SEQUENTIAL METHODS IN STATISTICS
BANACH CENTER PUBLICATIONS, VOLUME 16
PWN POLISH SCIENTIFIC PUBLISHERS
WARSAW 1985

SOME RESULTS ON EXPONENTIAL FAMILIES OF MARKOV PROCESSES

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1. Introduction

Exponential families of probability distributions on the real line have been defined by Koopman ([6]) and studied by many authors, see, e.g., [8] for some references. An extension of this notion to processes with independent increments was given by Magiera ([13]), and Franz and Winkler ([2]) independently, see also [15]. They defined such families by using the one-dimensional distributions of the processes under consideration. An analytical description in terms of the corresponding Lévy-characteristics was given in [9].

As an essential property of exponential families with independent increments we mention that the "last" observation X_t of the process is a sufficient statistics for the parameter of the family. This property has been used to define exponential families of Markov processes, which was done in [10]. There one can find also a description of those families in terms of the transition functions of the corresponding Markov processes. Examples of exponential families of Markov processes have been studied in [10], [11].

In this note we shall give a survey of the above mentioned results, for considerations from the statistical point of view see also [7], [14].

2. Definitions

2.1. By N we denote the set of nonnegative integers, by R the set of real numbers and by $\mathfrak{B}$ the σ-algebra of Borelian subsets of R. Let E be a σ-compact topological space, and $\mathfrak{E}$ the σ-algebra of its Borelian subsets, $\Omega := E^T$, $\mathfrak{A} := \mathfrak{E}^T$ with $T = [0, \infty)$, $X_t(\omega) = \omega_t$ $(\omega = (\omega_s)_{s \in T} \in \Omega)$. Then $\mathfrak{A}$ is generated by the variables X_s, $s \in T$. Define $\mathfrak{A}_t$ to be the σ-algebra generated by X_s, $s \in [0, t]$. Moreover, let $(P_x)_{x \in E}$ be a family of probability measures on $\mathfrak{A}$ such that $\pi := (X_t, \mathfrak{A}_t, P_x)$ is a *(conservative) Markov process* in the sense of [1]. This means that for all $x \in E$; $s, t \in T$ and every bounded, $\mathfrak{E}$-measurable, realvalued function f we have

$$E_x(f(X_{s+t})|\mathfrak{A}_t) = E_{X_t} f(X_s) \qquad (P_x\text{-a.s., } x \in E),$$

and that the function

$$(t, x, B) \to P(t, x, B) := P_x(X_t \in B) \qquad (x \in E,\ t > 0,\ B \in \mathfrak{E})$$

is a (conservative) *transition function* on $(E, \mathfrak{E})$.

By $\mathfrak{M}_E$ we denote the set of all (conservative) Markov processes π on $(E, \mathfrak{E})$ satisfying the following condition:

(A): *There exists a σ-finite measure μ on $\mathfrak{E}$ such that $\mu(U) > 0$ (U open, nonvoid), $P(t, x, \cdot)$ is equivalent to $\mu(\cdot)$ ($t > 0$, $x \in E$) and the derivation $\frac{P(t, x, dy)}{\mu(dy)}$ has a version continuous with respect to (t, x, y) ($t > 0$; $x, y \in E$).*

If $(E, \mathfrak{E}) \subseteq (R, \mathfrak{B})$ (E closed under operation of addition, $0 \in E$) and π belongs to $\mathfrak{M}_E$ with

$$P(t, x+z, B+\{z\}) = P(t, x, B), \qquad (t > 0;\ x, z \in E,\ B \in \mathfrak{E})$$

then π is called a *Markov process with independent increments* (shortly: *i.i.*). (We restrict ourselves to the one-dimensional case; an extension to R^n seems to be possible without serious problems.)

For every $\pi \in \mathfrak{M}_E$ with i.i. the process $(X_t)_{t \geq 0}$ has independent increments under $P := P_0$ and we have

$$P(X_t - X_s \in B) = P(t-s, 0, B) \qquad (0 \leq s < t,\ B \in \mathfrak{E}).$$

Conversely, every probability measure P on $\mathfrak{A}$ under which $(X_t)_{t \geq 0}$ has i.i. generates a Markov process π with i.i. which has the transition function

$$P(t, x, B) := P(X_t + x \in B) \qquad (t > 0,\ x \in E,\ B \in \mathfrak{E}).$$

In this sense we shall identify every process $(X_t)_{t \geq 0}$ on $(\Omega, \mathfrak{A}, P)$ having independent increments under P with a Markov process $\pi = (X_t, \mathfrak{A}_t, P_x)$ with i.i..

2.2. A family $(\pi_\vartheta)_{\vartheta \in \Theta}$ of Markov processes $\pi_\vartheta = (X_t, \mathfrak{A}_t, P_x^\vartheta) \in \mathfrak{M}_E$ on $(E, \mathfrak{E})$ is called an *exponential family of Markov processes* if the following conditions hold:

(i) Choose an arbitrary $x \in E$ and consider the family $(P_x^\vartheta)_{\vartheta \in \Theta}$. Then for every $t > 0$ the variable X_t is a *sufficient statistics* for ϑ with respect to $\mathfrak{A}_t$, i.e., for every $A \in \mathfrak{A}_t$ there exists a $(\mathfrak{E}, \mathfrak{B})$-measurable function φ_A such that

$$P_x^\vartheta(A|X_t) = \varphi_A(X_t) \qquad (P_x\text{-a.s., } \vartheta \in \Theta).$$

(ii) There exists a σ-finite measure μ on $\mathfrak{E}$ such that for every $\vartheta \in \Theta$ the measure μ_ϑ occurring in condition (A) above is equivalent to μ.

(iii) Θ contains at least two elements. If $\vartheta, \vartheta' \in \Theta$ with $\vartheta \neq \vartheta'$, then $\pi_\vartheta \neq \pi_{\vartheta'}$.

The set of all Markov processes π from $\mathfrak{M}_E$ which belong to some exponential family is called the *exponential class of Markov processes on* $(E, \mathfrak{E})$.

If an exponential family of Markov processes consists of Markov-processes with i.i. only, we shall speak of *exponential families of processes with i.i.*

2.3. Let us consider two examples.

a) Assume that $\mathscr{P} = (\pi_\vartheta)_{\vartheta \in \Theta}$ is a family of Markov processes with i.i. satisfying the following condition:

(B): *There exist a σ-finite measure μ on $\mathfrak{E}$ with $\mu(U) > 0$ (U open, nonvoid), a continuous function $h(x, t)$ ($x \in E$, $t > 0$) and real functions $A(\cdot)$, $B(\cdot)$ on Θ with $A(\vartheta) \neq A(\vartheta')$ ($\vartheta \neq \vartheta'$) such that*

$$P_0^\vartheta(X_t \in dx) = h(x, t)\exp\big(A(\vartheta)x + B(\vartheta)t\big)\mu(dx) \qquad (t > 0,\ x \in E,\ \vartheta \in \Theta).$$

Then $\mathscr{P}$ is an exponential family of processes with i.i.. (Obviously, (ii), (iii) of 2.2. are satisfied; to prove (i) see [2].) In particular, if π_ϑ is the *Wiener process on R* with trend coefficient $\vartheta \in R$ and diffusion coefficient σ^2 independent of ϑ, then (π_ϑ) is an exponential family of processes with i.i.. The same statement holds if π_ϑ is a *Poisson process* with intensity $\vartheta > 0$ (see, e.g., [2]; there are also further examples).

b) Let $\pi = (X_t, \mathfrak{A}_t, P_x)$ be a Markov process from $\mathfrak{M}_E$. Define Ξ_π to be the set of all pairs (α, g) such that α is a real number and g is a strictly positive continuous function on E with

$$P_t g(x) := \int_E P(t, x, dy) g(y) = \exp(\alpha t) g(x) \qquad (t > 0,\ x \in E).$$

Assume $(\alpha, g) \in \Xi_\pi$. Then by

$$P^{(\alpha, g)}(t, x, dy) := \exp(-\alpha t) P(t, x, dy)\frac{g(y)}{g(x)} \qquad (t > 0; x, y \in E)$$

a (conservative) transition function $P^{(\alpha,g)}$ on $(E, \mathfrak{E})$ is given. Thus there exists a Markov process $\pi^{(\alpha,g)} = (X_t, \mathfrak{A}_t, P_x^{(\alpha,g)})$ on $(E, \mathfrak{E})$ having $P^{(\alpha,g)}$ as its transition function.

Obviously, $\pi^{(\alpha,g)} \in \mathfrak{M}_E$. Define $M(\pi) := \{\pi^{(\alpha,g)} | (\alpha, g) \in \Xi_\pi\}$. We have $\pi \in M(\pi)$, because $\pi = \pi^{(0,1)}$ where $1(\cdot) \equiv 1$. If $M(\pi) \neq \{\pi\}$, i.e., if $M(\pi)$ has at least two elements, then $M(\pi)$ is an exponential family of Markov processes (see [10]). More concrete examples of exponential families of the form $M(\pi)$ are given below.

3. Exponential families of Markov processes

The following theorem characterizes those Markov processes π which belong to the exponential class and shows that $M(\pi)$, defined in example b) of the previous section, is the greatest exponential family of Markov processes to which π belongs.

THEOREM (see [10]). *The following statements hold*:

(i) *A Markov process* $\pi = (X_t, \mathfrak{A}_t, P_x) \in \mathfrak{M}_E$ *with state space* $(E, \mathfrak{E})$ *belongs to some exponential family of Markov processes if and only if there exist a real number* α *and a nonconstant strictly positive continuous function* $g(\cdot)$ *on* E *such that*

$$P_t g(x) = \exp(\alpha t) g(x) \qquad (t > 0;\ x \in E). \tag{1}$$

(ii) *If* π *belongs to some exponential family* $\mathscr{P}$ *of Markov processes, then* $\mathscr{P} \subseteq M(\pi)$.

EXAMPLES. a) (see [11]) Assume that π is a (conservative) birth-and-death process on N with birth-rates λ_i $(i \geqslant 0)$ and death-rates μ_i $(i \geqslant 1)$. Then it follows, from (1) that

$$\lambda_i g(i+1) - (\lambda_i + \mu_i) g(i) + \mu_i g(i-1) = \alpha g(i) \qquad (i \geqslant 1),$$

$$\lambda_0 g(1) - \lambda_0 g(0) = \alpha g(0).$$

If we suppose $g(0) = 1$, these equations have a uniquely determined solution for all real α, which we denote by $Q_i(\alpha)$ $(i \geqslant 0)$. It can be calculated recursively. The functions $Q(\alpha)$ $(\alpha \geqslant \alpha_0)$ satisfy (1) (i.e., π belongs to some exponential family of Markov processes) if and only if

$$\sum_{i=0}^{\infty} \pi_i \sum_{k=0}^{i-1} \frac{1}{\lambda_k \pi_k} = \infty \tag{2}$$

with

$$\pi_0 := 1, \qquad \pi_k := \frac{\lambda_0 \ldots \lambda_{k-1}}{\mu_1 \ldots \mu_k} \quad (k \geqslant 1).$$

(Recall that π is conservative if and only if $\sum_{i=0}^{\infty} \frac{1}{\lambda_i \pi_i} \sum_{k=0}^{i} \pi_k = \infty$.) One can show that there exists a nonpositive real number α_0 such that $Q_i(\alpha) > 0$ $(i \in N)$ if and only if $\alpha \geqslant \alpha_0$.

Suppose (2) holds. Then the largest exponential family of Markov processes $M(\pi)$ to which π belongs can be described as follows: $M(\pi)$ consists of birth-and-death processes $\pi^{(\alpha)}$, having the rates

$$\lambda_i^{(\alpha)} := \lambda_i \frac{Q_{i+1}(\alpha)}{Q_i(\alpha)} \quad (i \geqslant 0), \qquad \mu_i^{(\alpha)} := \mu_i \frac{Q_{i-1}(\alpha)}{Q_i(\alpha)} \quad (i \geqslant 1),$$

where α runs through $[\alpha_0, \infty)$.

The following example includes the previous one as a very special case.

b) (see [12]) Let m and p be nondecreasing functions on $[0, L)$ for some $L \leqslant \infty$ with $m(0) = p(0) = 0$, $0 < m(x) < m(L-0)$ $(x \in (0, L))$, p strictly increasing and continuous. Moreover, suppose that $\int_0^L m dp = \infty$ holds. Then there exists a uniquely determined conservative Markov process π on the state space $E := \{x \in [0, L) | x \text{ is a point of increase of } m\}$ which is reflected at zero and which has the infinitesimal operator $D := \frac{d}{dm}\frac{d}{dp}$ defined on an appropriate domain of continuous functions (see [12]). The process π is called the *quasidiffusion with speed measure m and scale p.*

If m is a step function with $E = N$, then π is a birth-and-death process; if m is strictly increasing, then π is a diffusion in the sense of [3]. Classical diffusions are obtained if m is strictly increasing and m, p *are* smooth enough (see [3]). Denote by $\varphi(x, \alpha)$ $(x \in E, \alpha \in R)$ the unique solution of

$$D\varphi = \alpha\varphi, \qquad \varphi(0, \alpha) = 1, \qquad \frac{d}{dp}\varphi(0, \alpha) = 0.$$

There exists a nonpositive α_0 such that $\varphi(\cdot, \alpha) > 0$ if and only if $\alpha \geqslant \alpha_0$.

Generalizing example a), one can show that π belongs to some exponential family of Markov processes if and only if $\int_0^L p dm = \infty$. If this holds, then $M(\pi)$ consists of all quasidiffusions $\pi^{(\alpha)}$ $(\alpha \geqslant \alpha_0)$ having the speed measure $m^{(\alpha)}$ and the scale $p^{(\alpha)}$ given by

$$dm^{(\alpha)} = \varphi^2(\cdot, \alpha) dm, \qquad dp^{(\alpha)} = \varphi^{-2}(\cdot, \alpha) dp.$$

c) Put $E := \{1, 2, \ldots, n\}$ and let π be an irreducible Markov chain on E. Then $\pi \in \mathfrak{M}_E$. The process π does not belong to any exponential family of Markov processes because a strictly positive solution $g(\cdot)$ of (1) exists for $\alpha = 0$ only and this solution is unique and a constant function by the Frobenius Theorem.

4. Exponential families of processes with independent increments

Let $\pi \in \mathfrak{M}_E$ with $E \subseteq R$ be a Markov process with i.i. Then by the definition of $\mathfrak{M}_E$ there exist a σ-finite measure μ on $\mathfrak{E}$ and a continuous function $h(x, t)$ $(x \in E, t > 0)$ such that

$$P_0(X_t \in dx) = h(x, t)\mu(dx) \qquad (x \in E, t > 0).$$

Furthermore, assume that π belongs to some exponential family $\mathscr{P} = (\pi_\vartheta)_{\vartheta \in \Theta}$ of processes with i.i.. Then by the theorem above we have $\mathscr{P} \subseteq M(\pi)$. Thus for every $\pi_\vartheta \in \mathscr{P}$ there exists a pair $(\alpha, g) = (\alpha(\vartheta), g_\vartheta) \in \Xi_\pi$ such that

$$P_\vartheta(t, x, dy) = \exp(-\alpha t)\frac{g(y)}{g(x)} P(t, x, dy) \qquad (x, y \in E; t > 0)$$

holds. Since π and π_ϑ have i.i., we get

$$g(y)g(x) = g(y-x)g(x) \qquad (x, y \in R),$$

assuming $g(0) = 1$, we obtain

$$g(x) = \exp(w(\vartheta)x) \qquad (x \in R)$$

for some real $v(\vartheta)$. Thus there exist functions

$$A(\vartheta) := w(\vartheta), \qquad B(\vartheta) = -\alpha(\vartheta) \qquad (\vartheta \in \Theta),$$

such that

$$P_0^\vartheta(X_t \in dx) = \exp(A(\vartheta)x + B(\vartheta)t)h(x, t)\mu(dx) \qquad (x \in E, t > 0, \vartheta \in \Theta).$$

Now, using example a) of 2.3, we have the following

PROPOSITION. *A family $\mathscr{P} = (\pi_\vartheta)_{\vartheta\in\Theta}$ of Markov processes with i.i. belonging to $\mathfrak{M}_E$ with $E \subseteq R$ is an exponential family of processes with independent increments if and only if there exist a σ-finite measure μ on E, a continuous function $h(x, t)$ $(x \in E, t > 0)$ and functions $A(\cdot)$, $B(\cdot)$ on Θ with $A(\vartheta) \neq A(\vartheta')$ $(\vartheta \neq \vartheta')$, card $\Theta \geqslant 2$ and*

$$P_0^\vartheta(X_t \in dx) = h(x, t)\exp(A(\vartheta)x + B(\vartheta)t)\mu(dx) \qquad (x \in E, t > 0, \vartheta \in \Theta).$$

This proposition characterizes exponential families of processes with i.i. by means of their one-dimensional distributions. Such a description was used in [13], [2] as a definition of exponential families.

Assume that $\pi \in \mathfrak{M}_E$ is a Markov process with i.i. and let $M(\pi) \neq \{\pi\}$. Denote by $I(\pi)$ the set of all Markov processes $\pi' \in M(\pi)$ which have i.i.. It may happen that $I(\pi) = M(\pi)$ (this holds, e.g., if π is a Poisson process on N) or that $I(\pi) \neq M(\pi)$ (this case occurs if π is a Wiener process). A description of $I(\pi)$ is given in the next proposition. To prepare it, let us recall some properties of processes with independent increments.

For every Markov process π with i.i. there exist real numbers γ, σ^2 with $\sigma^2 \geqslant 0$ and a σ-finite measure v on $R\setminus\{0\}$ with

$$\int_{R\setminus\{0\}} \frac{y^2}{1+y^2} v(dy) < \infty \tag{3}$$

such that

$$\int_R \exp(i\lambda x)P(t, 0, dx)$$

$$= \exp\left\{t\left[i\gamma\lambda - \frac{1}{2}\sigma^2\lambda^2 + \int_{R\setminus\{0\}} \left(\exp(i\lambda y) - 1 - \frac{i\lambda y}{1+y^2}\right)v(dy)\right]\right\}$$

$$(\lambda \in R, t > 0). \tag{4}$$

The triple (γ, σ^2, ν) is uniquely determined and we call it the *Lévy-characteristics* of π. For every triple (γ, σ^2, ν), where $\gamma \in R$, $\sigma^2 \geqslant 0$, and ν is a σ-finite measure on $R\setminus\{0\}$, satisfying (3), there exists a Markov process π with i.i. having (γ, σ^2, ν) as its Lévy-characteristics.

We define

$$R_\pi := \{u \in R \mid \int_R \exp(ux)\, P(t, 0, dx) < \infty \text{ for some } t > 0\}.$$

Then the equation

$$R_\pi = \left\{u \in R \mid \int_{R\setminus\{0\}} \exp(uy) \frac{y^2}{1+y^2}\, \nu(dy) < \infty\right\}$$

holds, and R_π is an interval of the real line which includes zero.

PROPOSITION (see [9]). *Let π be a Markov process with i.i.. Then π belongs to some exponential families of Markov processes with i.i. if and only if $R_\pi \neq \{0\}$.*

If $R_\pi \neq \{0\}$, then the largest exponential family $I(\pi)$ of processes with independent increments to which π belongs consists of all Markov processes $\pi^{(u)}$ $(u \in R_\pi)$ with independent increments, having the Lévy-characteristics $(\gamma_u, \sigma_u^2, \nu_u)$ with

$$\gamma_u = \gamma + u\sigma^2 + \int_{R\setminus\{0\}} \frac{y}{1+y^2} (\exp(uy) - 1)\, \nu(dy),$$

$$\sigma_u^2 = \sigma^2, \qquad d\nu_u(y) = \exp(uy)\, d\nu(y).$$

For the corresponding one-dimensional distributions we have

$$P_0^{(u)}(X_t \in dx) = \exp\big(ux - v(u)t\big) P_0(X_t \in dx) \qquad (u \in R_\pi,\ x \in E,\ t > 0)$$

with

$$v(u) = u\gamma + \frac{1}{2} u^2 \sigma^2 + \int_{R\setminus\{0\}} \left(\exp(uy) - 1 - \frac{uy}{1+y^2}\right) \nu(dy).$$

Using this proposition, one can construct a great many new examples, see [9].

For statistical investigations it is desirable to have a probabilistic interpretation of the parameter of the exponential family. We get it by a reparametrization of $I(\pi)$ as follows; for details see [6]:

Put $\vartheta := \vartheta(u) = v'(u)$, $\Theta := \{\vartheta(u) \mid u \in R\}$, and denote by $u = u(\vartheta)$ the inverse function of $\vartheta = \vartheta(u)$. Then, with the notation $A(\vartheta) := u(\vartheta)$, $B(\vartheta) :=$

$-v(u(\vartheta))$ $(\vartheta \in \Theta)$, we get

$$I(\pi) = \{\pi_\vartheta := \pi^{(u(\vartheta))} \mid \vartheta \in \Theta\},$$

where

$$P_0^{(u(\vartheta))}(X_t \in dx) = \exp(A(\vartheta)x + B(\vartheta)t) P_0(X_t \in dx) \qquad (\vartheta \in \Theta,\ x \in E,\ t > 0)$$

and

$$E_\vartheta X_t = \vartheta t \qquad (\vartheta \in \Theta,\ t > 0).$$

5. A generalized fundamental identity

Let π be a Markov process from $\mathfrak{M}_E$. Recall that a mapping τ from Ω into $[0, \infty]$ is called a *stopping time* if $\{\tau \leqslant t\} \in \mathfrak{A}_t$ $(t \geqslant 0)$. If τ is a stopping time, put $\mathfrak{A}_\tau := \{A \in \mathfrak{A} \mid A \cap \{\tau \leqslant t\} \in \mathfrak{A}_t,\ t \geqslant 0\}$. The following proposition gives a generalization of Wald's fundamental identity, which can be used in sequential statistics (see, e.g., [7]).

PROPOSITION. *For every* $(\alpha, g) \in \Xi_\pi$ *and every stopping time* τ *we have*

$$g(x) P_x^{(\alpha,g)}(A \cap \{\tau < \infty\}) = \int_{A \cap \{\tau < \infty\}} \exp(-\alpha\tau) g(X_\tau) dP_x \qquad (A \in \mathfrak{A}_\tau). \quad (5)$$

For the proof observe that $(\exp(-\alpha t) g(X_t), \mathfrak{A}_t)_{t \geqslant 0}$ is a positive martingale with respect to P_x $(x \in E)$ and apply the stopping theorem for martingales by using the stopping times $\tau_s := \tau \wedge s$ $(s > 0)$. This yields (5) for τ_s. By letting $s \uparrow \infty$ and after some calculations analogous to those made in [7] we get the proposition. (The proof can also be derived from the general theory of absolute continuity of measures, developed in [4].)

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Presented to the semester
Sequential Methods in Statistics
September 7–December 11, 1981