neighborhood of $P \cup D_k$ mod ∂P, [R]) where K'' and L'' are the second derived subdivisions of K and L. Then, $f^{-1}(N_2) = N_1$, $N_1 \cong \partial M \times I$ and $N_2 \cong (P \times I)/[(y, 0) = (y, t) | y \in \partial P, t \in I]$ (refer to [R], p. 25). By use of these product structures and adjoining to each an inner collar or an inner pinched collar, we can construct homotopies $F_t\colon M \to M$ and $G_t\colon Q \to Q$ with the following properties: $F_0 = \mathrm{id}_M$, F_1 is a PL homeomorphism $M \to \overline{M - N_1}$, $F_t(\partial M) \subset N_1$ for all t; $G_0 = \mathrm{id}_Q$, $G_1 | \overline{Q - N_2}\colon \overline{Q - N_2} \to Q$ is a PL homeomorphism and G_1 carries N_2 into P, G_t keeps ∂P fixed and $G_t(\partial Q) = \partial Q$ for all t. Then, $g = G_1 f F_1 \simeq G_1 f F_0 \simeq G_0 f F_0 = f$; consequently, g is a required PL embedding. Therefore, the theorem is proved.

§ 2. **Some technical lemmas.** In this section, we will prove some results about homotopy groups that will allow us to use Theorem 1.4 in the proof of Theorem 3.3 later.

For the notions of the homotopy groups, the homotopy exact sequences, ... of triads, we refer to [H]. Given two triads $(X; A, B) \subset (Y; C, D)$, the homomorphism

$$\pi_q(X; A, B) \to \pi_q(Y; C, D)$$

will be the inclusion-induced one if it is not specified otherwise.

For a map $f\colon K \to Y$, its *mapping cylinder* is defined to be

$$M(f) = (K \times [0, 1]) \cup_f Y,$$

where $(x, 1)$ is identified with $f(x)$ for each $x \in K$. Given two maps $f\colon K \to Y$ and $g\colon K \to X$ with $Y \subset X$, if $H\colon f \simeq g$ is a homotopy from f to g, let $J\colon M(f) \to M(g)$ be the map defined by

$$J([z]) = \begin{cases} [x, 2t] & \text{if } z = (x, t) \in K \times [0, \tfrac{1}{2}), \\ [H(x, 2t-1)] & \text{if } z = (x, t) \in K \times [\tfrac{1}{2}, 1], \\ [z] & \text{if } \quad z \in Y. \end{cases}$$

Observe that $J_*\colon \pi_q(M(f), K) \to \pi_q(M(g), K)$ is an isomorphism for each q if $Y = X$.

LEMMA 2.1. *Let Z be a compactum having the shape of a finite complex K. If Z is contained in a* PL *manifold M, then there is a sequence of compact-*PL*-manifold neighborhood $U_1 \supset U_2 \supset \ldots$ of Z, and a shape equivalence $\{f_n\}$, where $f_n\colon K \to U_n$, such that the induced homomorphism $J_*\colon \pi_q(M(f_{n+1}), K) \to \pi_q(M(f_n), K)$ is a zero map for each $n = 1, 2, \ldots$ and $q = 0, 1, \ldots$*

Proof. Referring to [M–S], we can define a sequence $\{U_n\}$ of compact-PL-manifold neighborhoods of Z that is an ANR-system associated to Z, and a shape equivalence $\{f_n\}\colon K \to \{U_n\}$ with a shape inverse $\{g_n\}\colon \{U_n\} \to K$ since K is a compact ANR. Moreover, we can assume that $g_n f_n \simeq \mathrm{id}_K$ and $f_n g_{n+1} \simeq i\colon U_{n+1} \subset U_n$ for each n. Therefore, for each q and n, we have the following:

(a) $0 \to \pi_q(K) \xrightarrow{(f_n)_*} \pi_q(U_n) \to \pi_q(M(f_n), K) \to 0$ is exact, and

(b) $\mathrm{Im}[\pi_q(U_{n+1}) \to \pi_q(U_n)] = \mathrm{Im}[(f_n)_*\colon \pi_q(K) \to \pi_q(U_n)]$, Now, from the following commutative ladder

$$\begin{tikzcd}
0 \arrow[r] & \pi_q(K) \arrow[r, "(f_{n+1})_*"] \arrow[d, "\mathrm{id}_*"] & \pi_q(U_{n+1}) \arrow[r] \arrow[d] & \pi_q(M(f_{n+1}), K) \arrow[r] \arrow[d, "J_*"] & 0 \\
0 \arrow[r] & \pi_q(K) \arrow[r, "(f_n)_*"] & \pi_q(U_n) \arrow[r] & \pi_q(M(f_n), K) \arrow[r] & 0\,,
\end{tikzcd}$$

using (a) and (b), we can prove that J_* is trivial.

LEMMA 2.2. *Assume that $\{U_n\}$, K and $\{f_n\}$ are defined from Lemma 2.1. Let r be an integer and P a subcomplex of K so that $\pi_j(K, P) = 0$ for all $j \leqslant r$. Then, for each $n = 1, 2, \ldots$, the map $J_*^P\colon \pi_j(M(f_{n+1}), P) \to \pi_j(M(f_n), P)$ is trivial for all $j \leqslant r$.*

Proof. For each n, it follows from the homotopy sequence of the triple $(M(f_n), K, P)$ that the homomorphism $\pi_j(M(f_n), P) \to \pi_j(M(f_n), K)$ is an isomorphism for all $j \leqslant r$. Therefore, J_*^P is trivial since J_* is trivial by Lemma 2.1.

The following lemma, which will be used in the proof of Lemma 2.4 below, is a shape version of Lemma 12.3 [Hd].

LEMMA 2.3. *Let K^k be a compact subpolyhedron of a* PL *manifold M^m. Given an integer $r > 0$, and a sequence of open neighborhoods $U_0 \supset U_1 \supset \ldots \supset U_r \supset N$ of K such that for each $j \leqslant r$ the homomorphism*

$$\tau_*\colon \pi_j(U_i - K, N - K) \to \pi_j(U_{i-1} - K, N - K)$$

is trivial for each $i = 1, \ldots, r$, then the homomorphism $\pi_j(U_r;\ U_r - K, N) \to \pi_j(U_0;\ U_0 - K, N)$ is trivial for $j \leqslant r + m - k - 1$.

Proof. Following the proof of Lemma 12.3 [Hd], let $f\colon (B^j;\ F_1, F_2) \to (U_r;\ U_r - K, N)$ be a nondegenerate PL map representing an element of $\pi_j(U_r;\ U_r - K, N)$ where F_1 and F_2 are PL $(j-1)$-cells such that $F_1 \cap F_2 = \partial F_1 = \partial F_2$ and that $F_1 \cup F_2 = \partial B^j$. Using a homotopy if necessary, we can assume that $f(B^j) \subset \mathring{U}_r$.

Define $X = f^{-1}(K)$. Then, $X \cap F_1 = \emptyset$ and $\dim X \leqslant j + k - m$. Let $\tilde{F}_2$ be the closure of the complement of a closed PL collar of ∂F_2 in F_2. Since $B^j \searrow \tilde{F}_2$, there is from Lemma 7.3 [Hd] a subpolyhedron C of B such that $X \subset C \searrow C \cap \tilde{F}_2$ and $\dim C \leqslant j + k - m + 1 \leqslant r$. Let P be a subpolyhedron of C such that $P \cap X = \emptyset$, $C - f^{-1}(N) \subset \mathrm{Int}_C P$, $f^{-1}(P) \cap \partial B = \emptyset$ and $(f|C)^{-1}(f(P)) = P$. Let $P_0 = \mathrm{Fr}_C P$, then $f(P) \subset U_r - K$ and $f(P_0) \subset N - K$.

Now, by use of the triviality of τ_*'s, we can construct by induction on the skeleta of $f(P)$ a homotopy (rel. $f(P_0)$) $H\colon f(P)\times I\to U_0-K$ such that $H(y,0)=y$ and $H(y,1)\in N-K$ for each $y\in f(P)$. Then, extend H over $f(C)$ via the identity. By use of $f^{-1}(P)\cap\partial B=\emptyset$ and with care, we can extend the homotopy $H_t f|C$ to a homotopy $f\simeq g\colon (B;F_1,F_2)\to(U_0;U_0-K,N)$ such that

(1) $g^{-1}(K)=f^{-1}(K)$, and (2) $g(C)\subset N$.

Let R be a second derived neighborhood of $\bar F_2\cup C$ in B with $R\cap F_1=\emptyset$ and $g(R)\subset N$. Since $\bar F_2\cup C\searrow \bar F_2$, R is a j-ball in B with $R\cap\partial B$ is its face. So, there is a strong deformation retraction $\beta\colon B\to\overline{B-R}$. It follows that $f\simeq g\simeq g\beta\colon (B;F_1,F_2)\to(U_0;U_0-K,N)$ with $g\beta(B)\subset U_0-K$. So, f represents the zero element of $\pi_j(U_0;U_0-K,N)$.

LEMMA 2.4. *Let L^k be a compact subpolyhedron of a* PL *manifold M^m with $k\leqslant m-3$ and $\dim(L\cap\partial M)\leqslant k-1$. Given an integer r, set $\alpha=2^r r^r$. Let $U_0\supset U_1\supset\ldots\supset U_\alpha$ be a sequence of open neighborhoods of L such that, for each $i=1,2,\ldots,\alpha$ and $j=0,1,\ldots,r$, the homomorphism $\pi_j(U_i,L)\to\pi_j(U_{i-1},L)$ is trivial. Let $N=\operatorname{Int}_{U_\alpha}W$ where W is a regular neighborhood of L in U_α meeting ∂U_α regularly. Then, the inclusion-induced homomorphism $\pi_q(U_\alpha;U_\alpha-L,N)\to\pi_q(U_0;U_0-L,N)$ and $\pi_j(U_\alpha-L,N-L)\to\pi_j(U_0-L,N-L)$ are trivial for each $q\leqslant r+m-k-1$, and $j\leqslant r$.*

Proof. We will prove by induction. Define $U_p^{(1)}=U_{pr}$ where $p=0,1,2,\ldots,\alpha_1=\alpha/r=2(2r)^{r-1}$. Inductively, $U_p^{(t)}=U_{2pr}^{(t-1)}$ where $p=0,1,2,\alpha_t=\alpha_{t-1}/2r=2(2r)^{r-t}$. Specially, $U_p^{(r)}=U_{2pr}^{(r-1)}$ where $p=0,1,2$; and $U_0^{(r)}=U_0$, $U_1^{(r)}=U_{\alpha/2}$, $U_2^{(r)}=U_\alpha$.

We will show that

$$(*)\qquad \pi_q(U_p^{(t)};U_p^{(t)}-L,N)\to\pi_q(U_{p-1}^{(t)};U_{p-1}^{t}-L,N)$$

is trivial for all $q\leqslant q_t=t+m-k-1$, $p=1,2,\ldots,\alpha_t$.

For $t=1$, since $k\leqslant m-3$, the homomorphism $\pi_1(U_i-L,N-L)\to\pi_1(U_i,N)$ is an isomorphism for each $i=0,1,\ldots,\alpha$. Consequently, the homomorphism $\pi_1(U_i-L,N-L)\to\pi_1(U_{i-1}-L,N-L)$ is trivial for each $i=1,2,\ldots,\alpha$. Then, applying Lemma 2.3 to each segment $\{U_i|pr-r\leqslant i\leqslant pr\}$ of $\{U_j|j=0,1,\ldots,\alpha\}$ (where $p=1,2,\ldots,\alpha_1$), we can infer that the homomorphism

$$\pi_q(U_p^{(1)};U_p^{(1)}-L,N)\to\pi_q(U_{p-1}^{(1)};U_{p-1}^{(1)}-L,N)$$

is trivial for each $q\leqslant q_1=1+m-k-1$.

Assume inductively that the statement $(*)$ has been proved up to $t=s$ and $q_s\leqslant r+1$. For each even integer p $(0\leqslant p\leqslant\alpha_s)$, by chasing on the natural ladder consisting of the first homotopy exact sequences of three consecutive triads $(U_p^{(s)};U_p^{(s)}-L,N)$, $(U_{p-1}^{(s)};U_{p-1}^{(s)}-L,N)$ and $(U_{p-2}^{(s)};U_{p-2}^{(s)}-L,N)$, we can show that the homomorphism

$$\pi_j(U_p^{(s)}-L,N-L)\to\pi_j(U_{p-2}^{(s)}-L,N-L)$$

is trivial for each $j\leqslant q_s-1$. Again, applying Lemma 2.3 to each segment

$$\{U_i^{(s)}|2(p-1)r\leqslant i\leqslant 2pr,\ \text{i even}\}$$

of the sequence $\{U_i^{(s)}|i=0,\ldots,\alpha_s\}$ (where $p=1,2,\ldots,\alpha_{s+1}$) we can infer that the homomorphism

$$\pi_q(U_p^{(s+1)};U_p^{(s+1)}-L,N)\to\pi_q(U_{p-1}^{(s+1)};U_{p-1}^{(s+1)}-L,N)$$

is trivial for all $q\leqslant q_{s+1}=(q_s-1)+m-k-1$ $(\geqslant q_s+1=(s+1)+m-k-1)$. Therefore, the proof of the inductive step is complete.

Now, from the case $t=r$, we have the homomorphisms

$$\pi_q(U_\alpha;U_\alpha-L,N)\to\pi_q(U_{\alpha/2};U_{\alpha/2}-L,N)$$

and

$$\pi_q(U_{\alpha/2};U_{\alpha/2}-L,N)\to\pi_q(U_0;U_0-L,N)$$

are trivial for all $q\leqslant q_r=r+m-k-1$ $(\geqslant r+1)$. Again, by chasing on the natural ladder consisting of the first homotopy exact sequences of these three triads, we can show that the homomorphism

$$\pi_j(U_\alpha-L,N-L)\to\pi_q(U_0-L,N-L)$$

is trivial for each $j\leqslant r$. Therefore, the lemma is proved. ■

LEMMA 2.5. (a) *In addition to the hypotheses of Lemma 2.2, assume that there is a polyhedron $L\subset U_{n+1}$ such that $f_{n+1}(P)\subset L$ and $f_{n+1}\colon P\to L$ is a homotopy equivalence, then the homomorphism $\pi_j(U_{n+1},L)\to\pi_j(U_n,L)$ is trivial for each $j\leqslant r$.*

(b) *Consequently, if $L\subset U_\alpha$ and N is a regular neighborhood of L in $\operatorname{Int}_M U_\alpha$ where $\alpha=2^r r^r$ as in Lemma 2.4, then for all $j\leqslant r+m-k-1$ $\operatorname{Im}(i_{\alpha,0})\subset\operatorname{Im}(j_0)$ where $j_0\colon\pi_j(U_0-L,N-L)\to\pi_j(U_0,N)$ and $i_{\alpha,0}\colon\pi_j(U_\alpha,N)\to\pi_j(U_0,N)$ are homomorphisms induced from the inclusions.*

Proof. (a) Since $f_n\simeq f_{n+1}$ in U_n, by use of the naturality of the map J defined at the beginning of this section, we can assume that $f_n|P=f_{n+1}|P$. Now, for each $i=n,n+1$, define a natural retraction $F_i\colon(M(f_i),P)\to(U_i,L)$ by

$$F_i(x)=x\qquad\text{if}\quad x\in U_i,$$

$$F_i(p,t)=f_i(p).\quad\text{if}\quad p\in P\ \text{and}\ t\in I.$$

Then, for each $j=1,2,\ldots$, it follows from Five-Lemma that

$$(F_i)_*\colon\pi_j(M(f_i),P)\to\pi_j(U_i,L)$$

is an isomorphism. Therefore, from Lemma 2.2, the homomorphism $\pi_j(U_{n+1},L)\to\pi_j(U_n,L)$ is trivial for each $j\leqslant r$.

(b) The lemma follows from using Lemma 2.4 and chasing on the following commutative diagram:

$$\begin{tikzcd}
\ldots \arrow[r] & \pi_j(U_\alpha-L, N-L) \arrow[r] \arrow[d] & \pi_j(U_\alpha, N) \arrow[r] \arrow[d, "i_{\alpha,0}"] & \pi_j(U_\alpha;\ U_\alpha-L, N) \arrow[r] \arrow[d, "0"] & \ldots \\
\ldots \arrow[r] & \pi_j(U_0-L, N-L) \arrow[r, "j_0"] & \pi_j(U_0, N) \arrow[r] & \pi_j(U_0;\ U_0-L, N) \arrow[r] & \ldots
\end{tikzcd}$$

where the horizontal lines are the first homotopy exact sequences of the triads $(U_\alpha;\ U_\alpha-L, N)$ and $(U_0;\ U_0-L, N)$, and the vertical maps are inclusion-induced homomorphisms.

§ 3. **Main results.** In this section, we will state and prove the main result, Theorem 3.3. All polyhedra will be compact.

Lemma 3.1. *Let N be a regular neighborhood of a polyhedron K^k in a* PL *manifold M^m, $\partial M = \emptyset$ and $k \leqslant m-3$. If $P^k \hookrightarrow \mathring{N}$ is a homotopy equivalence where P is a polyhedron in N, then $N-K \cong \partial N \times [0, 1) \cong N-P$.*

Proof. Let us give an outline of the proof. Let V be a regular neighborhood of P in $\mathring{N}$. We will show that for all q

(1) $\pi_q(\overline{N-V}, \partial V) = 0$, and (2) $\pi_q(\overline{N-V}, \partial N) = 0$.

Then, it follows from Theorem 7.11 [Hd] that $\partial N \times [0, 1) \cong N-V \cong N-P$. Therefore, the lemma will be proved.

To prove (1), observe that $\pi_i(N, P) = 0$ for all i. Hence, from the proof of Lemma 12.4 [Hd], it follows that $\pi_q(N-P, V-P) = 0$ for all q. So, $\pi_q(\overline{N-V}, \partial V) = 0$ since $V-P \cong \partial V \times [0, 1)$.

To prove (2), since $\pi_q(N, \mathring{V}) \cong \pi_q(N, P) = 0$ for all q, and since $k \leqslant m-3$, by Engulfing Theorem 7.4 [Hd], we can assume that $K \subset \mathring{V}$. Similarly, let W be a regular neighborhood of K in $\mathring{V}$ such that $P \subset \mathring{W}$. Then, by Corollary 2.16.2 [Hd], $\overline{N-W} \cong \partial N \times I$. In particular, there is a strong deformation $F\colon \overline{N-W} \times I \to \overline{N-W}$ from the identity to a retraction $F_1\colon \overline{N-W} \to \partial N$. Let $\varphi\colon V-P \to \partial V$ be a retraction. Then, $\varphi F|(\overline{N-V} \times I)$ defines a strong deformation from $\overline{N-V}$ onto ∂N. Hence, (2) is proved.

The following proposition will be used in the proof of Theorem 3.3, and generalized by Theorem 3.4 below. It is a stronger version of Corollary 3.4 [S], but a special case of Theorem 4.1 [S]. However, we take liberty to include here its simpler proof.

Proposition 3.2. *Let K_0^k and K_1^k be polyhedra in the interior of a* PL *manifold M^m, $m \geqslant 6$, which are polyhedral concordant by a polyhedron L^{k+1}. If $k \leqslant m-3$, then $M-K_0 \cong M-K_1$.*

Proof. Fix an $i = 0, 1$. Let N_i be a regular neighborhood of K_i in M. From Theorem 8.7 (2) [C], there is a finite complex $Q_i \supset K_i$ such that

$$\dim Q_i = \max\{3, \dim K_i\} \leqslant m-3$$

and the torsions of $K_i \subset L$ and $K_i \subset Q_i$ are equal. Therefore, by Theorem 24.2 [C], there is a map $\varphi_i\colon Q_i \to \mathring{N}_i$ such that $\bar{\varphi}_i\colon Q_i \xrightarrow{(\varphi_i, i)} N_i \times \{i\} \subset N_i \times \{i\} \cup L$ is a simple homotopy equivalence. Then, by Theorem 12.1 [Hd], we have the following commutative diagram:

$$\begin{tikzcd}
Q_i \arrow[rr, "\varphi_i"] \arrow[dr] & & \mathring{N}_i \\
& P_i \arrow[ur, hook] &
\end{tikzcd}$$

where P_i is a subpolyhedron of dimension $\leqslant k$ and $Q_i \to P_i$ is a simple homotopy equivalence. Now, by Lemma 7.3 [Hd], there is a subpolyhedron T_i of N_i such that $N_i \searrow K_i \cup T_i \equiv X_i \searrow K_i$, $P_i \subset T_i$ and $\dim T_i \leqslant k+1$.

Let $L^+ = (P_0 \times [-1, 0]) \cup (X_0 \times \{0\}) \cup L \cup (X_1 \times \{1\}) \cup (P_1 \times [1, 2])$. Then, it is straightforward to show that the inclusions $P_0 \times \{-1\} \subset L^+$ and $P_1 \times \{2\} \subset L^+$ are simple homotopy equivalences. Hence, $M-P_0 \cong M-P_1$ by Theorem 3.2 [S].

On the other hand, $M-P_i \cong M-K_i$ for each $i = 0, 1$, by Lemma 3.1 above. Therefore, the proposition is proved.

Theorem 3.3. *Let K_0^k and K_1^k be polyhedra in a* PL *manifold M^m, $m \geqslant 6$ and $\partial M = \emptyset$. If K_0 and K_1 are shape concordant by a compactum Z and if $k \leqslant m-3$, then $M-K_0 \cong M-K_1$.*

Sublemma. *There is a homotopy equivalence $g'\colon K_0 \times \{0\} \to K_1 \times \{1\}$ such that if U is an open neighborhood of Z in $M \times I$, then g' is homotopic to the inclusion $K_0 \times \{0\} \subset U$.*

Proof. By hypothesis, for each $\lambda = 0, 1$, the inclusion $i_\lambda\colon K_\lambda \times \{\lambda\} \to Z$ is a shape equivalence. Since K_1 is a compact absolute neighborhood retract, there is a map $h\colon Z \to K_1 \times \{1\}$ defining a shape inverse of i_1; consequently, the map $g' = hi_0\colon K_0 \times \{0\} \to K_1 \times \{1\}$ is a homotopy equivalence and the following diagram

$$\begin{tikzcd}
K_1\times\{1\} \arrow[dr, hook, "i_1"] \arrow[drr, hook] & & \\
& Z \arrow[r, hook] & U \\
K_0\times\{0\} \arrow[uu, "g'"] \arrow[ur, hook, "i_0"] \arrow[urr, hook] & &
\end{tikzcd}$$

is commutative in the shape category for every open neighborhood U of Z in $M \times I$. Therefore, g' is homotopic in U to the inclusion $K_0 \times \{0\} \subset U$.

Proof of Theorem 3.3. By Proposition 3.2, we only need to show that K_0 and K_1 are polyhedrally concordant by a polyhedron L of dimension at most $k+1$.

Let $g\colon K_0 \to K_1$ defined by $(g(x), 1) = g'(x, 0)$ where g' is a map obtained from Sublemma. Up to a homotopy, we can assume that g is a PL map. Let P_{-1}

$= (K_0\times\{0\})\cup(K_1\times\{1\})$; P a simplicial triangulation of the mapping cylinder $M(g)$ of g; and $\{A_i|i=0,1,\ldots,\omega\}$ the collection of all simplices of $P-P_{-1}$ in order of increasing dimension.

Let $U_0\supset U_1\supset\ldots$ be a sequence of compact-PL-manifold neighborhoods of Z in M defining an ANR-system associated to Z. For each n, since $g'\simeq i\colon K_0\times\{0\}\subset U_n$, there is a map $f_n\colon P\to U_n$ such that

(i) $f_n|P_{-1}=$ identity, and

(ii) $f_n([x,t])\in M\times\mathring{I}$ if $x\in K_0$, $t\in\mathring{I}$.

Furthermore, by general positioning ([Hd], p. 102), we can assume that each f_n is a nondegenerate PL map.

Observe that if $v\colon P\to K_1\times\{1\}$ is the natural deformation retraction, then $f_nv=f_{n+1}v$; hence, f_n and f_{n+1} are homotopic in U_n for each n. Consequently, $\{f_n\}$ is a shape equivalence since $\{f_nv\}$ is a shape equivalence. By choosing subsequences, we can assume that the sequence $U_0\supset U_1\supset\ldots$ satisfies Lemma 2.1. Set $\delta=2k+2-m$.

Let us consider the sequence $U_0\supset U_1\supset\ldots\supset U_{\omega(s_0+1)\alpha}$ where $s_0=\mu_0(1)+\mu_0(2)+\ldots+\mu_0(2(k+1)-(m+1)+1)$ (refer to Theorem 1.4) and where $\alpha=2^\delta\delta^\delta$ (refer to Lemma 2.4). For each $i=0,1,\ldots,\omega$, let $w_i=(\omega-i)(s_0+1)\alpha$. We will homotope f_{w_0} (rel. P_{-1}) to map $h\colon P\to U_0$ such that $h(P)\subset L\subset U_0$, $\dim L\leqslant k+1$, $L\cap(M\times\partial I)=P_{-1}$ and $h\colon P\to L$ is a homotopy equivalence. Therefore, K_0 and K_1 will be polyhedrally concordant by L, and the proof will be complete.

We proceed by induction. For each $i\geqslant 0$, let $P_i=P_{-1}\cup\{A_j|j\leqslant i\}$. Then, we will use the following inductive statement: $h_{-1}=f_{w_0}$ is homotopic (rel. P_{-1}) to a map $h_i\colon P\to U_{w_i}$ such that there is a compact subpolyhedron L_i of U_{w_i} satisfying the following properties:

$(a)_i$ $\dim L_i\leqslant k+1$,

$(b)_i$ $L_i\cap(M\times\partial I)=P_{-1}$,

$(c)_i$ $h_i(P_i)\subset L_i$, and

$(d)_i$ $h_i\colon P_i\to L_i$ is a (simple) homotopy equivalence.

When $i=0$, $P_0=P_{-1}\cup\{\text{point}\}$ and there is nothing to prove.

Assume that h_i has been defined for some $i<\omega$ and let $A=A_{i+1}$. Assume that $\dim A=r$; then $\pi_q(P,P_i)=0$ for all $q\leqslant r-1$. Hence, it follows from Lemma 2.2 that, for each $q\leqslant r-1$ and $j\leqslant w_i$, the homomorphism $\pi_q(M(f_j),P_i)\to\pi_q(M(f_{j-1}),P_i)$ is trivial; consequently, so is $\pi_q(U_j,L_i)\to\pi_q(U_{j-1},L_i)$, for $\pi_q(M(f_j),P_i)\cong\pi_q(M(h_{ij}),P_i)\cong\pi_q(U_j,L_i)$ since $f_j\simeq h_{ij}\equiv\tau_{ij}h_i$ (where $\tau_{ij}\colon U_{w_i}\subset U_j$) in U_j for each $j\leqslant w_i$ and since $h_{ij}|P_i=h_i|P_i\colon P_i\to L_i$ is a homotopy equivalence (see the proof of Lemma 2.5 (a)).

Let N be a regular neighborhood of L_i in U_{w_i} that meets $M\times\partial I$ regularly. For convenience, let $\operatorname{Fr}N$ denote $\operatorname{Fr}_{U_{w_i}}N$. Then, since $N-L_i\cong\operatorname{Fr}N\times[0,1)$ by use of Corollary 2.16.2 [Hd], we obtain:

(1) $\operatorname{Im}(i_*)\subset\operatorname{Im}(j_*)$ where

$$j_*\colon \pi_r(\overline{U_{w_i-\alpha}-N},\operatorname{Fr}N)\to\pi_r(U_{w_i-\alpha},N),$$
$$i_*\colon \pi_r(U_{w_i},N)\to\pi_r(U_{w_i-\alpha},N)$$

are inclusion-induced homomorphisms (refer to Lemma 2.5 (b));

(2) $\pi_j(\overline{U_{\lambda\alpha}-N},\operatorname{Fr}N)\to\pi_j(\overline{U_{(\lambda-1)\alpha}-N},\operatorname{Fr}N)$ is a zero map for each $j\leqslant 2r-(m+1)+1$ $(\leqslant r)$ and each λ with $w_{i-1}<\lambda\alpha\leqslant w_i-\alpha$ (from Lemma 2.4). Observe that λ takes s_0 different values.

Now, consider $h_i|A\colon(A,\partial A)\to(U_{w_i},N)$. By ($\alpha$), there is a map $\varrho\colon(A,\partial A)\to(\overline{U_{w_i-\alpha}-N},\operatorname{Fr}N)$ such that $\varrho\simeq h_i|(A,\partial A)$ in $(U_{w_i-\alpha},N)$ and $\varrho(A)\cap(M\times\partial I)=\emptyset$. Furthermore, by the property (2) and Theorem 1.4, there is a homotopy $H\colon(A\times I,\partial A\times I)\to(\overline{U_{w_{i-1}}-N},\operatorname{Fr}N)$ from ϱ to a PL embedding $\varphi\colon(A,\partial A)\to(\overline{U_{w_{i-1}}-N},\operatorname{Fr}N)$. By use of the homotopy extension property, we can extend φ to a map $\psi\colon P\to U_{w_{i-1}}$ such that $\psi|P_i\simeq h_i|P_i\colon P_i\to N$ and $\psi(P)\cap(M\times\partial I)=P_{-1}$. Then, $\psi|(P_i\cup A)\colon(P_i\cup A)\to N\cup\varphi(A)$ is a (simple) homotopy equivalence. Now, since $N\searrow L_i$ and $\varphi(A)\cap N=\varphi(A)\cap\operatorname{Fr}N\subset M\times\mathring{I}$, it follows from Lemma 1.2 that there is a subpolyhedron T of $M\times I$ with $\dim T\leqslant(r-1)+1=r\leqslant k+1$ such that $T\cap\partial(M\times I)\subset L_i$, $\varphi(A)\cap N\subset T$ and $N\searrow L_i\cup T$. So, $N\cup\varphi(A)\searrow L_i\cup T\cup\varphi(A)\equiv L_{i+1}$ say. Observe that L_{i+1} satisfies $(a)_{i+1}$ and $(b)_{i+1}$. Now, if $\theta\colon N\cup\varphi(A)\to L_{i+1}$ is a corresponding strong deformation retraction, define $h_{i+1}|P_{i+1}=\theta\psi|P_{i+1}$; then, using the homotopy extension property, we extend h_{i+1} over the whole P with $h_{i+1}\simeq h_i\simeq h_{-1}$ in $U_{w_{i-1}}$. Then, h_{i+1} satisfies $(c)_{i+1}$ and $(d)_{i+1}$. So, the proof of the inductive step is complete.

Now, define $h=h_\omega$ and $L=L_\omega$. Then, the properties $(b)_\omega$ and $(d)_\omega$ show that L defines a polyhedral concordance between K_0 and K_1. Therefore, the proof is complete.

THEOREM 3.4. *Let X_0 and X_1 be compacta in a PL manifold M^m, $\partial M=\emptyset$ and $m\geqslant 6$. Assume that X_0 and X_1 satisfy ILC in M and are shape concordant by a compactum $Z\subset M\times I$. If X_0 has the shape of a polyhedron K^k, $k\leqslant m-3$, then $M-X_0\simeq M-X_1$.*

Proof. Fix an $i=0,1$. From the proof of Corollary 1 of [I-S], there is a polyhedron $K_i^k\subset M$ $(k\leqslant m-3)$ and a regular neighborhood N_i of K_i in M such that $X_i\subset\mathring{N}_i$ is a shape equivalence and $N_i-K_i\simeq\partial N_i\times[0,1)\cong N_i-X_i$. Consequently,

$$(*)\qquad M-X_i\simeq M-K_i\,.$$

Define $Z^+=(K_0\times[-1,0])\cup(N_0\times\{0\})\cup Z\cup(N_1\times\{1\})\cup(K_1\times[1,2])$. Then, $K_0\times\{-1\}\subset Z^+$ and $K_1\times\{2\}\subset Z^+$ are shape equivalences. Hence, K_0 and K_1 are shape concordant; therefore, $M-K_0\cong M-K_1$ by Theorem 3.3. So, by (*), $M-X_0\cong M-X_1$; and the proof is now complete.

Remark. With a little care, when $\partial M\neq\emptyset$, we also can show that Theorems 3.3 and 3.4 also hold true if $X_0,X_1\subset\mathring{M}$.

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L-space without any uncountable 0-dimensional subspace

by

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Abstract. J. Roitman in [Ro] posed the following question: "Under CH is every left separated L-space of type ω_1 0-dimensional?". The paper contains a negative answer to this question. The construction gives also (under the assumption that there exists a cardinal $\varkappa$ s.t. $2^\varkappa = \varkappa^+$) an answer to two questions of Arkhangel'skiĭ [Ar; Problems 11 and 12, p. 81]: "Is every regular left separated space 0-dimensional?" and "Does every completely regular space have a dense 0-dimensional subspace?".

Terminology and notation. Our terminology related to topology and set theory follows [En1] and [Ku] respectively.

A topological space Z is called left (right) separated if there is a well-ordering $<$ of Z s.t. every initial segment of Z under $<$ is closed (open).

$$hL(X) = \sup\{|Z|\colon Z\subset X \text{ is right separated}\}+\omega.$$

By an L-space we will understand a regular space which is hereditarily Lindelöf and not hereditarily separable. Let us recall that a space X is heraditarily Lindelöf if and only if $hL(X) = \omega$.

Let $H(A, B)$ be the set of all finite functions from a set A into a set B and let $\mathscr{B}$ be a standard countable basis for the unit interval $I = \langle 0, 1\rangle$ not containing the empty set. For $\varepsilon \in H(A, \mathscr{B})$ the standard basic set in I^A given by ε is denoted by $[\varepsilon]$, i.e., $[\varepsilon] = \bigcap\{I^{A\setminus(a)}\times(\varepsilon(a))^{(a)}\colon a\in \operatorname{dom}\varepsilon\}$.

For a family $\mathscr{F}$ of subsets of a set X let $\tau(\mathscr{F})$ denote the topology on X determined by $\mathscr{F}$ as a subbasis.

Auxiliary lemmas. The first lemma is a generalization of the Hurewicz Theorem [En2; Thm. 1.8.20, p. 81], that the Hilbert cube I^ω is not the countable sum of 0-dimensional subspaces.

Let $\mathscr{D}$ be the family of all sets of the form $F\times I^{\omega\setminus A}$ where $A\in[\omega]^\omega$ and F is a 0-dimensional subset of I^A.

LEMMA 1. *If $D_n\in\mathscr{D}$ for $n<\omega$ then $I^\omega \neq \bigcup\{D_n\colon n<\omega\}$.*