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UPPER AND LOWER BOUNDS FOR MINIMAL NORM PROBLEMS UNDER LINEAR CONSTRAINTS

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1. Introduction

Upper and lower bounds for the performance index and the optimal control play an important role in the optimal control theory. These bounds are useful for estimating how close a sub-optimal control is to the optimal one, without actually calculating the latter. Further bounds can be successfully applied for the proof of bang-bang-ness of optimal controls (see [8]). The problem of finding bounds was first dealt with in [1]–[4], [10], [12], [13], [15]–[17] for lumped parameter systems and in [14], [19], [20] for distributed parameter systems. In the most papers, duality theorems and for the special case of optimal regulator problems estimations for the solution of the Riccati equations are used, in order to construct lower bounds.

Bounds for optimal control problems described by operator equations in Banach or Hilbert spaces were calculated by Aronoff–Leondes [3], Chan–Ho [11] and the author [6]–[8]. These bounds are applicable not only to lumped parameter but also to distributed parameter systems.

The aim of this paper is to generalize the results of papers [6], [7] and to compare with those of Aronoff–Leondes and Chan–Ho. In the following we consider the optimal control problem in normed linear spaces E_i (with the norm $\| \ \|_i$):

$$j(Q, u) = (\|Q-R\|_1^p + K\|u\|_2^p)^{1/p} \overset{!}{=} \underset{\left[\begin{smallmatrix}Q\\u\end{smallmatrix}\right]\in V}{\text{Infimum}} \tag{1}$$

and

$$j(Q, u) = \text{Maximum}\{\|Q-R\|_1, K\|u\|_2\} \overset{!}{=} \underset{\left[\begin{smallmatrix}Q\\u\end{smallmatrix}\right]\in V}{\text{Infimum}} \tag{2}$$

where $K \geqslant 0$, $1 \leqslant p < \infty$, $A \in L(E_1 \to E_3)$ and $B \in L(E_2 \to E_3)$ are linear operators with $D(A)$ and $D(B)$ dense in E_1 and E_2, resp., $R \in E_1$ and $f \in E_3$ are given elements and

$$V = \left\{ \begin{bmatrix} Q \\ u \end{bmatrix} \in E_1 \times E_2 \middle| \; AQ + Bu + f = 0, \; u \in U \subset E_2 \right\} \quad (U \text{ convex}).$$

If $A = I$, then we assume $B \in L(E_2 \to E_1)$ and $f \in E_1$.

Performance indexes of the forms (1) and (2) practically arise if we want to approximate a given state R and additionally to minimize the costs of control.

Obviously, problem (1) has the same solution as the problem

$$(1') \qquad j(Q, u) = \|Q - R\|_1^p + K\|u\|_2^p \overset{!}{=} \underset{\left[\begin{smallmatrix} Q \\ u \end{smallmatrix}\right] \in V}{\text{Infimum}}.$$

For our control problem we have chosen the form (1), in order to be able ro use the results of approximation theory. The problem

$$(3) \qquad \left\| \begin{bmatrix} Q \\ u \end{bmatrix} - \begin{bmatrix} R \\ 0 \end{bmatrix} \right\|_P \overset{!}{=} \underset{\left[\begin{smallmatrix} Q \\ u \end{smallmatrix}\right] \in V \subset E_1 \times E_2}{\text{Infimum}}$$

with

$$(4) \qquad \left\| \begin{bmatrix} Q \\ u \end{bmatrix} \right\|_P = (\|Q\|_1^p + K\|u\|_2^p)^{1/p} \qquad (1 \leqslant p < \infty)$$

and

$$(5) \qquad \left\| \begin{bmatrix} Q \\ u \end{bmatrix} \right\|_P = \text{Maximum}\,\{\|Q\|_1, K\|u\|_2\},$$

tespectively, is obviously an approximation problem in the product space $P = E_1 \times E_2$ (if $K > 0$) and contains our optimal control problems (1) and (2). Applying the results of approximation theory, we immediately obtain the following optimality and duality condition (for proofs see [7]):

THEOREM 1. $\begin{bmatrix} Q^0 \\ u^0 \end{bmatrix}$ *is a solution of problem* (3) *if and only if there exists* $l \in P' = (E_1 \times E_2)'$ (1) *with*

$$(6) \qquad \|l\|_{P'} \leqslant 1,$$

$$(7) \qquad l \begin{bmatrix} Q^0 - R \\ u^0 \end{bmatrix} = \left\| \begin{bmatrix} Q^0 \\ u^0 \end{bmatrix} - \begin{bmatrix} R \\ 0 \end{bmatrix} \right\|_P$$

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(1) We write the functional $l \in P'$ in the form $l \begin{bmatrix} Q \\ u \end{bmatrix} = l_1(Q) + K l_2(u)$ with $l_1 \in E_1'$, $l_2 \in E_2'$. For the norm $\|l\|_{P'}$ it follows then $\|l\|_{P'} = (\|l_1\|_{1'}^q + K\|l_2\|_{2'}^q)^{1/q}$ $(1/p + 1/q = 1$, $1 < p < \infty)$ and $\|l\|_{P'} = \text{Maximum}\,\{\|l_1\|_{1'}, \|l_2\|_{2'}\}$ $(p = 1)$, resp. if we take norm (4) on P. For norm (5) we obtain $\|l\|_{P'} = \|l_1\|_{1'} + \|l_2\|_{2'}$ ($\| \; \|_{i'}$ —norm on E_i').

and

(8) $$l\begin{bmatrix} Q^0-Q \\ u^0-u \end{bmatrix} \leqslant 0 \quad \forall \begin{bmatrix} Q \\ u \end{bmatrix} \in V.$$

THEOREM 2. *The following duality condition is valid:*

(9)
$$\operatorname*{Infimum}_{\left[\begin{smallmatrix} Q \\ u \end{smallmatrix}\right] \in V} \left\| \begin{bmatrix} Q \\ u \end{bmatrix} - \begin{bmatrix} R \\ 0 \end{bmatrix} \right\|_P = \operatorname*{Supremum}_{\|l\|_{P'} \leqslant 1} \Big(l_1(R) - \operatorname*{Supremum}_{\left[\begin{smallmatrix} Q \\ u \end{smallmatrix}\right] \in V} \big(l_1(Q) + K l_2(u) \big) \Big).$$

2. Lower bounds for problem (3)

The simple estimation $\|L(x-r)\|/\|L\| \leqslant \|x-r\|$ for bounded linear operators $L \neq 0$ leads to the lower bound

(10) $$\|f - Lr\|/\|L\| \leqslant \operatorname*{Infimum}_{Lx=f} \|x - r\|$$

without application of duality theorems. Applying inequality (10) to our problem (3) we obtain the following bounds:

LEMMA 1. *Under the additional assumption that the operators A and B are bounded (with $D(A) = E_1$, $D(B) = E_2$) we have the bounds:*

(11) $$\operatorname*{Infimum}_{AQ+Bu+f=0} (\|Q-R\|_1^p + K\|u\|_2^p)^{1/p} \geqslant \frac{\|AR+f\|_3}{(\|A\|^q + K^{1-q}\|B\|^q)^{1/q}}$$

for $1/p + 1/q = 1$ $(1 < p < \infty)$,

(12) $$\operatorname*{Infimum}_{AQ+Bu+f=0} (\|Q-R\|_1 + K\|u\|_2) \geqslant \frac{\|AR+f\|_3}{\text{Maximum}\,\{\|A\|,\ \|B\|/K\}}$$

and

(13) $$\operatorname*{Infimum}_{AQ+Bu+f=0} \text{Maximum}\,\{\|Q-R\|_1,\ K\|u\|_2\} \geqslant \frac{\|AR+f\|_3}{\|A\| + \|B\|/K}.$$

Proof. Defining $L \in L(E_1 \times E_2 \to E_3)$ by $L\begin{bmatrix} Q \\ u \end{bmatrix} := AQ + Bu$ and using the estimations $\|L\| \leqslant (\|A\|^q + K^{1-q}\|B\|^q)^{1/q}$ (for (11)), $\|L\| \leqslant$ Maximum $\{\|A\|, \|B\|/K\}$ (for (12)), and $\|L\| \leqslant \|A\| + \|B\|/K$ (for (13)), respectively, bounds (11)–(13) follow from (10). ■

Remarks. (1) In comparison with earlier results which only hold in Hilbert spaces, the advantage of the bounds derived in Lemma 1 lies in the fact that they are applicable to problems on arbitrary normed spaces.

(2) For $p = q = 2$ and the state equation

$$Q = Lu + f' \tag{14}$$

we obtain from (11), as a special case, the bound

$$\operatorname*{Infimum}_{u \in E_2} (\|Lu - r\|_1^2 + K\|u\|_2^2) \geqslant \frac{K\|r\|_1^2}{K + \|L\|^2} \qquad (r = R - f'),$$

given by Aronoff–Leondes [3] only for Hilbert spaces.

In the following we make use of duality theorems in order to be able to obtain better lower bounds than in Lemma 1. To this purpose we now assume the spaces E_i to be Hilbert spaces (denoted by H_i, with the inner product $(\, , \,)_i$).

In our considerations the inequality

$$\operatorname*{Infimum}_{AQ+Bu+f=0} j(Q, u) \geqslant \operatorname*{Supremum}_{l \in H_3} \operatorname*{Infimum}_{(l, AQ+Bu+f)_3 = 0} j(Q, u) \tag{15}$$

plays an essential role. This inequality is obviously valid for arbitrary functionals j and (also nonlinear) operators A and B (see also [1]).

For functional (3) it is easy to solve the problem

$$j(Q, u) \overset{!}{=} \operatorname*{Infimum}_{(l, AQ+Bu+f)_3 = 0}$$

for certain $l \in H_3$.

LEMMA 2. *Let* $l \in D(A^T) \cap D(B^T) \subset H_3$ *with* $\|A^T l\|_1 + \|B^T l\|_2 \neq 0$ *be given. Then we have*

$$\operatorname*{Minimum}_{(l, AQ+Bu+f)_3 = 0} \left\| \begin{bmatrix} Q \\ u \end{bmatrix} - \begin{bmatrix} R \\ 0 \end{bmatrix} \right\|_P = \frac{|(A^T l, R)_1 + (l, f)_3|}{\left\| \begin{bmatrix} A^T l \\ \dfrac{B^T l}{K} \end{bmatrix} \right\|_{P'}}. \tag{16}$$

Proof. Applying the well-known relation $\operatorname*{Minimum}_{f(x) = \beta} \|x - r\| = |f(r) - \beta| \times \|f\|^{-1}$ (see [22]) to our problem, we obtain (16) by setting

$$f\begin{bmatrix} Q \\ u \end{bmatrix} := (A^T l, Q)_1 + K(B^T l / K, u)_2, \quad x := \begin{bmatrix} Q \\ u \end{bmatrix}, \quad r := \begin{bmatrix} R \\ 0 \end{bmatrix}$$

and $\beta := -(l, f)_3$. ∎

Remarks. (1) For a given l it must be $(l, f)_3 = 0$ if $A^T l = 0$ and $B^T l = 0$. $Q^0 = R$, $u^0 = 0$ is then an optimal solution of (16).

(2) For quadratic problems it follows from (16) that

$$\operatorname*{Minimum}_{(l, AQ+Bu+f)_3 = 0} (\|Q - R\|_1^2 + K\|u\|_2^2) = \frac{|(A^T l, R)_1 + (l, f)_3|^2}{\|A^T l\|_1^2 + \|B^T l\|_2^2 / K}. \tag{16'}$$

Obviously,

$$\begin{bmatrix} Q^0 \\ u^0 \end{bmatrix} = -\frac{(A^T l, R)_1 + (l, f)_3}{\|A^T l\|_1^2 + \|B^T l\|_2^2/K} \begin{bmatrix} A^T l \\ \dfrac{B^T l}{K} \end{bmatrix} + \begin{bmatrix} R \\ 0 \end{bmatrix}$$

is an optimal solution of (16′).

From Lemma 2 and inequality (15) also follows a weak duality theorem.

THEOREM 3. *Under the same assumption as in Lemma 2 we have*

$$\text{(17)} \qquad \operatorname*{Infimum}_{AQ+Bu+f=0} \left\| \begin{bmatrix} Q \\ u \end{bmatrix} - \begin{bmatrix} R \\ 0 \end{bmatrix} \right\|_P \geqslant \operatorname*{Supremum}_{0 \neq l \in D(A^T) \cap D(B^T)} \frac{|(A^T l, R)_1 + (l, f)_3|}{\left\| \begin{bmatrix} A^T l \\ \dfrac{B^T l}{K} \end{bmatrix} \right\|_{P'}}.$$

Remarks. (1) Inequality (17) shows that the expression on the right-hand side of (17) is a lower bound for our optimal control problem also in the case when the operators A and B of the state equation are unbounded in contrast to the results of Aronoff–Leondes and Chan–Ho. Aronoff–Leondes only discuss state equations of the form $Q = Lu + f$ (L bounded) while Chan–Ho assume that the operators A and B in the general state equation $AQ + Bu + f = 0$ are bounded, and also do not give bounds of the form (18)–(20).

(2) Under the assumption that $R \in D(A)$, $AR + f \in D(A^T) \cap D(B^T)$ and

$$\|A^T(AR+f)\|_1 + \|B^T(AR+f)\|_2 > 0$$

we can estimate the supremum on the right-hand side of (17) by setting $l = AR + f$ and we obtain the following lower bounds:

(18)

$$\operatorname*{Infimum}_{AQ+Bu+f=0} (\|Q-R\|_1^p + K\|u\|_2^p)^{1/p} \geqslant \frac{\|AR+f\|_3^2}{(\|A^T(AR+f)\|_1^q + K^{1-q}\|B^T(AR+f)\|_2^q)^{1/q}}$$

$(1/p + 1/q = 1,\ 1 < p < \infty)$,

$$\text{(19)} \quad \operatorname*{Infimum}_{AQ+Bu+f=0} (\|Q-R\|_1 + K\|u\|_2) \geqslant \frac{\|AR+f\|_3^2}{\text{Maximum}\,\{\|A^T(AR+f)\|_1,\ \|B^T(AR+f)\|_2/K\}},$$

$$\text{(20)} \quad \operatorname*{Infimum}_{AQ+Bu+f=0} \text{Maximum}\,\{\|Q-R\|_1,\ K\|u\|_2\} \geqslant \frac{\|AR+f\|_3^2}{\|A^T(AR+f)\|_1 + \|B^T(AR+f)\|_2/K}.$$

These bounds are greater than or equal to the corresponding bounds (11)–(13). Bound (18) was derived for $p = 2$ in [6] and the bound (19) for $A = I$ in [7] using other methods.

(3) Obviously, for $A = I$ we have

$$\operatorname*{Infimum}_{Q+Bu+f=0} \left\| \begin{bmatrix} Q \\ u \end{bmatrix} - \begin{bmatrix} R \\ 0 \end{bmatrix} \right\|_P \leqslant \|R+f\|_1.$$

If now $R+f \in \operatorname{Ker} B^T$, then $\|R+f\|_1$ is also a lower bound (as follows from (17)), and therefore in this case we have the optimal control $u^0 = 0$. $u^0 = 0$ is also an optimal control for problem (1) if $K\|R+f\|_1 \geqslant \|B^T(R+f)\|_2$ and $p = 1$ (as follows from (19)). Now the question arises under which additional assumptions the equality holds in (17), i.e., when does Theorem 3 constitute a strong duality theorem.

THEOREM 4. *Let $j(Q, u)$ be convex and Gateaux differentiable (with the derivatives $j_Q(Q, u)$ and $j_u(Q, u)$). If we furthermore assume that there exists a solution (Q^0, u^0, l^0) of the equations*

$$\text{(21)} \qquad A^T l + j_Q(Q, u) = 0, \qquad B^T l + j_u(Q, u) = 0, \qquad AQ + Bu + f = 0,$$

then we have

$$\text{(22)} \qquad \operatorname*{Minimum}_{AQ+Bu+f=0} j(Q, u) = \operatorname*{Maximum}_{l \in H_3} \operatorname*{Minimum}_{(l, AQ+Bu+f)_3 = 0} j(Q, u).$$

Proof. Making use of

$$\begin{aligned} & j(Q, u) - j(Q^0, u^0) \\ & \quad = j(Q, u) + (l^0, AQ + Bu + f)_3 - j(Q^0, u^0) - (l^0, AQ^0 + Bu^0 + f)_3 \\ & \quad \geqslant \left(j_Q(Q^0, u^0) + A^T l^0, Q - Q^0\right)_1 + \left(j_u(Q^0, u^0) + B^T l^0, u - u^0\right)_2 = 0 \end{aligned}$$

$\forall \begin{bmatrix} Q \\ u \end{bmatrix}$ with $AQ + Bu + f = 0$, we see that $\begin{bmatrix} Q^0 \\ u^0 \end{bmatrix}$ is a solution of the primal problem of (22). $\begin{bmatrix} Q^0 \\ u^0 \end{bmatrix}$ also is a solution of the problem $\operatorname*{Infimum}_{(l^0, AQ+Bu+f)_3 = 0} j(Q, u)$ which we can analogously prove. Together with inequality (15) assertion (22) now follows. ■

Remarks. (1) Under the same assumption as in Theorem 4, a Lagrange duality theorem can also be derived:

(23)

$$\operatorname*{Minimum}_{AQ+Bu+f=0} j(Q, u) = \operatorname*{Maximum}_{l \in H_3} \operatorname*{Minimum}_{(Q,u) \in H_1 \times H_2} \left(j(Q, u) + (l, AQ + Bu + f)_3\right).$$

(2) For problem (1) with $p = 2$ equations (21) have the form

$$A^T l + Q - R = 0, \quad B^T l + Ku = 0, \quad AQ + Bu + f = 0, \tag{21'}$$

so that the assumptions of Theorem 4 are fulfilled if there exists a solution l of $AA^T l + BB^T l/K = AR + f$ (if $R \in D(A)$).

(3) The assumption of Theorem 4 can be weakened by replacing the Gateaux differentiability by the subdifferentiability. Equations (21) then have the form

$$A^T l + \partial j_Q(Q, u) = 0, \quad B^T l + \partial j_u(Q, u) = 0, \quad AQ + Bu + f = 0 \tag{21''}$$

with

$$\begin{bmatrix} \partial j_Q(Q, u) \\ \partial j_u(Q, u) \end{bmatrix} \in \partial j(Q, u).$$

3. Bounds for the quadratic problem

For the case $p = 2$, $A = I$, $B = -L$ (bounded) problem (1′) has the form

$$J(u) = \|Lu - r\|_1^2 + K\|u\|_2^2 \stackrel{!}{=} \underset{u \in U \subset H_2}{\text{Minimum}} \tag{24}$$

or

$$J(u) = \big((L^T L + KI)u, u\big)_2 - 2(u, L^T r)_2 + \|r\|_1^2 \stackrel{!}{=} \underset{u \in U \subset H_2}{\text{Minimum}}. \tag{24'}$$

For this special quadratic optimal control problem we can additionally derive bounds which characterize the deviation of an arbitrary control from the optimal control. It is well known that problem (24) for $K > 0$, $U = H_2$ has exactly one solution u^0 satisfying the operator equation

$$(L^T L + KI)u^0 = L^T r \tag{25}$$

with the self-adjoint, strongly positive operator $A = L^T L + KI$. Applying variational methods to the solution of the operator equation (25), the following estimations can be established.

THEOREM 5. *Let $U = H_2$, $K > 0$ and u^0 be the optimal control for problem* (24). *Then the following estimations for arbitrary controls $u \in H_2$ are valid:*

$$0 \leqslant J(u) - J(u^0) \leqslant \|(L^T L + KI)u - L^T r\|_2^2 / K, \tag{26}$$

$$\|u - u^0\|_2 \leqslant \|(L^T L + KI)u - L^T r\|_2 / K \tag{27}$$

and

$$\frac{\|L^T r\|_2}{K + \|L\|^2} \leqslant \|u^0\|_2 \leqslant \frac{\|L^T r\|_2}{K}. \tag{28}$$

Proof. Applying direct methods given in [23] to the operator equation $Au = f$ (with $A = L^TL+KI$, $f = L^Tr$), we obtain with

$$\|u\|_A^2 := (Au, u)_2 \quad \text{and} \quad (u, v)_A := (Au, v)_2 \quad \text{for}$$
$$\bar{J}(u) := (Au, u)_2 - 2(u, L^Tr)_2$$

the expression $\bar{J}(u) = \|u-u^0\|_A^2 - \|u^0\|_A^2$ where u^0 is the unique solution of (25). From $\bar{J}(u^0) = -\|u^0\|_A^2$ now follows $\bar{J}(u) = \bar{J}(u^0)+\|u-u^0\|_A^2$. The inequality $K\|u\|_2 \leqslant \|Au\|_2$ is obtained from $K\|u\|_2^2 \leqslant (Au, u)_2 \leqslant \|Au\|_2\|u\|_2$. Therefore, $\|u-u^0\|_A^2 \leqslant \|Au - L^Tr\|_2^2/K$. From $J(u) = \bar{J}(u) + \|r\|_1^2$ inequalities (26) and (27) now follow. Estimating the optimality condition (25) leads to inequality (28):

$$(28) \quad \|L^TL+KI\|\,\|u^0\|_2 \geqslant \|L^Tr\|_2 \quad \text{and} \quad \|u^0\|_2 \leqslant \|(L^TL+KI)^{-1}\|\|L^Tr\|_2 \leqslant \|L^Tr\|_2/K. \ \blacksquare$$

Remarks. (1) Inequalities (26) and (27) indicate the deviation of an arbitrary value from the optimal value depending on the "defect" of the operator equation (25). Bound (26) is also useful as a stopping condition for the application of iteration methods (gradient methods).

(2) Estimation (28) additionally gives bounds for the norm of optimal controls. Here the lower bound is also valid for $K = 0$ if in this case an optimal control exists. Making use of the spectral theory for linear operators the author presents in [8] the following bounds:

(28′)
$$\frac{\|L^Tr\|_2}{K+\|LL^Tr\|_1^2/\|L^Tr\|_2^2} \leqslant \|u^0\|_2 \leqslant \text{Minimum}\left\{\frac{\|r\|_1}{2K^{1/2}}, \frac{\|L^Tr\|_2}{K+\|L^Tr\|_2^2/\|r\|_1^2}\right\}.$$

(3) Minimizing problem (24′) subject to $u = \beta L^Tr$ yields an upper bound. This together with the lower bound from Chapter 2 gives

$$(29) \quad \|r\|_1^2 - \frac{\|L^Tr\|_2^2}{K+\|L^Tr\|_2^2/\|r\|_1^2} \leqslant J(u^0) \leqslant \|r\|_1^2 - \frac{\|L^Tr\|_2^2}{K+\|LL^Tr\|_1^2/\|L^Tr\|_2^2}.$$

(4) From the optimality condition (25) it follows that the optimal control u^0 is a fixed point of the mapping

$$(30) \qquad F(u) = (L^Tr - L^TLu)/K \quad (K > 0)$$

which is contractive for

$$(31) \qquad q := \|L\|^2/K < 1.$$

The use of the Banach fixed-point theorem now yields the estimation

$$0 \leqslant J(u^n) - J(u^0) \leqslant \|L^T r\|_2 \frac{q^{n-1}}{1-q} \|u^2 - u^1\|_2 + K\|u^n\|_2 \|u^{n+1} - u^n\|_2$$

for $u^{n+1} = F(u^n)$ ($n = 1, 2, \ldots$; u^1 arbitrary).

Applying a duality theorem of Schumacher [21], it is possible to calculate bounds for problem (24) with constraints of the form

$$U := \{u \in H_2 | \; \|u\|_2 \leqslant \beta\}. \tag{32}$$

THEOREM 6. *For the solution u^0 of problem* (24) *subject to the control-restriction* (32), *the following inequalities are valid for $u \in H_2$:*

$$\begin{aligned} &0 \leqslant J(u) - J(u^0) \\ &\leqslant \begin{cases} \|(L^T L + KI)u - L^T r\|_2^2 / K & \text{if } \|L^T Lu - L^T r\|_2 \leqslant K\beta, \\ 2\beta \|L^T Lu - L^T r\|_2 + K(\beta^2 - \|u\|_2^2) - \\ \qquad - 2(Lu - r, Lu)_1 & \text{else.} \end{cases} \end{aligned} \tag{33}$$

Proof. Schumacher [21] has given the duality theorem

$$J(u^0) = \operatorname*{Supremum}_{\lambda \geqslant 0, u \in H_2} S(u, \lambda)$$

with

$$\begin{aligned} S(u, \lambda) &= L(u, \lambda) - \|L'(u, \lambda)\|_2^2 / 4(C + \lambda), \\ L(u, \lambda) &:= J(u) + \lambda(\|u\|_2^2 - \beta^2) \end{aligned}$$

for strongly convex and differentiable functionals i.e.,

$$J(v) - J(u) \geqslant \big(J'(u), v - u\big)_2 + C\|v - u\|_2^2, \qquad C > 0.$$

For our problem this gives

$$J(u^0) = \operatorname*{Supremum}_{\lambda \geqslant 0, u \in H_2} \big(-\|L^T(Lu - r)\|_2^2 / (K + \lambda) - (Lu - r, Lu + r)_1 - \lambda\beta^2\big).$$

By maximization with respect to λ, bound (33) immediately follows. ∎

Remarks. (1) For $\beta \to \infty$, bound (33) obviously yields bound (26).

(2) Since for the optimal control u^0 the inequality $J(u) - J(u^0) \geqslant K\|u - u^0\|_2^2 \;\forall u \in U$ is fulfilled, estimation (33) also gives an estimation for $\|u - u^0\|_2^2 \;\forall u \in U$.

(3) From (28) we can see that the control problem (24) for $\|L^T r\|_2 \leqslant K\beta$ has the same solution for $U = H_2$ as for U from (32), and that for $\|L^T r\|_2 > \beta(K + \|L\|^2)$ these solutions are unequal.

4. Bounds for the case $K = 0$

For $K = 0$ all optimal control problems, described in Chapter 1, have the form

(34) $$\|Q - R\|_1 \overset{!}{=} \underset{AQ+Bu+f=0, u\in U\subset H_2}{\text{Infimum}}$$

and for $A = I$, $B = -L$

(34′) $$\|Lu - r\|_1 \overset{!}{=} \underset{u\in U\subset H_2}{\text{Infimum}} \qquad (r = R+f).$$

The bounds, presented in Chapters 2 and 3 for $K > 0$, are not valid for $K = 0$ (with exception of the lower bounds (28) and (28′)). For problems (34) and (34′) it is very difficult to calculate lower bounds since here the case Infimum $\|Q - R\|_1 = 0$ is possible. For instance, $\underset{u\in H_2}{\text{Infimum}} \|Lu - r\|_1 = 0$ if and only if $r \in \overline{R(L)}$. If $r \notin \overline{R(L)}$ then we have

(35) $$0 < \underset{u\in H_2}{\text{Infimum}} \|Lu - r\|_1 = \underset{l\in \mathrm{Ker} L^T, \|l\|_1 = 1}{\text{Maximum}} (l, r)_1.$$

This results from Theorem 2. Estimation (35) immediately yields the lower bound

(36) $$|(l, r)_1| / \|l\|_1$$

if there exists an $l \neq 0$ with $l \in \mathrm{Ker} L^T$ and $(l, r)_1 \neq 0$. An optimal control u^0 is again equal to 0 if $r \in \mathrm{Ker} L^T$.

For the control set (32) the Duality Theorem 2 has the form

(37) $$\underset{\|u\|_2 \leq \beta}{\text{Infimum}} \|Lu - r\|_1 = \underset{\|l\|_1 = 1}{\text{Supremum}} \left((l, r)_1 - \beta \|L^T l\|_2\right)$$

so that we can here also calculate lower bounds for the case where an $l \in H_1$ exists with $(l, r)_1 - \beta \|L^T l\|_2 > 0$.

To derive lower bounds for problem (34) is still more difficult In [6] the author gives such a bound under some additional assumptions

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