On special metrics characterizing topological properties

by

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Dedicated to Professor Tan-iti Nagata on his 60th birthday

Abstract. We shall discuss the characterizations of certain classes of infinite dimensional metrizable spaces by a special metric and consider the questions concerning special metrics raised by J. Nagata. A characterization of a strongly metrizable space is also obtained.

0. Introduction. It is an interesting problem to characterize a topological property of a metrizable space by means of a special metric. In dimension theory, the following theorem is already known.

THEOREM A. *For a metrizable space X, the following conditions are equivalent*:

(a) $\dim X \leqslant n$.

(b) *X admits a metric ϱ satisfying the following condition*:

$(0)_n$ *For every $\varepsilon > 0$, every point x of X and every $n+2$ many points $y_1, \ldots, y_{n+2}$ of X with $\varrho(S_{\varepsilon/2}(x), y_i) < \varepsilon$ for each $i = 1, \ldots, n+2$, there are distinct natural numbers i and j such that $\varrho(y_i, y_j) < \varepsilon$.*

(c) *X admits a metric ϱ satisfying the following condition*:

$(1)_n$ *For every point x of X and every $n+2$ many points $y_1, \ldots, y_{n+2}$ of X, there are distinct natural numbers i and j such that $\varrho(y_i, y_j) \leqslant \varrho(x, y_j)$.*

The equivalence of (a) and (b) of the above theorem was proved by J. Nagata [8] and simpler proofs were given by S. Buzási [3] and P. Assouad [1]. The equivalence of (a) and (c) of the theorem was proved by J. Nagata [9], [10] and P. A. Ostrand [14] independently. For the case of the separable metrizable spaces, the following theorem was obtained by J. de Groot in [5].

THEOREM B. *A separable metrizable space X has $\dim X \leqslant n$ if and only if X admits a totally bounded metric ϱ satisfying the following condition*:

$(2)_n$ *For every point x of X and every $n+2$ many points $y_1, \ldots, y_{n+2}$ of X, there are natural numbers i, j and k such that $i \neq j$ and $\varrho(y_i, y_j) \leqslant \varrho(x, y_k)$.*

In [11], J. Nagata posed the problem: *Extend Theorems A and B to infinite dimensional spaces.* He partially solve this problem as follows:

THEOREM C. *For a metrizable space X, the following conditions are equivalent*:

(a) *X is strongly countable-dimensional.*

(b) *X admits a metric ϱ satisfying the following condition*:

$(1)_\infty$ *For every point x of X, there is a natural number $n(x)$ such that for every $n(x)+2$ many points $y_1, \ldots, y_{n(x)+2}$ of X, there are distinct natural numbers i and j such that $\varrho(y_i, y_j) \leqslant \varrho(x, y_j)$.*

(c) *X admits a metric ϱ satisfying the following condition*:

$(2)_\infty$ *For every point x of X, there is a natural number $n(x)$ such that for every $n(x)+2$ many points $y_1, \ldots, y_{n(x)+2}$ of X, there are natural numbers i, j and k such that $i \neq j$ and $\varrho(y_i, y_j) \leqslant \varrho(x, y_k)$.*

In Section 1, we extend the above theorems to metrizable spaces that have both large transfinite dimension Ind and strong small transfinite dimension sind. Further modifications of $(0)_n$, $(1)_n$ and $(2)_n$ are discussed in Section 2. In Section 3, we characterize a strongly metrizable space by a special metric.

For a collection $\mathcal{U}$ of subsets of a set X, we denote $\mathcal{U}^{\#} = \bigcup\{U\colon U \in \mathcal{U}\}$. Furthermore, for a natural number n, $[\mathcal{U}]^1 = \mathcal{U}^* = \{\mathrm{St}(U, \mathcal{U})\colon U \in \mathcal{U}\}$ and $[\mathcal{U}]^{n+1} = ([\mathcal{U}]^n)^*$. Let N denote the set of all natural numbers and Q^* denote the set of all rational numbers of the form $2^{-m_1}+\ldots+2^{-m_t}$, where $m_1, \ldots, m_t$ are natural numbers satisfying $1 \leqslant m_1 < \ldots < m_t$. We refer the readers to [4] and [13] for un-defined terminology and basic results on dimension and infinite dimension theory.

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1. Characterizations of classes of infinite dimensional spaces. In this section, we consider the characterizations of two classes of infinite dimensional spaces. First, we characterize the class of strongly countable-dimensional spaces by extending the condition $(0)_n$.

1.1. THEOREM. *A metrizable space X is strongly countable-dimensional if and only if X admits a metric ϱ satisfying the following condition*:

$(0)_\infty$ *For every point x of X, there is an $n(x) \in N$ such that for every $\varepsilon > 0$ and every $n(x)+2$ many points $y_1, \ldots, y_{n(x)+2}$ of X with $\varrho(S_{\varepsilon/2}(x), y_i) < \varepsilon$ for each $i = 1, \ldots, n(x)+2$, there are distinct natural numbers i and j such that $\varrho(y_i, y_j) < \varepsilon$.*

Proof. Let ϱ be an admissible metric for X satisfying the condition $(0)_\infty$. For each point x of X, let $n(x)$ be the minimal number satisfying the condition $(0)_\infty$. For each $n \in N$, we put $F_n = \{x \in X\colon n(x) \leqslant n\}$. It is easy to see that each F_n is a closed set of X and $X = \bigcup\{F_n\colon n \in N\}$. By Theorem A, it follows that $\dim F_n \leqslant n$ for each $n \in N$. Hence X is strongly countable-dimensional. Conversely, let X be a strongly countable-dimensional space. Let $X = \bigcup\{F_n\colon n \in N\}$, where F_n is a closed set of X such that $\dim F_n \leqslant n$ and $F_n \subset F_{n+1}$ for each $n \in N$. By [11; Lemma 1], we can obtain a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$, of open covers of X which satisfies the following conditions:

(1) $[\mathcal{U}_{m+1}]^2$ refines $\mathcal{U}_m$ for each $m \in N$.

(2) $\{\mathrm{St}(x, \mathcal{U}_m)\colon m \in N\}$ is a neighborhood base at $x \in X$.

(3) For each $x \in F_n$ and each $m \in N$, $\mathrm{St}^2(x, \mathcal{U}_{m+1}^*)$ meets at most $n(n+1)/2$ members of $\mathcal{U}_m$.

For each $q = 2^{-m_1}+\ldots+2^{-m_t} \in Q^*$ and each $U \in \mathcal{U}_{m_1}$, we put

$S(U; m_1) = U$,

$S(U; m_1, \ldots, m_k) = \mathrm{St}^2(S(U; m_1, \ldots, m_{k-1}), \mathcal{U}_{m_k})$ for $2 \leqslant k \leqslant t$,

$S(U; q) = S(U; m_1, \ldots, m_t)$

and

$$\mathcal{S}(q) = \{S(U; q)\colon U \in \mathcal{U}_{m_1}\}.$$

We define a function $\varrho\colon X \times X \to [0, 1]$ as follows:
For $x, y \in X$,

$$\varrho(x, y) = \begin{cases} 1, & \text{if } y \notin \mathrm{St}(x, \mathcal{S}(q)) \text{ for every } q \in Q^*, \\ \inf\{q \in Q^*\colon y \in \mathrm{St}(x, \mathcal{S}(q))\}, & \text{otherwise}. \end{cases}$$

It follows that ϱ is an admissible metric for X (see the proof of [3; Theorem]). To prove that ϱ satisfies the condition $(0)_\infty$, for each point x of X, let n be the minimal number with $x \in F_n$ and $n(x) = n(n+1)/2-1$. Let $\varepsilon > 0$ and $y_1, \ldots, y_{n(x)+2} \in X$ with $\varrho(S_{\varepsilon/2}(x), y_i) < \varepsilon$ for each $i = 1, \ldots, n(x)+2$. For each $i = 1, \ldots, n(x)+2$, let x_i be a point of X such that $\varrho(x, x_i) < \varepsilon/2$ and $\varrho(x_i, y_i) < \varepsilon$. Put

$$\delta_i = \max\{2\varrho(x, x_i), \varrho(x_i, y_i)\}$$

and $\delta = \max\{\delta_i\colon i = 1, \ldots, n(x)+2\}$. Then there is a $q = 2^{-m_1}+\ldots+2^{-m_t} \in Q^*$ such that $\delta < q < \varepsilon$. Since $\varrho(x, x_i) < 2^{-(m_1+1)}+\ldots+2^{-(m_t+1)}$, there is a $V_i \in \mathcal{U}_{m_1+1}$ such that $x, x_i \in S(V_i;\ m_1+1, \ldots, m_t+1)$. Hence $x_i \in \mathrm{St}(x, \mathcal{U}_{m_1+1}^*)$. On the other hand, there is a $U_i \in \mathcal{U}_{m_1}$ such that $x_i, y_i \in S(U_i; m_1, \ldots, m_t)$. Therefore $\mathrm{St}(x_i, \mathcal{U}_{m_1+1}^*) \cap U_i \neq \emptyset$ and hence $\mathrm{St}^2(x, \mathcal{U}_{m_1+1}^*) \cap U_i \neq \emptyset$. By (3), there are distinct natural numbers i and j such that $U_i = U_j$. Then $y_i, y_j \in S(U_i; m_1, \ldots, m_t) = S(U_j; m_1, \ldots, m_t)$. Therefore, $\varrho(y_i, y_j) \leqslant q < \varepsilon$. This completes the proof.

Now, we extend Theorems A and B to another class of infinite dimensional spaces. We notice that for each ordinal number α, we can put $\alpha = \lambda(\alpha)+n(\alpha)$, where $\lambda(\alpha)$ is a limit ordinal number or 0 and $n(\alpha)$ is a non-negative integer.

1.2. DEFINITION ([2]). For a normal space X and a nonnegative integer n, we put $P_n(X) = \bigcup\{U\colon U$ is an open subspace of X such that $\mathrm{Ind}\,\overline{U} \leqslant n\}$. Let X be a normal space and α either an ordinal number $\geqslant 0$ or the integer -1. Then *strong small transfinite dimension sind of X* is defined as follows:

(si1) $\mathrm{sind}\,X = -1$ if and only if $X = \emptyset$.

(si2) $\mathrm{sind}\,X \leqslant \alpha$ if X is expressed in the form $X = \bigcup\{P_\xi\colon \xi < \alpha\}$, where $P_\xi = P_{n(\xi)}(X - \bigcup\{P_\eta\colon \eta < \lambda(\xi)\})$.

Furthermore, if $\mathrm{sind}\,X$ is defined, we say that *X has strong small transfinite dimension*.

Spaces that have strong small transfinite dimension are studied by P. Borst [2] and the author [6]. It should be noticed that L. Polkowski [17] introduced and discussed the similar class of spaces which are called small spaces. Recall from [16] that a normal space X *satisfies the condition* (K) if there is a compact subset K of X such that $\operatorname{Ind} F < \infty$ for every closed subset F of X which does not meet K.

1.3. LEMMA ([7; Propositions 2.2 and 2.3]). *For a metrizable space X, the following conditions are equivalent*:

(a) *X has both large transfinite dimension* Ind *and strong small transfinite dimension* sind.

(b) *X is a strongly countable-dimensional space satisfying the condition* (K).

(c) *X is a strongly countable-dimensional space which contains no discrete family $\{U_n: n \in N\}$ of open sets of X such that $\dim \overline{U}_n > n$ for each $n \in N$.*

1.4. THEOREM. *For a metrizable space X, the following conditions are equivalent*:

(a) *X has both large transfinite dimension* Ind *and strong small transfinite dimension* sind.

(b) *X admits a metric ϱ which satisfies the following condition*:

$(0)^*_\infty$ *For every closed discrete subspace F of X there is a natural number $n(F)$ such that for every $\varepsilon > 0$, every point x of F every $n(F)+2$ many points $y_1, \ldots, y_{n(F)+2}$ of X with $\varrho(S_{\varepsilon/2}(x), y_i) < \varepsilon$ for each $i = 1, \ldots, n(F)+2$, there are distinct natural numbers i and j such that $\varrho(y_i, y_j) < \varepsilon$.*

(c) *X admits a metric ϱ which satisfies the following condition*;

$(1)^*_\infty$ *For every closed discrete subspace F of X, there is a natural number $n(F)$ such that for every point x of F and every $n(F)+2$ many points $y_1, \ldots, y_{n(F)+2}$ of X, there are distinct natural numbers i and j such that $\varrho(y_i, y_j) \leqslant \varrho(x, y_j)$.*

(d) *X admits a metric ϱ which satisfies the following condition*:

$(2)^*_\infty$ *For every closed discrete subspace F of X, there is a natural number $n(F)$ such that for every point x of F and every $n(F)+2$ many points $y_1, \ldots, y_{n(F)+2}$ of X, there are natural numbers i, j and k such that $i \neq j$ and $\varrho(y_i, y_j) \leqslant \varrho(x, y_k)$.*

Proof. The implication (c)$\to$(d) is obvious. To prove the implication (d)$\to$(a), let ϱ be an admissible metric for X which satisfies the condition $(2)^*_\infty$. Since ϱ satisfies the condition $(2)_\infty$, by Theorem C, X is strongly countable-dimensional. Suppose that X contains a discrete family $\{U_n: n \in N\}$ of open sets of X such that $\dim \overline{U}_n \geqslant n$ for each $n \in N$. By Theorem A, for each $n \in N$, there are points $x^n, y^n_1, \ldots, y^n_{n+1}$ of $\overline{U}_n$ such that $\varrho(y^n_i, y^n_j) > \varrho(x^n, y^n_k)$ for each i, j and k with $1 \leqslant i, j, k \leqslant n+1$ and $i \neq j$. Put $F = \{x^n: n \in N\}$. Then F is a closed discrete subspace of X and ϱ does not satisfy the condition $(2)^*_\infty$ for F. Hence, by Lemma 1.3, X has both large transfinite dimension Ind and strong small transfinite dimension sind. In a similar fashion, by use of Theorem 1.1 instead of Theorem A, we can prove the implication (b)$\to$(a). Now, we prove the implication (a)$\to$(c). The basic idea of the proof is due to J. Nagata [11]. Let X be a metrizable space having both large transfinite dimension Ind and strong small transfinite dimension sind. By Lemma 1.3, there is a strongly countable-dimensional compact subspace K of X such that $\dim H < \infty$

for every closed subspace H of X which does not meet K. Let $K = \bigcup_{n=1}^{\infty} K_n$, where each K_n be a closed set of K such that $\dim K_n \leqslant n-1$ and $K_n \subset K_{n+1}$ for each $n \in N$. For each $n \in N$, we put $V_n = X - \overline{S_{1/n}(K)}$. Then V_n, $n \in N$, are open sets of X such that $X-K = \bigcup \{V_n: n \in N\}$, $\overline{V}_n \subset V_{n+1}$ and $\dim V_n < \infty$ for each $n \in N$. Furthermore, for every closed set H of X which does not meet K, there is an $n \in N$ such that $H \subset V_n$. We can assume that $\dim \overline{V}_n \leqslant n-1$ for each $n \in N$. Now, we need the following two lemmas.

1.5. LEMMA. *For every open cover $\mathcal{U}$ of X, there are a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$ of discrete families of open sets of X and an open cover $\mathcal{W}$ of X which satisfies the following conditions*:

(i) *$\bigcup \{\mathcal{V}_m: m \in N\}$ is a cover of X.*

(ii) *$\bigcup \{\mathcal{V}_m: m \in N\}$ refines $\mathcal{U}$.*

(iii) *W meets at most one member of $\mathcal{V}_k$ for $k \leqslant n^2$ and meets no member of $\mathcal{V}_k$ for $k > n^2$ for every $W \in \mathcal{W}$ with $W \cap K_n \neq \emptyset$.*

(iv) *W meets at most one member of $\mathcal{V}_k$ for $k \leqslant n(n+1)$ and meets no member of $\mathcal{V}_k$ for $k > n(n+1)$ for every $W \in \mathcal{W}$ with $W \cap \overline{V}_n \neq \emptyset$.*

Proof. Since $\dim K_n \leqslant n-1$ and $\dim V_n \leqslant n-1$, for each $n \in N$, there are families $\mathcal{G}^n_j$, $\mathcal{H}^n_j$, $j = 1, \ldots, n$, of open sets of X satisfying the following conditions:

(4) $\mathcal{G}^n_j$ and $\mathcal{H}^n_j$ are discrete in X for every $j = 1, \ldots, n$.

$$(5)\quad K_n \subset \bigcup_{j=1}^{n} \mathcal{G}^{n\#}_j \subset X - V_{n+1}.$$

$$(6)\quad \overline{V}_n \subset \bigcup_{j=1}^{n} \mathcal{H}^{n\#}_j \subset V_{n+1}.$$

$$(7)\quad \bigcup_{j=1}^{n} \mathcal{G}^n_j \text{ and } \bigcup_{j=1}^{n} \mathcal{H}^n_j \text{ refine } \mathcal{U}.$$

Let P_n and Q_n be open sets of X such that

$$(8)\quad K_n \subset P_n \subset \overline{P}_n \subset \bigcup_{j=1}^{n} \mathcal{G}^{n\#}_j, \text{ and}$$

$$(9)\quad \overline{V}_n \subset Q_n \subset \overline{Q}_n \subset \bigcup_{j=1}^{n} \mathcal{H}^{n\#}_j.$$

Put

$$(10)\quad \mathcal{V}_m = \begin{cases} \{G - \bigcup_{i=1}^{n-1} \overline{P}_i : G \in \mathcal{G}^n_j\}, & \text{if } m = n(n-1)+j,\ j \leqslant n, \\ \{H - (\bigcup_{i=1}^{n} \overline{P}_i \cup \bigcup_{i=1}^{n-1} \overline{Q}_i) : H \in \mathcal{H}^n_j\}, & \text{if } m = n^2+j,\ j \leqslant n. \end{cases}$$

Then $\mathcal{V}_m$ is a discrete family of open sets of X and refines $\mathcal{U}$. It is easy to see that $\bigcup \{\mathcal{V}_m: m \in N\}$ is a cover of X. For each point x of X, we put

$$n(x) = \begin{cases} \min\{n: n \in N \text{ and } x \in K_n\}, & \text{if } x \in K, \\ \min\{n: n \in N \text{ and } x \in \overline{V}_n\}, & \text{if } x \notin K. \end{cases}$$

There is an open neighborhood $W(x)$ of x such that

(11) if $x \in K$, then $W(x) \subset P_{n(x)} - K_{n(x)-1}$ and $W(x)$ meets at most one member of $\mathscr{V}_m$ for every $m \leqslant n(x)^2$ and if $x \in X-K$, then $W(x) \subset Q_{n(x)} - V_{n(x)-1}$ and $W(x)$ meets at most one member of $\mathscr{V}_m$ for every $m \leqslant n(x)^2 + n(x)$.

We put $\mathscr{W} = \{W(x): x \in X\}$. To prove the condition (iii), let $W(x) \in \mathscr{W}$ and $W(x) \cap K_n \neq \emptyset$. Then, by (6), (9) and (11), it follows that $x \in K$. Let us also notice that $n(x) < n$. Let $k > n(x)^2$. We consider the two cases:

Case 1; $k = m(m-1)+j$ for some $m, j \in N$ with $j \leqslant m$. Then it follows that $m > n(x)$. Hence, by (10), $\mathscr{V}_k^{\#} \cap \bar{P}_{n(x)} \subset \mathscr{V}_k^{\#} \cap \bigcup_{i=1}^{m-1} \bar{P}_i = \emptyset$.

Case 2; $k = m^2+j$ for some $m, j \in N$ with $j \leqslant m$. Then, it follows that $m > n(x)$. Hence $\mathscr{V}_k^{\#} \cap \bar{P}_{n(x)} \subset \mathscr{V}_k^{\#} \cap \bigcup_{i=1}^{m} \bar{P}_i = \emptyset$. Therefore, by (11), $\mathscr{V}_k^{\#} \cap W(x) \subset \mathscr{V}_k^{\#} \cap \cap P_{n(x)} = \emptyset$ in either case. If $k \leqslant n(x)^2$, then it is obvious that $W(x)$ meets at most one member of $\mathscr{V}_k$. Hence the condition (iii) is satisfied. In a similar fashion, we can prove the condition (iv) and hence the lemma is proved.

1.6. LEMMA. *For every $q \in Q^*$, there is an open cover $\mathscr{S}(q)$ which satisfies the following conditions*:

(i) $\mathscr{S}(q) = \bigcup_{i=1}^{\infty} \mathscr{S}^i(q)$, *where each $\mathscr{S}^i(q)$ is discrete in X.*

(ii) $\{\mathrm{St}(x, \mathscr{S}(q)): q \in Q^*\}$ *is a neighborhood base at $x \in X$.*

(iii) *Let $p, q \in Q^*$ and $p < q$. Then $\mathscr{S}(p)$ refines $\mathscr{S}(q)$.*

(iv) *Let $p, q \in Q^*$ and $p < q$. If $S_1 \in \mathscr{S}(p)$ and $S_2 \in \mathscr{S}(q)$, then $S_1 \cap S_2 = \emptyset$ or $S_1 \subset S_2$.*

(v) *Let $p, q \in Q^*$ and $p+q < 1$. Let $S_1 \in \mathscr{S}(p)$, $S_2 \in \mathscr{S}(q)$ and $S_1 \cap S_2 \neq \emptyset$. Then there is an $S_3 \in \mathscr{S}(p+q)$ such that $S_1 \cup S_2 \subset S_3$.*

(vi) *For every $q \in Q^*$ and every $S \in \bigcup \{\mathscr{S}^i(q): i > n(n+1)\}$, $S \cap K_n = \emptyset$ and $S \cap \bar{V}_n = \emptyset$.*

Proof. By use of Lemma 1.5, we can obtain a sequence of $\mathscr{U}_1, \mathscr{U}_2, \ldots$, open covers of $\bar{X}$ which satisfies the following conditions:

(12) $\mathscr{U}_j = \bigcup_{i=1}^{\infty} \mathscr{U}_j^i$, where each $\mathscr{U}_j^i$ is discrete in X.

(13) $\operatorname{mesh} \mathscr{U}_j = \sup\{\text{diameter } U: U \in \mathscr{U}_j\} < 1/j$ for each $j \in N$.

(14) $[\mathscr{U}_{j+1}]^{22}$ refines $\mathscr{U}_j$ for each $j \in N$.

(15) Let $j < k$ and $U^* \in [\mathscr{U}_k]^{22}$. If $U^* \cap K_n \neq \emptyset$, then U^* meets at most one member of $\mathscr{U}_j^i$ for $i \leqslant n^2$ and U^* meets no member of $\mathscr{U}_j^i$ for $i > n^2$. Furthermore, if $U^* \cap \bar{V}_n \neq \emptyset$, then U^* meets at most one member of $\mathscr{U}_j^i$ for $i \leqslant n(n+1)$ and meets no member of $\mathscr{U}_j^i$ for $i > n(n+1)$.

For $i, j \in N$, let $\mathscr{V}_j^i = \{\mathrm{St}^{22}(U, \mathscr{U}_{j+1}): U \in \mathscr{U}_j^i\}$ and $\mathscr{V}_j = \bigcup_{i=1}^{\infty} \mathscr{V}_j^i$. For a subset A of X and $k \in N \cup \{0\}$, we put

$$T^0(A, i, j) = \mathrm{St}(A, \mathscr{V}_j^i),$$

$$T^k(A, i, j) = \mathrm{St}(T^{k-1}(A, i, j), \mathscr{V}_{j+k}^i) \text{ for } k \geqslant 1, \text{ and}$$

$$T(A, i, j) = \bigcup_{k=0}^{\infty} T^k(A, i, j).$$

Furthermore, for $q = 2^{-m_1} + \ldots + 2^{-m_t} \in Q^*$ and $A \subset X$, we put

$$S(A, i, q) = \begin{cases} T(A, i, m_1+1), & \text{if } t = 1, \\ T(\mathrm{St}(S(A, i, q'), \mathscr{V}_{m_t}), i, m_t), & \text{if } t > 1, \end{cases}$$

where $q' = 2^{-m_1} + \ldots + 2^{-m_{t-1}} = q - 2^{-m_t}$.

For each $i \in N$, we put $\mathscr{S}^i(q) = \{S(U, i, q): U \in \mathscr{U}_{m_1}^i\}$ and $\mathscr{S}(q) = \bigcup_{i=1}^{\infty} \mathscr{S}^i(q)$. The proofs of (i)–(v) are quite similar to the finite dimensional case (see [13; Ch. V, § 3, (c)]). To prove (vi), let $S \in \bigcup \{\mathscr{S}^i(q): i > n(n+1)\}$ and $S = S(U, i, q)$ for some $U \in \mathscr{U}_{m_1}^i$, where $q = 2^{-m_1} + \ldots + 2^{-m_t} \in Q^*$. Assume that $S \cap K_n \neq \emptyset$. There is a finite sequence $\sigma_1, \sigma_2, \ldots, \sigma_r$, $r \leqslant t$, of families of open sets of X such that

$$\sigma_1 = \{U, S_{11}, S_{12}, \ldots\},$$
$$\sigma_2 = \{V_2, S_{20}, S_{21}, \ldots\},$$
$$\ldots$$
$$\sigma_r = \{V_r, S_{r0}, S_{r1}, \ldots\},$$

where $V_j \in \mathscr{V}_{m_j}$ for $j = 2, \ldots, r$, $S_{ju} \in \mathscr{V}_{m_j+u}^i$ or $S_{ju} = \emptyset$ for $j = 1, \ldots, r$ and $u \in N$, each nonempty member of a sequence meets its nonempty successor in the same sequence, $V_{j+1} \cap \sigma_j^{\#} \neq \emptyset$ and $\sigma_r^{\#} \cap K_n \neq \emptyset$. It is not difficult to see that there is a $W \in \mathscr{U}_{m_1+1}$ such that $\mathrm{St}^5(W, \mathscr{U}_{m_1+1}) \cap U \neq \emptyset$ and $\mathrm{St}^5(W, \mathscr{U}_{m_1+1}) \cap K_n \neq \emptyset$. Put $W^* = \mathrm{St}^{22}(W, \mathscr{U}_{m_1+1})$. Then $W^* \in [\mathscr{U}_{m_1+1}]^{22}$, $W^* \cap K_n \neq \emptyset$ and $W^* \cap U \neq \emptyset$. Since $i > n(n+1) > n^2$, by (15), $W^* \cap \mathscr{U}_{m_1}^{i\#} = \emptyset$ and hence $W^* \cap U = \emptyset$. This is a contradiction. In a similar fashion, we can prove that $S \cap \bar{V}_n = \emptyset$. This completes the proof of the lemma.

We continue the proof of the implication (a) → (c) of Theorem 1.4. Let $\mathscr{S}(q)$, $q \in Q^*$, be the open covers of X described in Lemma 1.6. We define a function $\varrho: X \times X \to [0, 1]$ as follows:

For $x, y \in X$,

$$\varrho(x, y) = \begin{cases} 1, & \text{if } y \notin \mathrm{St}(x, \mathscr{S}(q)) \text{ for every } q \in Q^*, \\ \inf\{q: q \in Q^* \text{ and } y \in \mathrm{St}(x, \mathscr{S}(q))\}, & \text{otherwise}. \end{cases}$$

It follows that ϱ is an admissible metric for X (cf., the proof of [13; Ch. V. § 3, (D)]). To prove that ϱ satisfies the condition $(1)_\infty^*$, let F be a closed discrete subset of X. Since K is compact, there is an $n_1 \in N$ such that $F \cap K \subset K_{n_1}$. On the other hand, there is an $n_2 \in N$ such that $F - K \subset V_{n_2}$. Put $n_0 = \max\{n_1, n_2\}$ and $n(F) = n_0(n_0+1) - 1$. Let x be a point of F and $y_1, \ldots, y_{n(F)+2}$ be points of X. We can assume that $\varrho(x, y_j) < 1$ for each $j = 1, \ldots, n(F)+2$. Let $\varepsilon > 0$. For each $j = 1, \ldots, n(F)+2$, there is a $q(j) \in Q^*$ such that $\varrho(x, y_j) < q(j) < \varrho(x, y_j) + \varepsilon$. Then, by the definition of ϱ, there is an $S_j \in \mathscr{S}(q(j))$ such that $x, y_j \in S_j$. Let $S_j \in \mathscr{S}^{i(j)}(q(j))$. Then, by (vi) of Lemma 1.6, there are distinct natural numbers j and k such that $i(j) = i(k)$. Assume that $q(j) \leqslant q(k)$. By (iv) of Lemma 1.6, we obtain $S_j \subset S_k$. Hence $y_j, y_k \in S_k$ and hence $\varrho(y_j, y_k) \leqslant q(k) < \varrho(x, y_k)+\varepsilon$. Therefore, there are distinct natural numbers j and k such that $\varrho(y_j, y_k) < \varrho(x, y_k)+1/n$ for each $n \in N$ and hence $\varrho(y_j, y_k) \leqslant \varrho(x, y_k)$. This completes the proof of the implication (a)$\to$(c) of Theorem 1.4. Finally, we prove the implication (a) $\to$ (b) of Theorem 1.4. We use the above notations. By Lemma 1.5, there is a sequence $\mathscr{U}_1, \mathscr{U}_2, \ldots$, of open covers of X which satisfies the conditions (1) and (2) of the proof of Theorem 1.1 and the following condition:

(16) For each $m \in N$, $\mathrm{St}^2(x, \mathscr{U}^*_{m+1})$ meets at most n^2 members of $\mathscr{U}_m$ if $x \in K_n$ and $\mathrm{St}^2(x, \mathscr{U}^*_{m+1})$ meets at most $n(n+1)$ members of $\mathscr{U}_m$ if $x \in \overline{V}_n$.

Then, by the similar arguments to the proof of Theorem 1.1, we can obtain the desired metric ϱ. This completes the proof of Theorem 1.4.

2. Further considerations of $(0)_n$, $(1)_n$ and $(2)_n$ and Nagata's questions. In [11], J. Nagata also considered the following conditions which are the modifications of $(1)_n$ and $(2)_n$:

$(1)_\omega$ For every point x of X and every sequence $y_1, y_2, \ldots$, of points of X, there are distinct natural numbers i and j such that $\varrho(y_i, y_j) \leqslant \varrho(x, y_j)$.

$(2)_\omega$ For every point x of X and every sequence $y_1, y_2, \ldots$, of points of X, there are natural numbers i, j and k such that $i \neq j$ and $\varrho(y_i, y_j) \leqslant \varrho(x, y_k)$.

He proved that for every metrizable space X, there is an admissible metric ϱ for X which satisfies the condition $(2)_\omega$ ([11]; Theorem 4) and raised the following question.

2.1. QUESTION ([11; Question 2]). *Is it possible to introduce a metric ϱ satisfying the condition $(1)_\omega$ into every metrizable space X?*

We partially answer this question as follows:

2.2. THEOREM. *Every metrizable space X admits a metric ϱ satisfying the following condition:*

(*) *For every point x of X and every sequence $y_1, y_2, \ldots$, of points of X such that $\{i\colon i \in N$ and $\varrho(x, y_i) \geqslant \delta\}$ is infinite for some $\delta > 0$, there are distinct natural numbers i and j such that $\varrho(y_i, y_j) \leqslant \varrho(x, y_j)$.*

Proof. Our metric ϱ is the same metric as J. Nagata defined in the proof of [11; Theorem 4]. We outline the construction of ϱ for the convenience of the reader. Let $\mathscr{U}_1, \mathscr{U}_2, \ldots$, be a sequence of open covers of X such that

(1) $[\mathscr{U}_{n+1}]^2$ refines $\mathscr{U}_n$ for each $n \in N$,

(2) $\{\mathrm{St}(x, \mathscr{U}_n)\colon n \in N\}$ is a neighborhood base at $x \in X$, and

(3) $\mathrm{St}^2(x, \mathscr{U}^*_{n+1})$ meets at most finitely many members of $\mathscr{U}_n$ for each $x \in X$ and each $n \in N$.

For each $q = 2^{-m_1}+\ldots+2^{-m_t} \in Q^*$ and $U \in \mathscr{U}_{m_1}$, we put

$$S(U; m_1) = U,$$
$$S(U; m_1, \ldots, m_k) = \mathrm{St}^2(S(U; m_1, \ldots, m_{k-1}), \mathscr{U}_{m_k}) \quad \text{for } 2 \leqslant k \leqslant t,$$
$$S(U; q) = S(U; m_1, \ldots, m_t)$$

and

$$\mathscr{S}(q) = \{S(U; q)\colon U \in \mathscr{U}_{m_1}\}.$$

Define a function $\varrho\colon X\times X \to [0, 1]$ as follows:
For $x, y \in X$,

$$\varrho(x, y) = \begin{cases} 1, & \text{if } y \notin \mathrm{St}(x, \mathscr{S}(q)) \text{ for every } q \in Q^*, \\ \inf\{q\colon q \in Q^* \text{ and } y \in \mathrm{St}(x, \mathscr{S}(q))\}, & \text{otherwise}. \end{cases}$$

It follows that ϱ is an admissible metric for X (cf., the proof of [3; Theorem]). It suffices to show that ϱ satisfies the condition (*). Let

$$M = \{i\colon i \in N \text{ and } \varrho(x, y_i) \geqslant \delta\}$$

and $i \in M$. We can suppose that $0 < \varrho(x, y_i) < 1$ for each $i \in M$. Let

$$\varrho(x, y_i) = \begin{cases} 2^{-m_1(i)}+\ldots+2^{-m_{t(i)}(i)}, & \text{if } \varrho(x, y_i) \in Q^*, \\ 2^{-m_1(i)}+2^{-m_2(i)}+\ldots, & \text{otherwise}, \end{cases}$$

where $m_s(i)$, $s \in N$, are natural numbers satisfying $1 \leqslant m_1(i) < m_2(i) < \ldots$

Since M is infinite, there is an $m \in N$ such that $M_1 = \{i\colon i \in M$ and $m_1(i) = m\}$ is infinite. For each $i \in M_1$ and $n \in N$, there are a $q(i, n) \in Q^*$ and a $U(i, n) \in \mathscr{U}_m$ such that

(4) $\varrho(x, y_i) < q(i, n) < \varrho(x, y_i)+1/n$ and $q(i, n+1) \leqslant q(i, n)$,

(5) $q(i, n) = q(j, n)$ if $\varrho(x, y_i) = \varrho(x, y_j)$ and $q(i, n) < q(j, n)$ if $\varrho(x, y_i) < \varrho(x, y_j)$.

(6) $S(U(i, n); q(i, n))$ contains both x and y_i for each $n \in N$.

It is easy to see that $S(U; q) \subset \mathrm{St}(U, \mathscr{U}^*_{m_1+1})$ for every $q = 2^{-m_1}+\ldots+2^{-m_t} \in Q^*$ and $U \in \mathscr{U}_{m_1}$. Hence, by (6), $x \in \mathrm{St}(U(i, n), \mathscr{U}^*_{m+1})$ and hence $\mathrm{St}(x, \mathscr{U}^*_{m+1}) \cap U(i, n) \neq \emptyset$. Therefore, by (3), there is a $U(i) \in \mathscr{U}_m$ such that $\{n\colon n \in N$ and $U(i) = U(i, n)\}$ is infinite. It should be noticed that $S(U(i); q(i, n))$ contains both x and y_i for all $n \in N$. Since $\mathrm{St}(x, \mathscr{U}^*_{m+1}) \cap U(i) \neq \emptyset$ and M_1 is infinite, by (3), there is a $U \in \mathscr{U}_m$ such that $\{i\colon i \in M_1$ and $U = U(i)\}$ is infinite. Hence there are distinct natural numbers i and j such that $U(i) = U(j) = U$. Assume that $\varrho(x, y_i) \leqslant \varrho(x, y_j)$. It is sufficient to consider only the following two cases; *case* (1); both $\varrho(x, y_i)$ and $\varrho(x, y_j)$ are the members of Q^* and *case* (2); neither $\varrho(x, y_i)$ nor $\varrho(x, y_j)$ is a member of Q^*. First, we consider case (1). If $\varrho(x, y_i) = \varrho(x, y_j)$, then, by (5), $y_i, y_j \in S(U; q(i, n)) = S(U; q(j, n))$ for all $n \in N$. Therefore, $\varrho(y_i, y_j) \leqslant q(j, n) < \varrho(x, y_j)+1/n$ for all $n \in N$. Thus $\varrho(y_i, y_j) \leqslant \varrho(x, y_j)$. Let $\varrho(x, y_i) < \varrho(x, y_j)$ and $\varepsilon = \varrho(x, y_j)-\varrho(x, y_i) > 0$. For each $n \in N$ with $1/n < \varepsilon$, $q(i, n) < \varrho(x, y_i)+1/n$

$<\varrho(x, y_j) < q(j, n)$. Thus $y_i \in S(U;\ q(i, n)) \subset S(U;\ q(j, n))$ and hence $\varrho(y_i, y_j) \leqslant q(j, n) < \varrho(x, y_j) + 1/n$ for every $n \in N$ with $1/n < \varepsilon$. Therefore, it follows that $\varrho(y_i, y_j) \leqslant \varrho(x, y_j)$. In a similar fashion, it can be seen that $\varrho(y_i, y_j) \leqslant \varrho(x, y_j)$ holds for the case (2). Hence, the theorem is proved.

2.3. Remark. The implication $(1)_\omega \to (*) \to (2)_\omega$ obviously holds. Therefore, our result is stronger than [11; Theorem 4].

Now, we consider the following condition which is a modification of $(0)_n$.

$(0)_\omega$ For every $\varepsilon > 0$, every point x of X and every sequence $y_1, y_2, \ldots$, of points of X with $\varrho(S_{\varepsilon/2}(x), y_i) < \varepsilon$ for each $i \in N$, there are distinct natural numbers i and j such that $\varrho(y_i, y_j) < \varepsilon$.

We can also show that the condition $(0)_\omega$ does not induce any topological property.

2.4. THEOREM. *Every metrizable space X admits a metric ϱ satisfying the condition $(0)_\omega$.*

Proof. The proof is similar to that of Theorem 1.1. Hence we describe a sketch of the proof. Let $\mathscr{U}_1, \mathscr{U}_2, \ldots$, be a sequence of open covers of X satisfying the condition (1) and (2) described in the proof of Theorem 1.1 and

(7) $\mathrm{St}^2(x, \mathscr{U}^*_{m+1})$ meets at most finitely many members of $\mathscr{U}_m$ for each $m \in N$ and $x \in X$.

Define a function ϱ: $X \times X \to [0, 1]$ as the same way in the proof of Theorem 1.1, but this time we use the above sequence $\mathscr{U}_1, \mathscr{U}_2, \ldots$, of open covers of X. Then we can obtain the desired metric.

In [12], J. Nagata asked whether it is possible to introduce a metric into a given metrizable space X such that for each $n \in N$, $\mathscr{B}_n = \{S_{1/n}(x)\colon x \in X_n\}$ is locally finite in X for some $X_n \subset X$ and $\bigcup \{\mathscr{B}_n\colon n \in N\}$ is a base for X. As a corollary to Theorem 2.4, we can answer this question affirmatively.

2.5. COROLLARY. *Every metrizable space X admits a metric ϱ such that for each $\varepsilon > 0$, there is a subset X_ε of X such that $\mathscr{B}_\varepsilon = \{S_\varepsilon(x)\colon x \in X_\varepsilon\}$ is a locally finite open cover of X. Therefore, $\bigcup \{\mathscr{B}_{1/n}\colon n \in N\}$ is a base for X.*

Proof. Let ϱ be an admissible metric for X which satisfies the condition $(0)_\omega$ and $\varepsilon > 0$. Let X_ε be the maximal subset of X such that $\varrho(x, y) \geqslant \varepsilon$ whenever $x, y \in X_\varepsilon$ with $x \neq y$. By the maximality of X_ε, it follows that $\mathscr{B}_\varepsilon = \{S_\varepsilon(x)\colon x \in X_\varepsilon\}$ is a cover of X. On the other hand, by the condition $(0)_\omega$, it follows that $S_{\varepsilon/2}(y)$ meets at most finitely many members of $\mathscr{B}_\varepsilon$ for each point y of X. Hence $\mathscr{B}_\varepsilon$ is locally finite in X. This completes the proof.

3. A characterization of a strongly metrizable space. In this section, we characterize a strongly metrizable space by a special metric. We begin with the definition.

3.1. DEFINITION. A collection $\mathscr{U}$ of a set X is called *star-finite* if U meets at most finitely many members of $\mathscr{U}$ for each $U \in \mathscr{U}$. A regular space X is said to be *strongly metrizable* if X has a base which is the countable union of star-finite open covers of X.

It is well known that every separable metrizable space is strongly metrizable and the equality $\operatorname{ind} X = \operatorname{Ind} X$ holds for every strongly metrizable space X. Further properties of a strongly metrizable space appear in [15].

3.2. LEMMA. *Let X be a strongly metrizable space. Then there is a sequence $\mathscr{U}_1, \mathscr{U}_2, \ldots$, of star-finite open covers of X such that $[\mathscr{U}_{n+1}]^2$ refines $\mathscr{U}_n$ for each $n \in N$ and $\{\mathrm{St}(x, \mathscr{U}_n)\colon n \in N\}$ is a neighborhood base at $x \in X$.*

Proof. We can assume that X is a subspace of $B(\tau) \times I^\omega$ for some infinite cardinal τ, where $B(\tau)$ is a Baire space of weight τ and I^ω is the Hilbert cube (see [15; Proposition 2.3.27]). Since $B(\tau) \times I^\omega$ is strongly paracompact, there is a sequence $\mathscr{V}_1, \mathscr{V}_2, \ldots$, of star-finite open covers of $B(\tau) \times I^\omega$ such that $[\mathscr{V}_{n+1}]^2$ refines $\mathscr{V}_n$ for each $n \in N$ and $\{\mathrm{St}(y, \mathscr{V}_n)\colon n \in N\}$ is a neighborhood base at $y \in B(\tau) \times I^\omega$. Put $\mathscr{U}_n = \{V \cap X\colon V \in \mathscr{V}_n\}$ for each $n \in N$. It is easy to see that the sequence $\mathscr{U}_1, \mathscr{U}_2, \ldots$, is the desired one.

3.3. THEOREM. *A metrizable space X is strongly metrizable if and only if X admits a metric ϱ satisfying the following condition:*

($\#$) *For every $\varepsilon > 0$, every point x of X and every sequence $y_1, y_2, \ldots$, of points of X with $\varrho(S_\varepsilon(x), y_i) < \varepsilon$ for each $i \in N$, there are distinct natural numbers i and j such that $\varrho(y_i, y_j) < \varepsilon$.*

Proof. Let ϱ be an admissible metric for X which satisfies the condition ($\#$). For each $\varepsilon > 0$, let X_ε be the maximal subset of X such that $\varrho(x, y) > \varepsilon$ whenever $x, y \in X$ with $x \neq y$. By the maximality of X_ε, it follows that $\mathscr{B}_\varepsilon = \{S_\varepsilon(x)\colon x \in X_\varepsilon\}$ is a cover of X. On the other hand, by the condition ($\#$), we can see that $\mathscr{B}_\varepsilon$ is star-finite. Hence $\mathscr{B} = \bigcup \{\mathscr{B}_{1/n}\colon n \in N\}$ is a base for X which is the union of countably many star-finite open covers of X and hence X is strongly metrizable. Conversely, let X be a strongly metrizable space and $\mathscr{U}_1, \mathscr{U}_2, \ldots$, a sequence of star-finite open covers of X constructed in Lemma 3.2. It is easy to see that

(1) $\mathrm{St}^2(x, \mathscr{U}^*_{n+1})$ meets at most finitely many members of $\mathscr{U}_n$ for each $x \in X$ and each $n \in N$.

For each $q = 2^{-m_1} + \ldots + 2^{-m_t} \in Q^*$ and each $U \in \mathscr{U}_{m_1}$, we put $S(U;\ m_1) = U$,

$$S(U;\ m_1, \ldots, m_k) = \mathrm{St}^2(S(U;\ m_1, \ldots, m_{k-1}), \mathscr{U}_{m_k}) \quad \text{for } 2 \leqslant k \leqslant t,$$

$$S(U;\ q) = S(U;\ m_1, \ldots, m_t), \quad \text{and} \quad \mathscr{S}(q) = \{S(U;\ q)\colon U \in \mathscr{U}_{m_1}\}.$$

Let us notice the following:

(2) Let $p = 2^{-m_1} + \ldots + 2^{-m_t} \in Q^*$, $q = 2^{-n_1} + \ldots + 2^{-n_t} \in Q^*$ and $p < q$. Then $\mathscr{S}(p)$ refines $\mathscr{S}(q)$. Furthermore, if $m_1 = n_1$, then $S(U;\ p) \subset S(U;\ q)$ for each $U \in \mathscr{U}_{m_1}$.

Define a function ϱ: $X \times X \to [0, 1]$ as follows:
For $x, y \in X$,

$$\varrho(x, y) = \begin{cases} 1, & \text{if } y \notin \mathrm{St}(x, \mathscr{S}(q)) \text{ for every } q \in Q^*, \\ \inf\{q\colon q \in Q^* \text{ and } y \in \mathrm{St}(x, \mathscr{S}(q))\}, & \text{otherwise}. \end{cases}$$

Then, ϱ is an admissible metric for X. To prove that ϱ satisfies the condition (#), let $\varepsilon > 0$, $x, y_1, y_2, \ldots \in X$ with $\varrho(S_\varepsilon(x), y_i) < \varepsilon$ for each $i \in N$. For each $i \in N$, let x_i be a point of X such that $\varrho(x, x_i) < \varepsilon$ and $\varrho(x_i, y_i) < \varepsilon$. Put $\delta_i = \max\{\varrho(x, x_i), \varrho(x_i, y_i)\}$. We can assume that $\varepsilon/4 \leqslant \delta_i < \varepsilon$. For each $i \in N$, let $q_i = 2^{-m_1(i)} + \ldots + 2^{-m_{t(i)}(i)} \in Q^*$ such that $\delta_i < q_i < \varepsilon$. Since $\varrho(x, x_i) < q_i$ and $\varrho(x_i, y_i) < q_i$, there are $V_i \in \mathscr{U}_{m_1(i)}$ and $U_i \in \mathscr{U}_{m_1(i)}$ such that $x, x_1 \in S(V_i;\, q_i)$ and $x_i, y_i \in S(U_i;\, q_i)$. Since $\varepsilon/4 \leqslant q_i$ for each $i \in N$, there is an $m_0 \in N$ such that $M_1 = \{i\colon i \in N \text{ and } m_1(i) = m_0\}$ is infinite. For each $i \in M_1$, $x \in S(V_i;\, q_i) \subset \mathrm{St}(V_i, \mathscr{U}_{m_0})$ and $x_i \in S(V_i;\, q_i) \cap S(U_i;\, q_i) \subset \mathrm{St}(V_i, \mathscr{U}_{m_0}) \cap \mathrm{St}(U_i, \mathscr{U}_{m_0})$. Hence $U_i \cap \mathrm{St}^2(x, \mathscr{U}^*_{m_0}) \neq \emptyset$. By (1), there is a $U_0 \in \mathscr{U}_{m_0}$ such that $M_2 = \{i\colon i \in M_1 \text{ and } U_i = U_0\}$ is infinite. Let i and j be members of M_2 such that $i \neq j$. Suppose that $q_i \leqslant q_j$. It is obvious that $y_j \in S(U_j;\, q_j) \in \mathscr{S}(q_j)$. On the other hand, by (2), $y_i \in S(U_i;\, q_i) \subset S(U_j;\, q_j) \in \mathscr{S}(q_j)$. Hence $y_i \in \mathrm{St}(y_j, \mathscr{S}(q_j))$ and hence $\varrho(y_i, y_j) \leqslant q_j < \varepsilon$. This completes the proof of Theorem 3.3.

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