so enthält jede andere Nebendiagonale mindestens 2 Elemente, die kongruent 0 modulo 2^{t-1} sind, deren Produkt also kongruent 0 mod $2^{2(t-1)}$ und damit auch kongruent 0 modulo z^t ist. $\left|\left(\frac{\alpha_{ij}^n m_i^n}{m_j'^n}\right)\right| \equiv \pm\, n$-te Potenz mod 2^t, je nachdem ob die modulo 2^{t-1} nichttriviale Nebendiagonale zu einer geraden oder ungeraden Permutation gehört.

(Für $s = 1$ ist die Determinante trivialerweise eine n-te Potenz.)

Falls für einen Primteiler p von ggT keine zwei Primpotenzanteile in der Folge der Drehnenner dieselben sind, bestimmt die Lage der nichttrivialen Nebendiagonale die Permutation dieser Primpotenzanteile.

Im Zusammenhang mit Formel (3) folgt somit aus (α) und (β) die Aussage (2b) des Satzes.

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Embedding inverse limits of interval maps as attractors

by

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Abstract. We prove that the inverse limit of the map $4x(1-x)$ of the interval $[0, 1]$ onto itself can be embedded as an attractor into a C^∞ diffeomorphism of any manifold of dimension at least 3 and into a homeomorphims of any manifold of dimension 2.

1. Inverse limits of dynamical systems. We start by recalling some topological facts.

Let I be a compact space and $T\colon I \to I$ a continuous map. We may regard our system (I, T) as an inverse system $\ldots \xrightarrow{T} I \xrightarrow{T} I \xrightarrow{T} I$ and consider its inverse limit. It is a subset K of the product of an infinite number of copies of I: $\prod_0^\infty I$, defined as

$$K = \{(t_n)_{n=0}^\infty \colon T(t_n) = t_{n-1} \text{ for } n = 1, 2, 3, \ldots\}.$$

Denote by Ψ_n the projection of K to the nth coordinate: $\Psi_n((t_i)_{i=0}^\infty) = t_n$. There exists a unique map $\tau\colon K \to K$ such that $\Psi_n \circ \tau = T \circ \Psi_n$ for $n = 0, 1, 2, \ldots$ It is given by $\tau((t_n)_{n=0}^\infty) = (T(t_n))_{n=0}^\infty$. Notice that since $T(t_n) = t_{n-1}$, the nth coordinate of $\tau((t_n)_{n=0}^\infty)$ is equal to t_{n-1} (for $n \geqslant 1$). We consider K as a topological space with topology induced by the product topology in $\prod_0^\infty I$. The map τ is then a homeomorphism. We call τ the *inverse limit* of T. This notion is an analogue of the notion of a natural extension of a measure preserving endomorphism.

If $\tilde{\tau}\colon \tilde{K} \to \tilde{K}$ is a homeomorphism and $\tilde{\Psi}\colon \tilde{K} \to I$ a continuous map such that $\tilde{\Psi} \circ \tilde{\tau} = T \circ \tilde{\Psi}$, then there exists a unique map $\Phi\colon \tilde{K} \to K$ such that $\Phi \circ \tilde{\tau} = \tau \circ \Phi$ and $\tilde{\Psi} = \Psi_0 \circ \Phi$. This Φ is continuous. Thus, the inverse limit is the simplest homeomorphism having T as a factor. This property can be used as a characterization of an inverse limit, up to conjugacy.

2. Problems and results. We want to embed an inverse limit of a continuous map of an interval into itself into a diffeomorphism (homeomorphism) of a manifold onto itself, as an attractor. This means that for a given map of an interval we want to find a diffeomorphism (homeomorphism) with an attractor such that this diffeomorphism (homeomorphism) restricted to the attractor is topologically conjugate to the inverse limit of our interval map.

Since there are several nonequivalent definitions of an attractor, we have to say which one we choose. We say that a set C is an *attractor* for a map $f\colon M\to M$ if:

(i) there exists an open set $U\subset M$ such that $\mathrm{cl}(f(U))\subset U$ and $C=\bigcap_{n\geqslant 0} f^n(U)$,

(ii) $f|_C$ is topologically transitive.

Because of the condition (ii), we need to consider only transitive maps of an interval (an inverse limit of a map is transitive if and only if the map itself is transitive). Hence the results of [1] are of no help here.

In this paper we consider the simplest transitive continuous interval map, namely

(1) $$T\colon I\to I,\quad \text{where}\quad I=[0,1]\quad\text{and}\quad T(t)=1-|2t-1|\,.$$

We obtain the following results:

THEOREM A. *The inverse limit of T can be embedded as an attractor into a C^∞ diffeomorphism of any manifold of dimension at least* 3.

In fact, our example is even better than C^∞ (see Section 7).

THEOREM B. *The inverse limit of T can be embedded as an attractor into a homeomorphism of any manifold of dimension at least* 2.

Clearly, the results are the same for all maps conjugate to T (e.g. $x\mapsto 4x(1-x)$). Apart of some obvious modifications of the examples given here, the problem in the general case seems to be more difficult (especially if the orbit of some turning point is dense), but we do not consider it in this paper.

We shall construct two examples. The first one will be used to prove Theorem A, the second one to prove Theorem B.

The attractors of diffeomorphisms are studied often from the point of view of ergodic theory. One takes the Lebesgue measure restricted to U, looks at its images under the iterates of f and studies limits of subsequences of their averages. In our example the limit of these averages is a measure concentrated at one point. We show reasons why this phenomenon seems to be difficult to avoid.

3. Geometric description of the first example. We take a set

(2) $$A=[-1,1]\times[-\tfrac{1}{3},\tfrac{1}{3}]\subset \boldsymbol{R}^2.$$

Then we take a smooth map $\varphi\colon A\to\boldsymbol{R}^2$ which is one-to-one on the set

(3) $$A_0=(-1,1)\times[-\tfrac{1}{3},\tfrac{1}{3}],$$

glues together the segments $\{-1\}\times[0,\frac{1}{3}]$ with $\{-1\}\times[-\frac{1}{3},0]$ and similarly $\{1\}\times[0,\frac{1}{3}]$ with $\{1\}\times[-\frac{1}{3},0]$ (see Fig. 1). As a consequence, the points of the form $\varphi(\pm 1,t)$, $t\in(-\frac{1}{3},\frac{1}{3})$, are interior points of $\varphi(A)$.

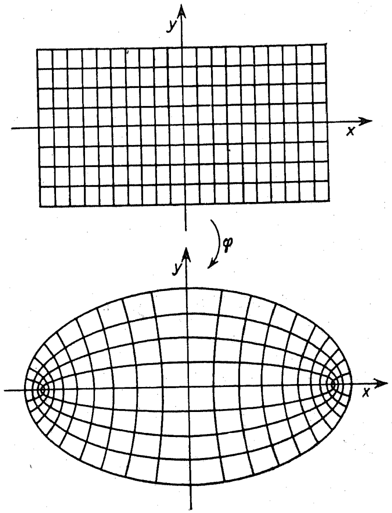

Fig. 1. The sets A and $\varphi(A)$

Now we take a set

(4) $$B=A\times[-1,1]\subset\boldsymbol{R}^3$$

and define a map $\psi\colon B\to\boldsymbol{R}^3$ by

(5) $$\psi(x,y,z)=(\varphi(x,y),z)\,.$$

Next we define a map $g=(g_1,g_2,g_3)\colon B\to B$ and then $f\colon\psi(B)\to\psi(B)$ will be its factor under ψ. The map g is stretching in the direction of x axis (the graph of this stretching part is shown on Fig. 2) and leaves invariant the foliation consisting of 2-dimensional leaves parallel to the y,z-plane. In the directions of y and z axes the map is a contraction and is chosen in such a way that g is one-to-one and stays smooth after factorizing through ψ. The images of the left and right halves of B are shown on Fig. 3 (x and y directions) and Fig. 4 (x and z directions).

The details of the construction are given in the next section. I am giving the formulae instead of saying „there exists a C^∞ map such that ..." even if the existence of such maps (functions) is evident. This is because of two reasons. The first reason

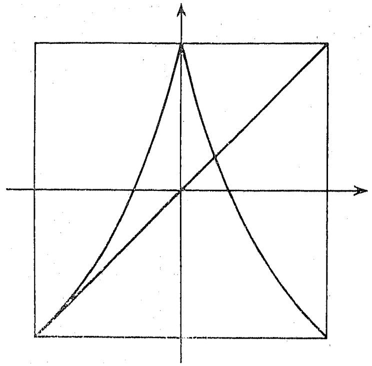

Fig. 2. The graph of g_1

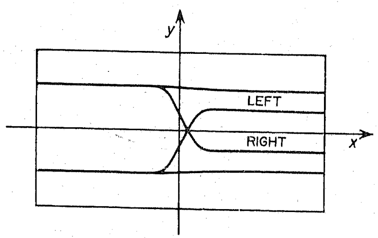

Fig. 3. The image of g_2

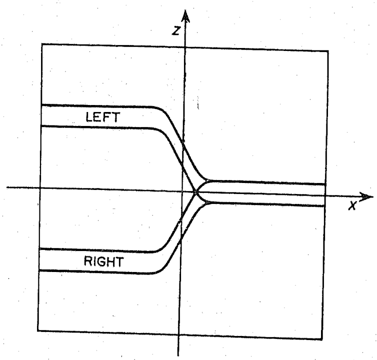

Fig. 4. The image of g_3

is that I want to obtain a map with the properties described in Remarks 1 and 2 (Section 7). The second reason is that I want to give an example which can be examined also with computers (if someone finds it interesting to do so) [1].

4. Details of the first example. We define a function $\sigma\colon \boldsymbol{R}\to\boldsymbol{R}$ by

$$\sigma(t) = \begin{cases} 1 & \text{for } t\leqslant -1, \\ 1-2\exp\left(\dfrac{1}{2}-\dfrac{1}{t+1}\right) & \text{for } t>-1. \end{cases}$$

It is a standard thing to check that σ is nonincreasing and of class C^∞.

Next, we define a function $\xi\colon \boldsymbol{R}\to\boldsymbol{R}$ by $\xi(t) = \frac{1}{2}[\sigma(\sigma(-5t))-\sigma(\sigma(5t))]$.

Lemma 1. *The function ξ has the following properties:*

(i) *ξ is nonincreasing,*
(ii) *ξ is of class C^∞,*
(iii) *$\xi(t) = 1$ for $t\leqslant -\frac{1}{5}$,*
(iv) *$\xi(t) = -1$ for $t\geqslant \frac{1}{5}$,*
(v) *ξ is odd,*
(vi) *$-16<\xi'(t)\leqslant 0$ for every t.*

Proof. (ii) follows from the definition of ξ and smoothness of σ.

To prove (iii) and (iv), notice that if $t\geqslant 1$ then $\sigma(t)\leqslant\sigma(1) = -1$ and $\sigma(\sigma(t)) = 1$ and if $t\leqslant -1$ then $\sigma(t) = 1$ and $\sigma(\sigma(t)) = \sigma(1) = -1$. Consequently, if $t\leqslant -\frac{1}{5}$ then $\xi(t) = \frac{1}{2}[1-(-1)] = 1$ and if $t\geqslant\frac{1}{5}$ then $\xi(t) = \frac{1}{2}[(-1)-1] = -1$.

(v) holds since $\xi(-t) = \frac{1}{2}[\sigma(\sigma(5t))-\sigma(\sigma(-5t))] = -\xi(t)$.

To prove (vi), we have to make some computations. Since σ' is nonpositive, $(\sigma\circ\sigma)'$ is nonnegative, and consequently ξ' is nonpositive. For $t\leqslant -1$, $\sigma'(t) = 0$. For $t>-1$ we have:

$$\sigma'(t) = -2\exp\left(\frac{1}{2}-\frac{1}{t+1}\right)\cdot\left(\frac{1}{t+1}\right)^2,$$

$$\sigma''(t) = -2\exp\left(\frac{1}{2}-\frac{1}{t+1}\right)\cdot\left[\left(\frac{1}{t+1}\right)^4-2\left(\frac{1}{t+1}\right)^3\right].$$

Therefore $\sigma'(t)$ attains its minimum if $t+1 = \frac{1}{2}$. We have $\sigma'(-\frac{1}{2}) = -8\exp(-\frac{3}{2})$, and hence

$$|\sigma'(t)|\leqslant 8\exp(-\tfrac{3}{2}) \quad \text{for all } t.$$

Thus

$$|(\sigma\circ\sigma)'(t)|\leqslant 64\exp(-3) \quad \text{for all } t,$$

[1] Note, that to find a trajectory $p, f(p), f^2(p), \ldots$ it is enough to find q such that $\psi(q) = p$, then to find $q, g(q), g^2(q), \ldots$, and take their images by ψ.

and consequently

$$\left|\frac{d}{dt}\big(\sigma(\sigma(\pm 5t))\big)\right| \leqslant 320\exp(-3) \quad \text{for all } t.$$

Since $\exp(3) > 2.715^3 > 20$, we get $|\xi'(t)| \leqslant 16$ for all t. This ends the proof of (vi).

(i) follows from (vi). ■

Now we take the set A given by (2) and define a map $\varphi\colon A \to \mathbf{R}^2$ by (cf. Remark 2 in Section 7)

$$\varphi(x, y) = \big(2x + (x^2 - y^2)\xi(x),\ 2y + 2xy\xi(x)\big). \tag{6}$$

LEMMA 2. *The map φ restricted to the set A_0 (given by (3)) is one-to-one.*

Proof. We shall show first that if $(x, y) \in A_0$ then

$$\frac{\partial\varphi_1}{\partial x}(x, y) > 0 \tag{7}$$

and

$$4\frac{\partial\varphi_1}{\partial x}(x, y)\cdot\frac{\partial\varphi_2}{\partial y}(x, y) > \left[\frac{\partial\varphi_1}{\partial y}(x, y) + \frac{\partial\varphi_2}{\partial x}(x, y)\right]^2, \tag{8}$$

where φ_i is the ith coordinate of φ.

We have:

$$\frac{\partial\varphi_1}{\partial x}(x, y) = 2 + 2x\xi(x) + (x^2 - y^2)\xi'(x),$$

$$\frac{\partial\varphi_2}{\partial y}(x, y) = 2 + 2x\xi(x),$$

$$\frac{\partial\varphi_1}{\partial y}(x, y) + \frac{\partial\varphi_2}{\partial x}(x, y) = 2xy\xi'(x).$$

If $\frac{1}{5} \leqslant |x| < 1$ then $\xi(x) = -\operatorname{sgn} x$ and $\xi'(x) = 0$, and hence:

$$\frac{\partial\varphi_1}{\partial x}(x, y) = 2 - 2|x| > 0,$$

$$\frac{\partial\varphi_2}{\partial y}(x, y) = 2 - 2|x| > 0,$$

$$\frac{\partial\varphi_1}{\partial y}(x, y) + \frac{\partial\varphi_2}{\partial x}(x, y) = 0.$$

Thus, (7) and (8) are satisfied in this case.

Assume now that $|x| < \frac{1}{5}$. We have

$$(x^2 - y^2)\xi'(x) \geqslant x^2\xi'(x) \geqslant (\tfrac{1}{5})^2\cdot(-16) = -\tfrac{16}{25} \quad \text{and} \quad x\xi(x) > -\tfrac{1}{5}.$$

Therefore:

$$\frac{\partial\varphi_1}{\partial x}(x, y) > 2 - \frac{2}{5} - \frac{16}{25} = \frac{24}{25}, \qquad \frac{\partial\varphi_2}{\partial y}(x, y) > 2 - \frac{2}{5} = \frac{8}{5},$$

$$\frac{\partial\varphi_1}{\partial y}(x, y) + \frac{\partial\varphi_2}{\partial x}(x, y) < 2\cdot\frac{1}{5}\cdot\frac{1}{3}\cdot 16 = \frac{32}{15}.$$

Hence, (7) is satisfied, and since $4\cdot\frac{24}{25}\cdot\frac{8}{5} > (\frac{32}{15})^2$, (8) is also satisfied.

Now we go back to the general case. Denote by $D\varphi_{(x,y)}$ the derivative of φ at $(x, y) \in A_0$. Take a vector $(\alpha, \beta) \in \mathbf{R}^2 \setminus \{(0, 0)\}$. Then the scalar product

$$(\alpha, \beta)\cdot D\varphi_{(x,y)}(\alpha, \beta) = \frac{\partial\varphi_1}{\partial x}(x, y)\alpha^2 + \left[\frac{\partial\varphi_1}{\partial y}(x, y) + \frac{\partial\varphi_2}{\partial x}(x, y)\right]\alpha\beta + \frac{\partial\varphi_2}{\partial y}(x, y)\beta^2$$

is positive by (7) and (8) (in other words, the matrix $D\varphi_{(x,y)}$ is positive definite). Take a point $(x_0, y_0) \in A_0$ and set $p(t) = (x_0 + t\alpha, y_0 + t\beta)$. As long as $p(t)$ stays in A_0, the function $t \mapsto (\alpha, \beta)\cdot[\varphi(p(t)) - \varphi(p(0))]$ is increasing. Hence, if $t \neq 0$, then $\varphi(p(t)) \neq \varphi(p(0))$. Since A_0 is convex, this proves that φ is one-to-one in A_0. ■

It is easy to notice that φ glues together certain segments, as described in Section 3, so the following property also holds:

LEMMA 3. *All points of the form $\varphi(1, t)$ and $\varphi(-1, t)$, where $t \in (-\frac{1}{3}, \frac{1}{3})$, are the interior points of $\varphi(A)$.* ■

We take the set B given by (4) and define a map $g\colon B \to \mathbf{R}^3$ by

$$g(x, y, z) = \big(g_1(x), g_2(x, y), g_3(x, z)\big), \tag{9}$$

where (cf. Remark 2 in Section 7)

$$g_1(x) = 1 - |x^3 + 3x^2\xi(x) + 4x|, \tag{10}$$

$$g_2(x, y) = s(x)\left[(y - y^3)\frac{1 + \xi(g_1(x))}{2} + \frac{1}{5}(y + 1)\frac{1 - \xi(g_1(x))}{2}\right], \tag{11}$$

where

$$s(x) = \begin{cases} -1 & \text{if } x \geqslant 0, \\ \ \ 1 & \text{if } x < 0 \end{cases} \tag{12}$$

$$g_3(x, z) = \frac{1}{14}z + s(x)\frac{1 + \xi(g_1(x))}{4}. \tag{13}$$

LEMMA 4. *We have:*

(i) $g_1(-1) = g_1(1) = -1$ *and* $g_1(0) = 1$,

(ii) $g_1'(-1) = 1$, $g_1'(1) = -1$, $g_1'(x) > 1$ *for* $x \in (-1, 0)$ *and* $g_1'(x) < -1$ *for* $x \in (0, 1)$,

(iii) $g_1([-1, 1]) = [-1, 1]$.

Proof. (i) follows immediately from (10) and Lemma 1 (iii), (iv) and (v).

To prove (ii), set $\eta(x) = x^3+3x^2\xi(x)+4x$. We have

$$\eta'(x) = 1+3\big(x^2+2x\xi(x)+1+x^2\xi'(x)\big)\,. \tag{14}$$

If $|x|\geqslant\frac{1}{5}$ then $x\xi(x) = -|x|$ and $\xi'(x) = 0$. Hence,

$$\eta'(x) = 1+3(1-|x|)^2 \quad \text{if} \quad |x|\geqslant\tfrac{1}{5}\,. \tag{15}$$

Assume now that $|x|<\frac{1}{5}$. We have then $x\xi(x)\geqslant-|x|$ and $\xi'(x)>-16$ and hence by (14),

$$\eta'(x)>1+3(x^2-2|x|+1-16x^2) = 1+3\,[(1-|x|)^2-16x^2]$$
$$>1+3\,[(\tfrac{4}{5})^2-16\cdot(\tfrac{1}{5})^2] = 1\,.$$

From this and (15) we get in a general case

$$\eta'(x)\begin{cases} =1 & \text{for} \quad x\in\{-1,1\} \\ >1 & \text{for} \quad x\in(-1,1)\,. \end{cases} \tag{16}$$

Since $\eta(0) = 0$, we have $\operatorname{sgn}\eta(x) = \operatorname{sgn}x$ for all $x\in[-1,1]$, and consequently $g_1(x) = 1-\eta(x)\operatorname{sgn}x$. Hence,

$$g_1'(x) = -\eta'(x)\operatorname{sgn}x\,. \tag{17}$$

Now (ii) follows from (16) and (17).

From (ii) it follows that g_1 is monotone on $[-1,0]$ and on $[0,1]$. Together with (i) this gives (iii). ■

Lemma 5. (i) *For every $x\in[-1,1]$ and $a,b\in[-\frac{1}{3},\frac{1}{3}]$, we have* $|g_2(x,a)-g_2(x,b)| \leqslant|a-b|-\frac{1}{4}|a-b|^3$.

(ii) *For every $x\in[-1,1]$ and $y\in[-\frac{1}{3},\frac{1}{3}]$, we have* $|g_2(x,y)|\leqslant\frac{8}{27}$.

Proof. To prove (i), set $t = \dfrac{1+\xi(g_1(x))}{2}$. Since $t\in[0,1]$, we have

$$|g_2(x,a)-g_2(x,b)| = |(a-a^3)t+\tfrac{1}{5}(a+1)(1-t)-(b-b^3)t-\tfrac{1}{5}(b+1)(1-t)|$$
$$= |[(a-a^3)-(b-b^3)]t+\tfrac{1}{5}(a-b)(1-t)|$$
$$\leqslant \max\big(|(a-a^3)-(b-b^3)|,\ \tfrac{1}{5}|a-b|\big)\,.$$

But

$$a^2+ab+b^2 = \tfrac{3}{4}(a^2+2ab+b^2)+\tfrac{1}{4}(a^2-2ab+b^2)\geqslant\tfrac{1}{4}|a-b|^2,$$

and therefore, taking into account that $a^2+ab+b^2\leqslant 3\cdot(\frac{1}{3})^2<1$, we get

$$|(a-a^3)-(b-b^3)| = |a-b|[1-(a^2+ab+b^2)]\leqslant|a-b|-\tfrac{1}{4}|a-b|^3.$$

Also $\frac{1}{5}|a-b|<|a-b|-\frac{1}{4}|a-b|^3$ since $|a-b|^2<\frac{16}{5}$. This ends the proof of (i).

Now we prove (ii). Since

$$(y-y^3)' = 1-3y^2\geqslant1-3(\tfrac{1}{3})^2 = \tfrac{2}{3} \quad \text{for} \quad |y|\leqslant\tfrac{1}{3}, \tag{18}$$

we have

$$|y-y^3|\leqslant\tfrac{1}{3}-(\tfrac{1}{3})^3 = \tfrac{8}{27} \quad \text{for} \quad |y|\leqslant\tfrac{1}{3}\,. \tag{19}$$

Also $|\frac{1}{5}(y+1)|\leqslant\frac{1}{5}\cdot(\frac{1}{3}+1) = \frac{4}{15}<\frac{8}{27}$. Hence,

$$|g_2(x,y)| = |(y-y^3)t+\tfrac{1}{5}(y+1)(1-t)|\leqslant\tfrac{8}{27} \quad \text{for} \quad |y|\leqslant\tfrac{1}{3}\,. \ ■$$

Lemma 6. (i) *For every $x,a,b\in[-1,1]$, we have* $|g_3(x,a)-g_3(x,b)| = \frac{1}{14}|a-b|$.

(ii) *For every $x,z\in[-1,1]$, we have* $|g_3(x,z)|\leqslant\frac{4}{7}$.

Proof. (i) follows immediately from (13). To prove (ii), we estimate: $|g_3(x,z)| \leqslant\frac{1}{14}+\frac{2}{4} = \frac{4}{7}$. ■

From Lemmata 4(iii), 5(ii) and 6(ii), it follows (since $\frac{8}{27}<\frac{1}{3}$ and $\frac{4}{7}<1$):

Lemma 7. *We have $g(B)\subset B$.* ■

We continue to investigate the properties of g.

Lemma 8. *The map g is one-to-one.*

Proof. The map g_1 is one-to-one on both $[-1,0]$ and $(0,1]$. For a fixed x, the map $g_2(x,\cdot)$ is one-to-one since $(y-y^3)'>0$ (see (18)) and $[\frac{1}{5}(y+1)]'>0$. Also $g_3(x,\cdot)$ is one-to-one (it follows immediately from (13)). Hence, g is one-to-one on $[-1,0]\times[-\frac{1}{3},\frac{1}{3}]\times[-1,1]$ and on $(0,1]\times[-\frac{1}{3},\frac{1}{3}]\times[-1,1]$. Therefore it remains to prove that if $(x,y,z)\in[-1,0]\times[-\frac{1}{3},\frac{1}{3}]\times[-1,1]$ and $(\tilde{x},\tilde{y},\tilde{z}) \in(0,1]\times[-\frac{1}{3},\frac{1}{3}]\times[-1,1]$, then $g(x,y,z)\neq g(\tilde{x},\tilde{y},\tilde{z})$.

Suppose, in contrary, that $g(x,y,z) = g(\tilde{x},\tilde{y},\tilde{z})$. Since $g_1(x) = g_1(\tilde{x})$, $s(x) = -s(\tilde{x})$ and $g_2(x,y) = g_2(\tilde{x},\tilde{y})$, we have (denote $t = \big(1+\xi(g_1(x))\big)/2$):

$$0 = [(y-y^3)t+\tfrac{1}{5}(y+1)(1-t)]+[(\tilde{y}-\tilde{y}^3)t+\tfrac{1}{5}(\tilde{y}+1)(1-t)]\,. \tag{20}$$

Also $g_3(x,z) = g_3(\tilde{x},\tilde{z})$, and hence $\frac{1}{14}z+\frac{1}{2}t = \frac{1}{14}\tilde{z}-\frac{1}{2}t$. Thus, $t = \frac{1}{14}(\tilde{z}-z)$, and consequently $t\leqslant\frac{1}{7}$. Therefore (we use also (19)),

$$(y-y^3)t+\tfrac{1}{5}(y+1)(1-t)\geqslant-\tfrac{8}{27}\cdot\tfrac{1}{7}+\tfrac{1}{5}\cdot\tfrac{2}{3}\cdot\tfrac{6}{7} = \tfrac{68}{945}\,.$$

For $\tilde{y}$ instead of y, we obtain the same estimate. But this contradicts (20). ■

Lemma 9. *For $p,q\in B$, we have $\psi(p) = \psi(q)$ if and only if $\psi(g(p)) = \psi(g(q))$.*

Proof. Assume that $\psi(p) = \psi(q)$. By Lemma 2 and the remark following it, we have either $p = (-1,u,v)$ and $q = (-1,-u,v)$, or $p = (1,u,v)$ and $q = (1,-u,v)$ for some $u\in[-\frac{1}{3},\frac{1}{3}]$, $v\in[-1,1]$. One can easily compute that in the first case

$$g(p) = (-1,\ u-u^3,\ \tfrac{1}{14}v+\tfrac{1}{2}) \quad \text{and} \quad g(q) = (-1,\ u^3-u,\ \tfrac{1}{14}v+\tfrac{1}{2})\,,$$

and in the second case

$$g(p) = (-1,\ u^3-u,\ \tfrac{1}{14}v-\tfrac{1}{2}) \quad \text{and} \quad g(q) = (-1,\ u-u^3,\ \tfrac{1}{14}v-\tfrac{1}{2})\,.$$

In both cases, $\psi(g(p)) = \psi(g(q))$.

Assume now that $\psi(g(p)) = \psi(g(q))$. By the same arguments as before, we have either $g(p) = (-1, u, v)$ and $g(q) = (-1, -u, v)$ or $g(p) = (1, u, v)$ and $g(q) = (1, -u, v)$ for some $u \in [-\frac{1}{3}, \frac{1}{3}]$ and $v \in [-1, 1]$. In the first case, the first coordinate of p and q has to be either -1 or 1. If one of them is -1 and the other one is 1, we get a contradiction with (13). If both of them are 1 or both of them are -1, we can use the arguments, that for a given x, the maps $g_2(x, \cdot)$ and $g_3(x, \cdot)$ are one-to-one (see the beginning of the proof of Lemma 8) and that if $|x| = 1$ then the map $g_2(x, \cdot)$ is odd. We conclude that the second coordinates of p and q have the same absolute value but opposite signs and the third coordinates of p and q are equal. Consequently, $\psi(p) = \psi(q)$.

In the second case, the first coordinate of both p and q is 0, and we simply use the argument that for a given x, $g_2(x, \cdot)$ and $g_3(x, \cdot)$ are one-to-one. Hence, $p = q$. ■

In view of Lemmata 7 and 9, there exists a unique map $f\colon \psi(B) \to \psi(B)$ such that

(21) $$\psi \circ g = f \circ \psi .$$

Lemma 10. *The map $f\colon \psi(B) \to \psi(B)$ is a diffeomorphism onto its image.*

Proof. By Lemmata 8 and 9, f is one-to-one.

From the computations in the proof of Lemma 2, we see that the Jacobian of ψ is nonzero at all points of the set $B_0 = A_0 \times [-1, 1]$. By (9), the Jacobian of g is equal to $g_1'(x) \cdot \frac{\partial g_2}{\partial y}(x, y) \cdot \frac{\partial g_3}{\partial z}(x, z)$. From Lemma 4 we have $g_1'(x) \neq 0$ for every $x \in [-1, 1]$ (at $x = 0$ we consider one-sided derivatives). From (11) we compute that

$$\frac{\partial g_2}{\partial y}(x, y) = s(x)\left[(1-3y^2)\,\frac{1+\xi(g_1(x))}{2} + \frac{1}{5}\cdot\frac{1-\xi(g_1(x))}{2}\right]$$

which is nonzero for every $x \in [-1, 1]$, $y \in [-\frac{1}{3}, \frac{1}{3}]$, since both $1-3y^2$ and $\frac{1}{5}$ are positive. From (13) we compute that $\frac{\partial g_3}{\partial z}(x, z) = \frac{1}{14} \neq 0$. Hence, the Jacobian of g is nonzero on the whole B. Therefoie it remains to check the behaviour of f at the regions where either p or $f(p)$ is close to $\psi(B \setminus B_0)$.

The first of these regions is a neighbourhood of $\psi(\{-1\} \times [-\frac{1}{3}, \frac{1}{3}] \times [-1, 1])$. If x is close to -1, then $\xi(x) = 1$, $\varphi(x, y) = (2x+x^2-y^2, 2y+2xy)$, $g_1(x) = 1+x^3+3x^2+4x$, $g_2(x, y) = y-y^3$ and $g_3(x, z) = \frac{1}{14}z+\frac{1}{2}$. Setting $u = x+1$, we get: $\psi(x, y, z) = (u^2-y^2-1, 2uy, z)$ and $g(x, y, z) = (u^3+u-1, y-y^3, \frac{1}{14}z+\frac{1}{2})$.

We claim that in this region

(22) $$f(w, t, z) = ((w+1)[(w+1)^2+\tfrac{3}{4}t^2+1]+2[(w+1)^2+\tfrac{1}{2}t^2]-1,\; -t[\tfrac{1}{4}t^2-(w+1)-1],\; \tfrac{1}{14}z+\tfrac{1}{2}).$$

Indeed, for f defined as above, we have

$$\begin{aligned} f(\psi(x, y, z)) &= ((u^2-y^2)[(u^2-y^2)^2+\tfrac{3}{4}(2uy)^2+1]+2[(u^2-y^2)^2+\tfrac{1}{2}(2uy)^2]-1,\\ &\qquad -2uy[\tfrac{1}{4}(2uy)^2-(u^2-y^2)-1],\; \tfrac{1}{14}z+\tfrac{1}{2})\\ &= ((u^2-y^2)(u^4+u^2y^2+y^4+1)+2(u^4+y^4)-1,\\ &\qquad -2uy(u^2y^2-u^2+y^2-1),\; \tfrac{1}{14}z+\tfrac{1}{2})\\ &= ((u^3+u)^2-(y-y^3)^2-1,\; 2(u^3+u)(y-y^3),\; \tfrac{1}{14}z+\tfrac{1}{2})\\ &= \psi(g(x, y, z)), \end{aligned}$$

i.e. (21) holds. Since there is only one f satisfying (21), (22) holds.

The formula (22) shows that f is of class C^∞. We shall check that its Jacobian is nonzero. We have to check it in a neighbourhood of $\psi(\{-1\} \times [-\frac{1}{3}, \frac{1}{3}] \times [-1, 1])$, but this set is compact and therefore it is enough to check it on this set. If $x = -1$ then $u = 0$ and thus $\psi(-1, y, z) = (-y^2-1, 0, z)$. Set $v = w+1$. We have to compute the Jacobian of f for $v \in [-\frac{1}{9}, 0]$, $t = 0$, $z \in [-1, 1]$. It is equal to

$$\begin{vmatrix} 3v^2+1+4v & 0 & 0 \\ 0 & v+1 & 0 \\ 0 & 0 & \frac{1}{14} \end{vmatrix} = \tfrac{1}{14}(3v+1)(v+1)^2 .$$

This is positive for $v \in [-\frac{1}{9}, 0]$. This ends checking in the first region.

The second region is a neighbourhood of the set $\psi(\{1\} \times [-\frac{1}{3}, \frac{1}{3}] \times [-1,1])$. In this region we have x close to 1, $\xi(x) = -1$, $\tilde{\varphi}(x, y) = (2x-x^2+y^2, 2y-2xy)$, $\tilde{g}_1(x) = 1-x^3+3x^2-4x$, $\tilde{g}_2(x, y) = y^3-y$, $\tilde{g}_3(x, z) = \frac{1}{14}z-\frac{1}{2}$ (we use tildes to distinguish these formulae from those valid for the first region). To compare the formulae from the first and second regions, we define two symmetries: $\vartheta_1(x, y, z) = (-x, -y, -z)$, $\vartheta_2(x, y, z) = (x, y, -z)$. We have:

$$\begin{aligned} \vartheta_1^{-1} \circ \psi \circ \vartheta_1(x, y, z) &= \vartheta_1(-2x+x^2-y^2, -2y+2xy, -z)\\ &= (2x-x^2+y^2, 2y-2xy, z) = \tilde{\psi}(x, y, z), \end{aligned}$$

$$\begin{aligned} \vartheta_2^{-1} \circ \psi \circ \vartheta_2(x, y, z) &= \vartheta_2(2x+x^2-y^2, 2y+2xy, -z)\\ &= (2x+x^2-y^2, 2y+2xy, z) = \psi(x, y, z), \end{aligned}$$

$$\begin{aligned} \vartheta_2^{-1} \circ g \circ \vartheta_1(x, y, z) &= \vartheta_2((1-x)^3+(1-x)-1, y^3-y, -\tfrac{1}{14}z+\tfrac{1}{2})\\ &= (1-3x+3x^2-x^3+1-x-1, y^3-y, \tfrac{1}{14}z-\tfrac{1}{2}) = \tilde{g}(x, y, z). \end{aligned}$$

Hence, if we set

(23) $$\tilde{f} = \vartheta_2^{-1} \circ f \circ \vartheta_1 ,$$

then, taking into account that if x is close to 1, then $g_1(x)$ is close to -1, we have:

$$\begin{aligned} \tilde{f} \circ \tilde{\psi} &= \vartheta_2^{-1} \circ f \circ \vartheta_1 \circ \vartheta_1^{-1} \circ \psi \circ \vartheta_1 = \vartheta_2^{-1} \circ f \circ \psi \circ \vartheta_1\\ &= \vartheta_2^{-1} \circ \psi \circ g \circ \vartheta_1 = \vartheta_2^{-1} \circ \psi \circ \vartheta_2 \circ \vartheta_2^{-1} \circ g \circ \vartheta_1 = \psi \circ \tilde{g} . \end{aligned}$$

Therefore (23) describes f in the second region (as $\tilde{f}$). Since clearly ϑ_1 and ϑ_2 are diffeomorphisms, this ends checking in the second region.

The third (and last) region is a neighbourhood of the set

$$\psi(\{0\}\times[-\tfrac{1}{3},\tfrac{1}{3}]\times[-1,1]).$$

Since it is contained in $\psi(B_0)$, ψ^{-1} is a diffeomorphism there, and consequently it is enough to prove that $\psi\circ g$ is a diffeomorphism in a neighbourhood of the set $\{0\}\times[-\frac{1}{3},\frac{1}{3}]\times[-1,1]$. If x is close to 0, then $g_1(x)$ is close to 1 and consequently $\xi(g_1(x)) = -1$. Hence, $g_2(x,y) = \frac{1}{5}s(x)(y+1)$ and $g_3(x,z) = \frac{1}{14}z$. If u is close to 1 then $\xi(u) = -1$ and hence $\psi(u,v,w) = (2u-(u^2-v^2), 2v-2uv, w)$. Denote, as in the proof of Lemma 4, $\eta(x) = x^3+3x^2\xi(x)+4x$. Then we have

$$\begin{aligned}(\psi\circ g)(x,y,z) &= \psi(1-|\eta(x)|, \tfrac{1}{5}s(x)(y+1), \tfrac{1}{14}z)\\ &= (2-2|\eta(x)|-1+2|\eta(x)|-(\eta(x))^2+\tfrac{1}{25}(y+1)^2,\\ &\qquad \tfrac{2}{5}s(x)(y+1)|\eta(x)|, \tfrac{1}{14}z).\end{aligned}$$

By (16), $\eta'(x)>0$. Since $\eta(0) = 0$, we have $\operatorname{sgn}\eta(x) = \operatorname{sgn}x$ and hence $s(x)\cdot|\eta(x)| = -\eta(x)$. Thus, $(\psi\circ g)(x,y,z) = \big(1-(\eta(x))^2+\frac{1}{25}(y+1)^2, -\frac{2}{5}(y+1)\eta(x), \frac{1}{14}z\big)$. Consequently, $\psi\circ g$ is of class C^∞. It remains to show that the Jacobian of $\psi\circ g$ is nonzero for $x = 0$, $y\in[-\frac{1}{3},\frac{1}{3}]$, $z\in[-1,1]$. But at such point this Jacobian is equal to

$$\begin{vmatrix} 0 & \frac{2}{25}(y+1) & 0\\ -\frac{2}{5}(y+1)\eta'(x) & 0 & 0\\ 0 & 0 & \frac{1}{14}\end{vmatrix} = \frac{2}{875}(y+1)^2\eta'(x)>0.$$

This ends checking in the third region and the whole proof. ■

LEMMA 11. *We have* $f(\psi(B))\subset\operatorname{int}\psi(B)$.

Proof. By Lemma 5 (ii) and Lemma 6 (ii), we have $g(B)\subset[-1,1]\times[-\frac{8}{27},\frac{8}{27}]\times[-\frac{4}{7},\frac{4}{7}]$. As in the beginning of the proof of Lemma 10, we see that the Jacobian of ψ is nonzero at all points of B_0. Hence, all points of $\psi(\operatorname{int}B)$ are the interior points of $\psi(B)$. But by Lemma 3 and in view of (4) and (5), also the points of the form $\psi(1,p)$ and $\psi(-1,p)$, where $p\in(-\frac{1}{3},\frac{1}{3})\times(-1,1)$, are in the interior of $\psi(B)$. Hence (since $\frac{8}{27}<\frac{1}{3}$ and $\frac{4}{7}<1$), all points of $\psi(g(B))$ are in $\operatorname{int}\psi(B)$. ■

Denote by $\pi\colon B\to[-1,1]$ the projection onto the first coordinate:

$$\pi(x,y,z) = x. \tag{24}$$

For every $p\in\psi(B)$ set

$$h(p) = \pi(\psi^{-1}(p)). \tag{25}$$

Notice that if $\psi^{-1}(p)$ consists of more than one point, they have the same first coordinate. Therefore, the above formula defines a map $h\colon\psi(B)\to[-1,1]$. Clearly, h is continuous.

LEMMA 12. *We have* $h\circ f = g_1\circ h$.

Proof. By (9) and (24), we have

$$\pi\circ g = g_1\circ\pi. \tag{26}$$

From this, (21) and (25), we obtain

$$h\circ f = \pi\circ\psi^{-1}\circ f = \pi\circ g\circ\psi^{-1} = g_1\circ\pi\circ\psi^{-1} = g_1\circ h. \ ■$$

Set

$$C = \bigcap_{n\geqslant 0} f^n(\psi(B)). \tag{27}$$

LEMMA 13. *Let $(x_n)_{n=0}^\infty$ be a sequence of points of $[-1,1]$ such that $g_1(x_n) = x_{n-1}$ for $n = 1,2,3,\ldots$ Then there exists exactly one point $p\in C$ such that $h(f^{-n}(p)) = x_n$ for all $n\geqslant 0$.*

Proof. Set $Q_n = g^n(\pi^{-1}(x_n))$. For $n = 1,2,3,\ldots$, we have, by (26), $g(\pi^{-1}(x_n))\subset\pi^{-1}(g_1(x_n)) = \pi^{-1}(x_{n-1})$, and hence $Q_n\subset Q_{n-1}$. Since for every $n\geqslant 0$, $\pi^{-1}(x_n)$ is a closed rectangle with sides parallel to the y and z axes, it is easy to deduce from (9) and the continuity of $g_2(x,\cdot)$ and $g_3(x,\cdot)$ that each Q_n is also such rectangle. By Lemma 5 (i), the length of Q_n along the y axis is at most $\zeta^n(\frac{2}{3})$, where $\zeta(t) = t-\frac{1}{4}t^3$, and by Lemma 6 (i), the length of Q_n along the z axis is at most $(\frac{1}{14})^n\cdot 2$. But $|\zeta'(t)| = |1-\frac{3}{4}t^2|<1$ for $t\in[0,\frac{2}{3}]$ and $\zeta(0) = 0$, and hence $\lim_n\zeta^n(\frac{2}{3}) = 0$. Also $\lim_n(\frac{1}{14})^n\cdot 2 = 0$. Consequently, the set $\bigcap_{n\geqslant 0} Q_n$ consists of a single point. We call this point q.

We shall show that the point $p = \psi(q)$ satisfies the conditions from the conclusion of the lemma. For $n = 0,1,2,\ldots$ we have

$$p = \psi(q)\in\psi\big(g^n(\pi^{-1}(x_n))\big) = f^n\big(\psi(\pi^{-1}(x_n))\big) = f^n\big(h^{-1}(x_n)\big)$$

and therefore $p\in C$ and $h(f^{-n}(p)) = x_n$.

Suppose now that some point $\tilde{p}\in C$ satisfies $h(f^{-n}(\tilde{p})) = x_n$ for all $n\geqslant 0$. Then we have $\tilde{p}\in f^n(h^{-1}(x_n)) = \psi\big(g^n(\pi^{-1}(x_n))\big) = \psi(Q_n)$ for all $n\geqslant 0$. Since $(Q_n)_{n=0}^\infty$ is a descending sequence of compact sets, we have $\bigcap_{n\geqslant 0}\psi(Q_n) = \psi(\bigcap_{n\geqslant 0} Q_n)$, and therefore $\tilde{p} = p$. ■

LEMMA 14. *We have* $h(C) = [-1,1]$.

Proof. Since $g_1([-1,1]) = [-1,1]$, for every $x\in[-1,1]$ we can find a sequence $(x_n)_{n=0}^\infty$ satisfying the hypothesis of Lemma 13 and such that $x_0 = x$. For p from the conclusion of Lemma 13, we have $p\in C$ and $h(p) = h(f^{-0}(p)) = x_0 = x$. ■

LEMMA 15. *The map $g_1\colon[-1,1]\to[-1,1]$ is conjugate to the map $T\colon I\to I$ (given by (1)).*

Proof. Both maps expand lengths of all subintervals on which they are one-to-one. Therefore they have no homtervals (intervals, on which all iterates are one-to-one). Hence, since they have the same kneading invariant, they are conjugate (cf. [2]). ■

PROPOSITION 1. *The map $f|_C\colon C \to C$ is topologically conjugate to the inverse limit of T.*

Proof. Denote by $H\colon [-1,1] \to I$ the conjugacy between g_1 and T. Let $\tau\colon K \to K$ (the inverse limit of T) and $\Psi_n\colon K \to I$ $(n = 0, 1, 2, \ldots)$ be as in Section 1. By Lemma 12, $h \circ f|_C = g_1 \circ h$, and hence $(H \circ h) \circ f|_C = T \circ (H \circ h)$. Since $f|_C$ is a homeomorphism and $H \circ h$ is continuous, then, as in Section 1 (with $\tilde{K} = C$, $\tilde{\tau} = f|_C$ and $\tilde{\Psi} = H \circ h$), there exists a continuous map $\Phi\colon C \to K$ such that

$$\Phi \circ f|_C = \tau \circ \Phi \tag{28}$$

and

$$H \circ h = \Psi_0 \circ \Phi . \tag{29}$$

Take a point $q = (t_n)_{n=0}^{\infty} \in K$ and set $x_n = H^{-1}(t_n)$, $n = 0, 1, 2, \ldots$ For $n = 1, 2, 3, \ldots$, since $T(t_n) = t_{n-1}$, we have $g_1(x_n) = x_{n-1}$. By Lemma 13, there is exactly one point $p \in C$ such that $h(f^n(p)) = x_n$ for all $n \geqslant 0$. The above condition is equivalent to $(H \circ h)(f^{-n}(p)) = t_n$ for all $n \geqslant 0$. We have by (28) and (29): $H \circ h \circ f^{-n} = \Psi_0 \circ \Phi \circ f^{-n} = \Psi_0 \circ \tau^{-n} \circ \Phi$, and since $\Psi_0 \circ \tau^{-n} = \Psi_n$, we get $H \circ h \circ f^{-n} = \Psi_n \circ \Phi$ for all $n \geqslant 0$. Therefore, there is exactly one point $p \in C$ such that $\Psi_n(\Phi(p)) = t_n$ for all $n \geqslant 0$. Since q is the unique point of K such that $\Psi_n(q) = t_n$ for all $n \geqslant 0$, we have $\Phi(p) = q$. This proves that Φ is one-to-one and onto. Since Φ is continuous and C is compact, it follows that Φ is a homeomorphism from C onto K. In view of (28). Φ is a conjugacy between $f|_C$ and τ. ■

PROPOSITION 2. *The set C is an attractor for $f\colon \psi(B) \to \psi(B)$.*

Proof. Set $U = \operatorname{int}\psi(B)$. Since the set $f(\psi(B))$ is compact, we have $\operatorname{cl} f(U) \subset f(\psi(B))$, and by Lemma 11, $\operatorname{cl} f(U) \subset U$. Again by Lemma 11, $f(\psi(B)) \subset U \subset \psi(B)$, and in view of (27) we get $C = \bigcap_{n\geqslant 0} f^n(U)$.

Since T is transitive, so is its inverse limit (a lift of a dense orbit is a dense orbit). Hence, by Proposition 1, $f|_C$ is also transitive. ■

Proof of Theorem A. Let M be a manifold of dimension $n \geqslant 3$. We take a set $D = \psi(B) \times [-1, 1]^{n-3}$ and a C^∞ map $F\colon D \to D$ given by $F(p, q) = (\psi(p), \frac{1}{2}q)$. The map F is a diffeomorphism of D onto its image, and since $F^n(p, q) = (\psi^n(p), \frac{1}{2^n}q)$, by Propositions 1 and 2, F has an attractor $C \times \{0\}^{n-3}$ such that F restricted to this attractor is conjugate to the inverse limit ot T. But D can be C^∞ embedded into M and F can be extended to a C^∞ diffeomorphism of M onto itself in a standard way (notice that f preserves the orientation — this is computed for instance in the proof of Lemma 10 — and so does F). ■

5. Images of the Lebesgue measure in the first example. We denote by λ_i the Lebesgue measure in $\mathbf{R}^i$. Since ψ is a diffeomorphism on B_0, the image of $\lambda_3|_B$ under ψ is equivalent to $\lambda_3|_{\psi(B)}$. Hence, instead of investigating the images of $\lambda_3|_{\psi(B)}$ under f^n, we may investigate the images of $\lambda_3|_B$ under g^n $(n = 0, 1, 2, \ldots)$. Further, since the foliation in directions of y and z axes is invariant for g, it is enough to investigate the images of $\lambda_1|_{[-1,1]}$ under the iterates of g_1. But it is easy to check that g_1 satisfies the conditions given by Thaler in [5], sufficient for the existence of an ergodic invariant infinite measure equivalent to $\lambda_1|_{[-1,1]}$ with the density having a singularity only at -1. Therefore the averages of those images converge to the measure concentrated at -1. Consequently, the averages of the images of $\lambda_3|_{\psi(B)}$ under the iterates of f converge to the measure concentrated at $(-1, 0, \frac{7}{13})$ (a corresponding fixed point of f). The same can be told about the averages of images of a measure concentrated at one point, for almost every point.

Nevertheless, since there is an infinite ergodic measure invariant for g_1 and equivalent to $\lambda_1|_{[-1,1]}$, for almost every point $p \in \psi(B)$ the closure of its orbit contains the whole C. The convergence of the averages of the measures $\delta_p, \delta_{f(p)}, \delta_{f^2(p)}, \ldots$ to a measure concentrated at one point means only that the trajectory of p will stay for longer and longer times close to this point and go away for relatively short periods (although, for a typical point, these periods are not bounded from above).

From the ergodic theory point of view, more interesting situation would be if those averages have converged to a measure with a whole C as a support. The reason why this does not happen, is that $g_1'(-1) = 1$, or equivalently, that $Df_{(-1,0,\frac{7}{13})}$ has no eigenvalue of absolute value larger than 1. We shall show that this cannot be changed.

PROPOSITION 3. *Let V be an open subset of $\mathbf{R}^3$ and let $F\colon V \to V$ be a C^1 diffeomorphism with an attractor C such that τ is conjugate to $F|_C$ by $G\colon K \to C$. Then $DF_{G(\Theta)}$ (where $\Theta = (0)_{n=0}^{\infty} \in K$) has no eigenvalue of absolute value larger than* 1.

Before proving Proposition 3, we shall prove two lemmata.

LEMMA 16. *Let $Y = \Psi_0^{-1}([0, \frac{1}{2}))$ and let Z be the connected component of Y containing Θ. Then $\Psi_0|_Z\colon Z \to [0, \frac{1}{2})$ is a homeomorphism.*

Proof. Set $X = \{(t_n)_{n=0}^{\infty}\colon t_0 \in [0, \frac{1}{2}),\ t_n = 2^{-n}t_0 \text{ for } n = 1, 2, 3, \ldots\}$. Clearly, $X \subset Y$, $\Theta \in X$ and $\Psi_0|_X\colon X \to [0, \frac{1}{2})$ is a homeomorphism. Hence, $X \subset Z$ and it is enough to show that $Z \setminus X = \emptyset$.

Take a point $u = (u_n)_{u=0}^{\infty} \in Y \setminus X$. Let k be the smallest positive integer such that $u_k \neq 2^{-k}u_0$. Then the sets

$$\{(t_n)_{n=0}^{\infty} \in K\colon t_0 \in [0, \tfrac{1}{2}),\ t_n = 2^{-n}t_0 \text{ for } n = 1, 2, \ldots, k\}$$

and

$$\{(t_n)_{n=0}^{\infty} \in K\colon t_0 \in [0, \tfrac{1}{2}),\ t_n = 2^{-n}t_0 \text{ for } n = 1, 2, \ldots, k-1 \text{ and } t_k = 1 - 2^{-k}t_0\}$$

are both open in Y (since $2^{-k}t_0 < 2^{-k+1}$), disjoint, and contain Θ and u respectively. Hence, $u \notin Z$. ■

LEMMA 17. *Let $F\colon V \to V$ be a continuous map. If $U \subset V$ and $p \in V$ are such that for every $n \geqslant 0$ there exists $p_n \in U$ such that $F^n(p_n) = p$, then $p \in \bigcap_{n\geqslant 0} F^n(U)$.*

Proof. Since $p_n \in U$ and $F^n(p_n) = p$, we have $p \in F^n(U)$ for every $n \geqslant 0$. Hence, $p \in \bigcap_{n\geqslant 0} F^n(U)$. ■

Proof of Proposition 3. Suppose that $DF_{G(\Theta)}$ has an eigenvalue of absolute value larger than 1. Then F has a local (strong) unstable manifold at $G(\Theta)$ of positive dimension (cf. [4], Chapter IX, Theorem 5.1.). This manifold, intersected with any neighbourhood of $G(\Theta)$ contains a set homeomorphic to an open interval and containing $G(\Theta)$. By Lemma 17, this set is contained in C. But this contradicts Lemma 16. ■

6. The second example. The method of constructing the second example is very similar to that of the first one. There are two main differences: instead of 3 dimensions, we shall use only 2, and instead of a diffeomorphism we shall obtain only a homeomorphism.

To stress the similarity we shall use similar notation.

We take the set A given by (2) and define $g: A \to A$ by

$$g(x, y) = (g_1(x), g_2(x, y)),$$

where $g_1(x) = 1-|2x|$ and $g_2(x, y) = s(x)\cdot\frac{1}{5}(y+1)$ (s is given by (12)).

Now we have to define new φ. Close to $\{1\}\times[-\frac{1}{3}, \frac{1}{3}]$ we use formula (6). Since for x close to 1 we have $\xi(x) = -1$, we get

$$\varphi(x, y) = (2x-x^2+y^2, 2y-2xy) \text{ in a neighbourhood of } \{1\}\times[-\tfrac{1}{3}, \tfrac{1}{3}]. \tag{30}$$

Since glues together $\{1\}\times[0, \frac{1}{3}]$ with $\{1\}\times[-\frac{1}{3}, 0]$, to define φ in a neighbourhood of $\{-1\}\times[-\frac{1}{3}, \frac{1}{3}]$, we have to look at the images of $\{1\}\times[-\frac{1}{3}, 0]$ and $\{1\}\times[0, \frac{1}{3}]$ under g^n. It is easy to see that they are $\{-1\}\times[\frac{1}{4}-\frac{31}{12\cdot 5^n}, \frac{1}{4}-\frac{27}{12\cdot 5^n}]$ and $\{-1\}\times[\frac{1}{4}-\frac{27}{12\cdot 5^n}, \frac{1}{4}-\frac{23}{12\cdot 5^n}]$ respectively ($n = 1, 2, 3, \ldots$). Since g^n is linear on $\{-1\}\times[-\frac{1}{3}, \frac{1}{3}]$ if we impose the following conditions on φ:

(i) φ glues together slightly larger segments, namely $\{-1\}\times[\frac{1}{4}-\frac{11}{4\cdot 5^n}, \frac{1}{4}-\frac{9}{4\cdot 5^n}]$ with $\{-1\}\times[\frac{1}{4}-\frac{9}{4\cdot 5^n}, \frac{1}{4}-\frac{7}{4\cdot 5^n}]$ ($n = 1, 2, 3, \ldots$),

(ii) φ is a homeomorphism (onto its image) on the rest of A (except also $\{1\}\times[-\frac{1}{3}, \frac{1}{3}]$, where we already know that φ glues two segments),

(iii) φ is continuous on the whole A,

then, as in the first example, there exists a unique $f: \varphi(A) \to \varphi(A)$ such that $f\circ\varphi = \varphi\circ g$. It is clear that a map φ satisfying (30) and (i) — (iii) exists. The set $\varphi(A)$ looks like on Fig. 5.

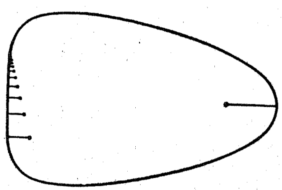

Fig. 5. An approximate shape of the set $\varphi(A)$ in the second example

Now one can follow the scheme of Section 4. Since the proofs are mostly very similar and often much easier than in Section 4 (one does not need any proofs of smoothness), we omit this part of deduction, stating only the final result:

PROPOSITION 4. *The map $f: \varphi(A) \to \varphi(A)$ is a homeomorphism onto its image, preserves the orientation and has an attractor C such that $f|_C: C \to C$ is conjugate to τ.* ■

Theorem B is an immediate corollary to the above proposition.

7. Remarks.

Remark 1. In the first example, from the definition of σ, ξ, φ, ψ and g it follows that:

σ is real analytic except at the point $t = -1$,

ξ is real analytic except at two points given by $|t| = \frac{1}{5}$,

φ is real analytic except on two lines given by $|x| = \frac{1}{5}$,

ψ is real analytic except on two planes given by $|x| = \frac{1}{5}$,

g is real analytic except on 7 planes given by $x = 0$ (1 plane), $|x| = \frac{1}{5}$ (2 planes) and $|g_1(x)| = \frac{1}{5}$ (4 planes).

Therefore, by the definition of f and the end of the proof of Lemma 10, the map f is real analytic except on 6 surfaces (the images under ψ of the planes given by $|x| = \frac{1}{5}$ and $|g_1(x)| = \frac{1}{5}$).

Remark 2. In the first example, from (22) and (23) it follows that in the neighbourhoods of the sets

$$\psi(\{-1\}\times[-\tfrac{1}{3}, \tfrac{1}{3}]\times[-1, 1]) \quad\text{and}\quad \psi(\{1\}\times[-\tfrac{1}{3}, \tfrac{1}{3}]\times[-1, 1])$$

(i.e. the sets on which the hyperbolicity of f fails) the map f is given by polynomials (of at most third degree). To understand better the action of f in these neighbourhoods, one should make a suitable coordinate changes (translations by $(-1, 0)$ and $(1, 0)$ respectively). Then in the first neighbourhood (more exactly — its projection into $\boldsymbol{R}^2$), (g_1, g_2) has the form $(x, y) \to (x+x^3, y-y^3)$ and φ has the form $(x+iy) \to (x+iy)^2$ (here we identify $\boldsymbol{R}^2$ with $\boldsymbol{C}$).

The formulae in the second case are similar.

Remark 3. In the first example, the stable and unstable foliations look like in the case of pseudo-Anosov maps (cf. Figure 1). Hence, one can hope that some methods similar to those of M. Gerber [3] can be used to find real analytic examples.

Remark 4. In the second example, a question arises whether it can be made smooth. I do not see any principal obstructions, but the complicated structure of the stable and unstable foliations makes the problem more difficult than in the three-dimensional case.

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Essential mappings and transfinite dimension

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Abstract. We construct a compact metrizable space with inductive dimension $\omega+1$ that admits no essential mappings into Henderson's $(\omega+1)$-dimensional absolute retract $J^{\omega+1}$.

1. Introduction. A continuous mapping $f\colon X \to I^n = [0, 1]^n$ is called *essential* if there is no continuous extension $g\colon X \to \partial I^n$ of $f|f^{-1}(\partial I^n)$, where ∂I^n is the geometric boundary of I^n. The following characterization is well known (see e.g. Engelking [1], 3.2.10).

1.1 THEOREM. *A normal space has* $\dim \geqslant n$ *iff it admits an essential mapping into* I^n.

D. W. Henderson [2] has attempted to extend this result to transfinite inductive dimension.

1.2 DEFINITION. $\operatorname{Ind}(\emptyset) = -1$. Let α be an ordinal and X a normal space. $\operatorname{Ind}(X) \leqslant \alpha$ if every pair of disjoint closed subsets of X can be separated by a closed set with $\operatorname{Ind} < \alpha$ (S *separates* A and B in X if $X \setminus S$ is the union of disjoint open sets U and V with $A \subset U$ and $B \subset V$).

1.3 DEFINITION (Henderson). For each countable ordinal α we define a compact metric space J^α, its „boundary" T^α and a point $p^\alpha \in T^\alpha$.

(i) if α is finite then $J^\alpha = I^\alpha$, $T^\alpha = \partial I^\alpha$ and $p^\alpha = (0, 0, \ldots, 0)$.

(ii) If we have a successor $\alpha+1$ we define $J^{\alpha+1} = J^\alpha \times I$, $T^{\alpha+1} = (T^\alpha \times I) \cup \cup (J^\alpha \times \{0, 1\})$ and $p^{\alpha+1} = (p^\alpha, 0)$.

(iii) If α is a limit, put $K^\beta = J^\beta \cup L^\beta$ for every $\beta < \alpha$, where L^β is a half open arc such that $L^\beta \cap J^\beta = \{p^\beta\}$ = (the end-point of L^β). J^α is defined as the one-point compactification of the discrete sum $\bigoplus_{\beta<\alpha} K^\beta$; $T^\alpha = J^\alpha \setminus \bigcup_{\beta<\alpha} (J^\beta \setminus T^\beta)$ and p^α is the compactifying point.

A continuous mapping f from a space X into J^α is called *essential* if every continuous $g\colon X \to J^\alpha$ that satisfies $g|f^{-1}(T^\alpha) = f|f^{-1}(T^\alpha)$ is an onto mapping.

The following two theorems are due to Henderson [2].

1.4 THEOREM. *J^α is an absolute retract and* $\operatorname{Ind}(J^\alpha) = \alpha$.

1.5 THEOREM. *If there is an essential mapping from a normal space X into J^α then* $\operatorname{Ind}(X) \geqslant \alpha$ *or* $\operatorname{Ind}(X)$ *does not exist.*