Strongly retractive Boolean algebras

by

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Dedicated to the memory of V. A. Rokhlin

Abstract. A Boolean algebra is called *retractive* iff every ideal is the kernel of some endomorphism. The author observed that many retractive algebras actually have a slightly stronger property which is defined and studied in this paper. As a byproduct we obtain a new proof of the known result that the free product of two infinite Boolean algebras cannot be embedded into an interval algebra, unless it is countable.

A Boolean algebra (BA for short) is called *retractive* iff for every ideal I there is a subalgebra containing exactly one representative of each congruence class modulo I. To state the same in more advanced terminology: A BA is *retractive* iff every ideal is the kernel of some idempotent endomorphism. For information on retractive BA's the reader may consult [5] (with motivations and some history) or [1], § 6 (a list of results and open problems).

Via Stone duality retractiveness of BA's translates into the following property of topological spaces which will be called *retractiveness* as well: A BA is *retractive* iff for every nonempty closed subset F of its Stone space X there is a retraction $f\colon X \to F$, i.e. a continuous mapping which is the identity on F.

It seems that the first to use this concept was Sierpiński who proved that all subspaces of the irrationals are retractive [7]. A slight modification of his proof works for all zero-dimensional separable metric spaces [4], p. 169. Actually Sierpiński proved slightly more than retractiveness. He constructed retractions $f\colon X \to F$ with the additional property that every point x that is moved by f has a neighbourhood U which f sends entirely to the same point as x, i.e. $f(U) = \{f(x)\}$.

More concisely: $f^{-1}(y)\setminus\{y\}$ is open for each $y \in X$. Below such retractions will be called *strong*.

This paper is devoted to the study of those BA's, called *strongly retractive*, whose Stone spaces admit a strong retraction onto each of their nonempty closed subsets.

First we give a Boolean algebraic translation of this concept. Improving on a result of Rubin's and using his idea we then prove that all subalgebras of interval algebras are strongly retractive. Finally we observe that the free product of two infinite BA's is either countable or not strongly retractive. This gives a new proof of the known fact ([5] or rather [9]) that the free product of infinite BA's cannot be embedded into an interval algebra unless it is countable. Under MA or CH the observation also yields examples of retractive but not strongly retractive BA's.

The Boolean operations will be denoted by $\wedge$, $\vee$, $-$, the constants by $0, 1$. $a \leqslant b$ stands for $a \wedge b = a$, and $a-b$ abbreviates $a \wedge (-b)$. For the set-theoretical concepts we use the symbols $\cap$, $\cup$, $\setminus$, $\subset$ and $\emptyset$. When dealing with an algebra of sets both denotations are possible and will be used alternatively according to what the author feels more suggestive.

1. Dualization. To describe strong retractiveness in Boolean algebraic terms we need the following notion.

DEFINITION. Suppose B is a subalgebra of A. Then $a \in A$ is called *stable with respect to B* iff for all $b \in B$ either $a \leqslant b$ or $a \wedge b = 0$, i.e. a is not split by b.

THEOREM 1. *A BA A is strongly retractive iff for each ideal $I \subset A$ there is a subalgebra $B \subset A$ such that*

(1) $B \cap I = \{0\}$,

(2) *A is generated by $B \cup I$, and*

(3) *I is generated (as an ideal) by its stable w.r.t. B elements.*

Proof. To prove necessity conceive A as consisting of all clopen subsets of its Stone space X.

If $I \subset A$ is an ideal, then $U = \bigcup I$ is an open subset of X. Hence there is a strong retraction $f\colon X \to X\setminus U$. For every $a \in A$ the set $a \setminus U$ is clopen in $X\setminus U$. $f^{-1}(a\setminus U)$ is, therefore, clopen in X and hence an element of A.

Put $B = \{f^{-1}(a\setminus U)\colon a \in A\}$.

The easy verification that B is a subalgebra and $B \cap I = \{0\}$ is left to the reader.

To see that $B \cup I$ generates A consider any $a \in A$. Since f is the identity on $X\setminus U$ it follows that

$$a\setminus f^{-1}(a\setminus U) \subset U \quad \text{and} \quad f^{-1}(a\setminus U)\setminus a \subset U.$$

Consequently both sets are elements of I and

$$a = [f^{-1}(a\setminus U)\setminus (f^{-1}(a\setminus U)\setminus a)] \cup [a\setminus f^{-1}(a\setminus U)]$$

is in the subalgebra generated by B and I.

To say that I is generated by its stable elements is to say that every point of U has a stable neighbourhood. Consider any $x \in U$. Since f is strong, there is some neighbourhood V of x such that $f(V) = \{f(x)\}$. Then, clearly, $V \cap f^{-1}(a\setminus U) = \emptyset$ if $f(x) \notin a\setminus U$, and $V \subset f^{-1}(a\setminus U)$ if $f(x) \in a\setminus U$. So V is stable.

For the proof of the other direction it is more convenient to consider X as consisting of all ultrafilters of A with a base of the topology formed by all sets $\{p \in X\colon a \in p\}$, where a runs through A. Then for every closed subset $F \subset X$ there is some ideal $I \subset A$ such that $F = \{p \in X\colon p \cap I = \emptyset\}$. Using the subalgebra B that exists by assumption, we have to construct a strong retraction $f\colon X \to F$. Put $f(p) = \{(a-i) \vee j\colon a \in p \cap B,\ i, j \in I\}$. The reader will easily verify that $f(p)$ is indeed an ultrafilter of A.

Suppose $(a-i) \vee j \in f(p)$. Then $a \in q$ implies $(a-i) \vee j \in f(q)$. This proves continuity. For $p \in F$, i.e., $p \cap I = \emptyset$, we have to prove $f(p) = p$. Since both are ultrafilters, $p \subset f(p)$ will do. Consider $c \in p$. $B \cup I$ generates A, so $c = (b-i) \vee j$ for some $b \in B$, and $i, j \in I$. We have to prove $b \in p$. From $p \cap I = \emptyset$ we have $j \notin p$. Together with $(b-i) \vee j \in p$ this implies $b-i \in p$. From $b-i \leqslant b$ we then conclude $b \in p$. It remains to show that f is strong.

Suppose $p \cap I \neq \emptyset$. By assumption, I is generated by its stable elements. So, there is some stable $i_0 \in p \cap I$. It will be sufficient to show that $i_0 \in q$ implies $f(q) = f(p)$. This, in turn, is an obvious consequence of the identity

$$f(p) = \{(b-i) \vee j\colon i_0 \leqslant b \in B,\ i, j \in I\}. \text{ If } i_0 \leqslant b \text{ and } i_0 \in p, \text{ then } b \in p.$$

Hence the right-hand side is always contained in $f(p)$ whether i_0 is stable or not.

Suppose, on the other hand, $(b-i) \vee j \in f(p)$, i.e. $b \in p \cap B$ and $i, j \in I$. Then $b \in p$ and $i_0 \in p$ imply $b \wedge i_0 \neq 0$. From the stability of i_0 we have $i_0 \leqslant b$, as desired.

2. Strong retractiveness and interval algebras. Consider any linearly ordered set R. By a left-closed, right-open interval we mean a set of one of the following forms:

$$[\alpha, \beta) = \{\gamma \in R\colon \alpha \leqslant \gamma < \beta\}, \quad [-\infty, \beta) = \{\gamma \in R\colon \gamma < \beta\} \text{ or}$$
$$[\alpha, \infty) = \{\gamma \in R\colon \alpha \leqslant \gamma\}.$$

The finite unions of such intervals together with $\emptyset$ form a BA under the set-theoretic operations. It will be called *interval algebra* on R and denoted by $B(R)$. Every nonzero element $a \in B(R)$ has a unique representation

$$a = [\alpha_1, \beta_1) \cup [\alpha_2, \beta_2) \cup \ldots \cup [\alpha_n, \beta_n),$$

with

$$-\infty \leqslant \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_n < \beta_n \leqslant +\infty,$$

Call it the *canonical from of a*.

The elements $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n$ will be called *endpoints of a*. If necessary, we shall speak of right and left endpoints.

If Q is a subset of R, then $B(Q)$ is canonically isomorphic to the subalgebra of $B(R)$ consisting of those elements whose endpoints are in $Q \cup \{\pm\infty\}$. We identify $B(Q)$ with that subalgebra.

More details on interval algebras can be found in [1], [5], and [6].

In [6] M. Rubin proves that every subalgebra of an interval algebra is retractive (Theorem 5.1). His construction even yields strong retractiveness, as we shall now see.

THEOREM 2. *If a BA is embeddable in an interval algebra, then it is strongly retractive.*

Proof. Suppose $A \subset B(R)$ and consider any ideal $I \subset A$. We have to find a subalgebra $B \subset A$ with the properties indicated in Theorem 1.

Choose a maximal subset Q of R such that $B(Q) \cap I = \{0\}$. Rubin's argument [6], Theorem 5.1, ensures us that $B = B(Q) \cap A$ is a subalgebra of A such that $B \cap I = \{0\}$ and A is generated by $B \cup I$.

To prove that I is generated by its stable elements we first give a description of them.

If $a = [\alpha_1, \beta_1) \cup \ldots \cup [\alpha_n, \beta_n)$ is the canonical form, denote by $||a||$ the set

$$\{\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n\} \setminus (Q \cup \{\pm\infty\}).$$

Call $[\alpha_i, \beta_i)$ an essential part of a, if either α_i or β_i is in $||a||$. Note that every nonzero element of I has an essential part, because $B(Q) \cap I = \{0\}$.

LEMMA 1. *If $a \in I$ is not stable w.r.t. B, then there is some $b \in B$ such that $||a \wedge b||$ is not empty, but strictly contained in $||a||$.*

Note that the endpoints of $a \wedge b$ and $a-b$ are endpoints of either a or b. Since $b \in B(Q)$, we must have $||a \wedge b|| \cup ||a-b|| \subset ||a||$.

Proof of the Lemma. If $a \in I$ is not stable, then there is some $c \in B$ with $a \wedge c \neq 0$ and $a-c \neq 0$. We show that either c or $-c$ does the job. $||a \wedge c|| \neq \emptyset$ and $||a-c|| \neq \emptyset$ since both elements are in I and $B \cap I = \{0\}$.

It remains to see that there is some element in $||a||$ which is either not in $||a \wedge c||$ or not in $||a-c||$. We distinguish two cases.

(1) c does not split any essential part of a.

Suppose $[\alpha, \beta)$ is an essential part of a and $[\alpha, \beta) \subset c$. Then neither α nor β is an endpoint of $a-c$, but one of them is in $||a||$. If $[\alpha, \beta) \cap c = \emptyset$, then neither α nor β is an endpoint of $a \wedge c$.

(2) c does split the essential part $[\alpha, \beta)$ of a.

Then the following implications are obvious (draw pictures and note that no element of $||a||$ can be an endpoint of c) and prove our assertion.

If $\alpha \in ||a||$ and $\alpha \in c$, then $\alpha \notin ||a-c||$.
If $\alpha \in ||a||$ and $\alpha \notin c$, then $\alpha \notin ||a \wedge c||$.
If $\beta \in ||a||$ and $\beta \in c$, then $\beta \notin ||a-c||$.
If $\beta \in ||a||$ and $\beta \notin c$, then $\beta \notin ||a \wedge c||$.

To end the proof of the theorem consider any $a \in I$. Using induction on the cardinality of $||a||$ we show that a is covered by stable elements. $||a|| = \emptyset$ implies $a \in B$. Then $a = 0$ which is stable. The lemma tells us that a is stable if $||a||$ consists of one element.

Suppose $||a||$ has $n > 1$ elements. If a is not itself stable, then the lemma yields some $b \in B$ with $||a \wedge b||$ having at least one, say γ, but less than n elements. By induction hypothesis $a \wedge b$ is covered by stable elements. It remains to show that so is $a-b$. Since b has no endpoints outside $Q \cup \{\pm\infty\}$, γ must be an "inner point" of b. Being an endpoint of a, γ will, therefore, not be an endpoint of $a-b$. Consequently $||a-b||$ has less elements than $||a||$ and the induction hypothesis implies that $a-b$ is covered by stable elements.

4. Strong retractiveness of free products. The question whether the free product of two uncountable BA's can be retractive is still open. As far as strong retractiveness is concerned the situation is quite simple.

THEOREM 3. *If A and B are infinite* BA's *and $A * B$ is strongly retractive, then both algebras must be countable.*

Proof. The proof becomes much clearer if we use the topological language. Let X and Y be the Stone spaces of A and B, respectively. Then $A * B$ has the Stone space $X \times Y$.

Being (strongly) retractive, this space must be hereditarily normal (easy exercise, or [3]). Hence X and Y are hereditarily Lindelöf (Katetov's theorem [2,2.7.15]) and, therefore, first countable. In particular, there is a convergent sequence in Y, i.e. $\omega+1$ embeds into Y. Since closed subspaces of strongly retractive spaces are strongly retractive, we have reduced the theorem to the following special case:

If $X \times (\omega+1)$ is strongly retractive, then X is second countable.

Consider a strong retraction f: $X \times (\omega+1) \to X \times \{\omega\}$. For fixed $x \in X$ and $n \in \omega$ denote by U_{nx} the subset of X for which $U_{nx} \times \{n\} = f^{-1}(x, \omega) \cap X \times \{n\}$. f being strong, each U_{nx} is clopen. For n fixed the sets U_{nx} are pairwise disjoint and cover X. Thus only finitely many of them can be nonempty. Then the collection $\{U_{nx}: n \in \omega, x \in X, U_{nx} \neq \emptyset\}$ is countable. We show that it constitutes a base of X.

Suppose $V \subset X$ is nonempty and clopen. We claim that there is a number N such that $U_{nx} \subset V$ holds for all $n > N$ and $x \in V$.

If this were not the case, we could find an increasing sequence $(n_k)_{k\in\omega}$ of natural numbers and points $x_k \notin V$ such that $f(x_k, n_k) \in V \times \{\omega\}$. By compactness and first countability we can assume that the sequence $(x_k)_{k\in\omega}$ converges to some point y, which cannot be in V. Then, on the one hand,

$$f(y, \omega) = \lim f(x_k, n_k) \in V \times \{\omega\}$$

and, on the other hand,

$$f(y, \omega) = (y, \omega) \notin V \times \{\omega\}.$$

The contradiction proves the claim.

Since $X \setminus V$ is as clopen as V, for sufficiently large n we have $U_{nx} \subset X \setminus V$ for all $x \notin V$. This shows $V = \bigcup \{U_{nx}: x \in V\}$ for all sufficiently large n, and we are done.

COROLLARY. *No free product of an uncountable* BA *and an infinite* BA *can be embedded into an interval algebra.*

This was known before. Rotman [6] derived it from a rather deep topological theorem of Treybig and Ward [9].

5. Examples. To complete our discussion of strong retractiveness we would like to have examples of retractive, but not strongly retractive BA's and examples of strongly retractive BA's that are not embeddable in interval algebras. So far, no such examples are known in ZFC. If, however, we are ready to accept additional axioms, we can get examples from the literature.

Rubin proved [6], 6.7. (a), that $B(S) * B(\omega)$ is retractive for a Suslin line S. Under either MA or CH he also indicated uncountable subsets Q of the real line such that $B(Q) * B(\omega)$ is retractive [6], 6.7. (b, c). By Theorem 3 none of these free products can be strongly retractive.

The second type of examples arises from two constructions due to Rubin (his SC algebra in [6]) and to Shelah [8]. Rubin used $\diamondsuit_{\omega_1}$ and Shelah CH. Both algebras are not embeddable in interval algebras, but have the following two properties:

(1) Every factor algebra by a dense ideal is countable.

(2) Every ideal is countably generated.

Property (1) is sufficient to guarantee retractiveness. Together with (2) it gives strong retractiveness. This is an easy consequence of the following lemma (compare the argument in [6], 4.3. (c)).

LEMMA 2. *If I is countably generated and A/I is countable, then there is a subalgebra B as required in Theorem* 1.

Proof. Take enumeration $(i_n)_{n\in\omega}$ of a set generating I and $(a_n)_{n\in\omega}$ such that $A/I = \{a_n/I\colon n\in\omega\}$. For $n\in\omega$ and $\varepsilon\in\{0,1\}^{n+1}$ put $i_n^\varepsilon = (i_n - \bigvee_{m=0}^{n-1} i_m)\wedge \bigwedge_{m=0}^{n} \varepsilon_m a_m$, where, as usual, $1a = a$ and $0a = -a$. Then the elements i_n^ε generate I and every a_n splits just finitely many of them. Consequently, one can choose $c_n\in a_n/I$ such that c_n does not split any i_n^ε. That means that I has a set of generators that are stable w.r.t. the subalgebra C generated by all c_n. The rest of the proof goes as in the mere retractive case. Inductively one picks $b_n\in C\cap c_n/I$ such that the subalgebra generated by $\{b_0, b_1, \ldots, b_n\}$ has only 0 in common with I. The subalgebra generated by all b_n then is as desired.

I would like to end the paper with a problem. Find an example of a non-retractive BA which has, however, the weaker property that every ideal is the kernel of some endomorphism. Speaking topologically, every closed subset of the Stone space is a continuous image.

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