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# The "local" law of the iterated logarithm for processes related to Lévy's stochastic area process

by

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**Abstract.** For a class of stochastic processes which includes Lévy's stochastic area process as a special case we prove the law of the iterated logarithm at zero. Based on this result we also prove the law for the $l^1$-norm of a finite number of independent copies of area processes.

**1. Introduction.** Let $(W_t) = (X_t, Y_t)$ be a 2-dimensional Brownian motion and let

$$L_t = \int_0^t \langle JW_s, dW_s\rangle, \qquad \text{where} \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbf{R}^2$. The "area process" $(2^{-1} L_t)$ was introduced by Lévy in 1939 (cf. [9], for further references see [7]) but was only sporadically studied until the mid-seventies [5, 8, 10, 12]. As has been noticed by Gaveau [5, see also 1], the process $(L_t)$ or, more precisely, the diffusion process $(W_t, L_t)$, is a useful tool for solving certain problems which naturally occur in analysis and differential geometry. Dugué's recent note [4] shows that the process $(L_t)$ also plays a certain role in statistics, viz. in hypothesis testing; for a special parameter estimation problem of 2-dimensional Gauss–Markov processes it was previously used in [10, Ch. 17.4]. Asymptotic fine properties of its sample paths were investigated in [2] and [6] (cf. also [11]). In these papers a law of the iterated logarithm at infinity is proved for classes of stochastic processes which include $(L_t)$ as a special case. In [2], processes

$$Z_t^{\alpha,\beta} = \alpha \int_0^t X_s\, dY_s + \beta \int_0^t Y_s\, dX_s, \qquad t > 0,$$

are investigated while in [6] processes $L_t^A = \int_0^t \langle AW_s, dW_s\rangle$, where $A$ is a $d \times d$ skew symmetric matrix, $d \geqslant 2$, are dealt with.

In this paper we shall prove the "lil" (law of the iterated logarithm) *at*

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*zero* for the latter class of processes. The method of proof to derive the "lil" at zero differs from the one used in [2] to derive the result at infinity. To be more specific, we derive the "upper class result" from Doob's submartingale inequality combined with the formula for the characteristic function of $(L_t)$, whereas we derive the "lower class result" from a variant of the second Borel–Cantelli lemma and good lower bounds for certain tail probabilities of the conditional increments of $(L_t)$. Moreover, we prove the "lil" for the $l^1$-norm of a finite number of independent copies of processes $(L_t)$.

**2. Auxiliary results.** For the sake of completeness we list some of the basic properties of $(L_t)$ which will be used in the following.

PROPOSITION 2.1. (1) (Scaling property). $(\bar{L}_t) = (cL_{t/c})$, $c > 0$, *is the area process associated with the Wiener process* $(\bar{W}_t) = (\sqrt{c}\, W_{t/c})$.

(2) (Symmetry property). $(-L_t)$ *is an area process.*

(3) *For any* $t \geqslant 0$, $h > 0$, $L_{t+h} - L_t = \tilde{L}_h + \langle JW_t, \tilde{W}_h\rangle$, *where* $(\tilde{L}_s)$ *denotes the area process associated with the Wiener process* $(\tilde{W}_s) = (W_{s+t} - W_t)$, $s \geqslant 0$.

(4) *For any* $c > 0$, $(\exp[cL_t])$ *is a submartingale.*

(5) $E[\exp(i\lambda L_t)] = \mathrm{ch}^{-1}(\lambda t) = (2t)^{-1} \int_{-\infty}^{\infty} e^{i\lambda x} \mathrm{ch}^{-1}(\tfrac{1}{2}\pi x/t)\, dx$, $\lambda \in \boldsymbol{R}$, $t > 0$ (cf. [9]).

(6) $E\ [e^{i\lambda L_t}/W_t = x] = \dfrac{t\lambda}{\mathrm{sh}(t\lambda)} \exp\left[\dfrac{|x|^2}{2t}(1 - t\lambda \coth(t\lambda)\right]$ (cf. e.g. [9]).

(7) $E[\exp(i\lambda[L_t + \langle\gamma, W_t\rangle])] = \mathrm{ch}^{-1}(\lambda t)\exp[-\tfrac{1}{2}\lambda\ \mathrm{th}(\lambda t)|\gamma|^2]$ (combine (5) and (6)).

Since $(L_t)$ does not have independent increments the following variant of the Borel–Cantelli lemma turns out to be useful in the proof of Theorem 3.1. Its proof is, with but a minor modification, a duplicate of the proof of the original version of the Lemma.

LEMMA 2.2. *Let* $(\Omega, \mathfrak{F}, P)$ *be a probability space. Let* $(A_n)$ *be a sequence of events and let* $(\mathfrak{F}_n)$ *be a sequence of sub-$\sigma$-algebras of* $\mathfrak{F}$ *such that*

$$\mathfrak{F}_{n+1} \supset \sigma(A_{n+1}, A_{n+2}, \ldots), \qquad n \in \boldsymbol{N}.$$

*If* $(\alpha_n)$ *is a sequence of positive real numbers such that*

(i) $P[A_n | \mathfrak{F}_{n+1}] \geqslant \alpha_n$ *P-a.s.*,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

*then* $P[\limsup_{n\to\infty} A_n] = 1$.

Let $\mathcal{M}_t = \sigma(W_s, s \leqslant t)$; an immediate consequence of Proposition 2.1 (3), (5) and of the independence of the increments $\tilde{W}_s = W_{t+s} - W_t$, $s, t \geqslant 0$, given $\mathcal{M}_t$, is a formula for the conditional Laplace transform of the increments of $(L_t)$.



PROPOSITION 2.3. *For any* $0 \leqslant r < s$, $T = s - r$, *and for any* $\lambda \geqslant 0$ *such that* $\lambda T < \pi/2$,

$$E[\exp(\lambda[L_s - L_r]|\mathcal{M}_r)] = \cos^{-1}(\lambda T)\exp[\tfrac{1}{2}\lambda\,\mathrm{tg}(\lambda t)|W_r|^2]. \tag{8}$$

To derive a lower bound for tail probabilities of the conditional increments of $(L_t)$ we shall examine the second factor of the product which appears on the right-hand side of (7).

LEMMA 2.4. *Let* $\alpha, T \geqslant 0$ *and* $\varphi(z) = \exp(-\alpha z\ \mathrm{th}(zT))$, $z \in \boldsymbol{R}$. *The transformation* $\varphi$ *is the characteristic function of an (infinitely divisible) symmetric distribution function.*

Proof. Since $\varphi$ is real-valued, symmetry of the distribution function associated with $\varphi$ is clear provided that we have already shown that $\varphi$ is a characteristic function. So we shall prove that for any $\beta \geqslant 0$

$$\Psi_\beta(z) = (1 + z\ \mathrm{th}(\beta z))^{-1}$$

is a characteristic function. Hence,

$$\left(1 + \frac{z\ \mathrm{th}\left(\frac{z}{n} n\right)}{n}\right)^{-1}, \qquad n \in \boldsymbol{N},$$

is a characteristic function too, and the assertion follows from the continuity theorem. Now consider the product representation of $\mathrm{ch}(\beta z)$ and $\mathrm{ch}(\beta z) + z\mathrm{sh}(\beta z)$, respectively. Since these functions are entire functions of order 1, real-valued and symmetric (about zero) for $z \in \boldsymbol{R}$, it follows from Hadamard's factorization theorem (e.g. [3, p. 291]) that $\Psi_\beta$ is given by

$$\Psi_\beta(z) = \prod_{k=1}^{\infty} \frac{\gamma_k^2}{\alpha_k^2} \frac{(\alpha_k^2 + z^2)}{(\gamma_k^2 + z^2)},$$

where $\pm i\alpha_k$ and $\pm i\gamma_k$ denote the purely imaginary zeros of $\mathrm{ch}(\beta z)$ and $\mathrm{ch}(\beta z) + z\ \mathrm{sh}(\beta z)$, respectively. Note that $|\gamma_k| < |\alpha_k|$ for all $k \in \boldsymbol{N}$. Hence,

$$\Psi_\beta(z) = \prod_{k=1}^{\infty}\left[\frac{\gamma_k^2}{\alpha_k^2} + \left(1 - \frac{\gamma_k^2}{\alpha_k^2}\right)\frac{\gamma_k^2}{\gamma_k^2 + z^2}\right]. \tag{9}$$

Each factor in (9) corresponds to a mixture (convex combination with coefficients $\gamma_k^2/\alpha_k^2$ and $1 - \gamma_k^2/\alpha_k^2$) of the Heaviside distribution function (at zero) and a bilateral exponential distribution function. Since $\Psi_\beta$ is continuous and an infinite product of characteristic functions it follows that $\Psi_\beta$ is a characteristic function. ∎

Remark 2.5. Another way of proving the assertion is by obtaining the Kolmogorov representation of $\varphi$, viz.

$$\log \varphi(z) = \int_{-\infty}^{\infty} \frac{\cos(tx)-1}{x^2}\, dK(x),$$

where $dK(x) = \frac{\pi}{2} \frac{x^2 \operatorname{ch}(\frac{1}{2}\pi x)}{\operatorname{sh}^2(\frac{1}{2}\pi x)}\, dx$.

LEMMA 2.6. *For any* $0 \leqslant r < s$, $T = s-r$ *and* $\xi \in \mathbf{R}^+$,

$$P[L_s - L_r \geqslant \xi | \mathcal{M}_r] \geqslant (4T)^{-1} \int_{\xi}^{\infty} \operatorname{ch}^{-1}(\tfrac{1}{2}\pi x/T)\, dx \geqslant (2\pi)^{-1} \exp(\tfrac{1}{2}\pi\xi/T). \tag{10}$$

Proof. By Proposition 2.1 (3), (5),

$$\begin{aligned} P[L_s - L_r \geqslant \xi | \mathcal{M}_r] &= P[\tilde{L}_T + \langle JW_r, \tilde{W}_T\rangle \geqslant \xi | \mathcal{M}_r] \\ &= P[\tilde{L}_T + \langle \gamma, \tilde{W}_T\rangle \geqslant \xi] \quad P\text{-a.s.}, \end{aligned}$$

where $\gamma = JW_r(\cdot)$. By Lemma 2.4 there are two independent random variables $U_1$ and $U_2$ with characteristic function $\operatorname{ch}^{-1}(zT)$ and $\exp[-\frac{1}{2}|\gamma|^2 \times z \operatorname{th}(zT)]$, respectively, such that

$$P[\tilde{L}_T + \langle \gamma, \tilde{W}_T\rangle \geqslant \xi] = P[U_1 + U_2 \geqslant \xi].$$

Since $U_1$ and $U_2$ are independent and the distribution function of $U_2$ is symmetric (about zero) we have

$$\begin{aligned} P[U_1+U_2 \geqslant \xi] &= P[U_1+U_2 \geqslant \xi | U_2 \geqslant 0]\, P[U_2 \geqslant 0] \\ &\quad + P[U_1+U_2 \geqslant \xi | U_2 \leqslant 0]\, P[U_2 \leqslant 0] \\ &\geqslant P[U_1+U_2 \geqslant \xi | U_2 \geqslant 0]\, P[U_2 \geqslant 0] \geqslant \tfrac{1}{2} P[U_1 \geqslant \xi]. \end{aligned}$$

Together with Proposition 2.1 (3) we obtain (10). ∎

## 3. Main result.

THEOREM 3.1.

$$\limsup_{t \searrow 0} \frac{L_t}{(2/\pi)\, t \log\log(1/t)} = 1 \quad P\text{-a.s.}, \tag{11}$$

$$\liminf_{t \searrow 0} \frac{L_t}{(2/\pi)\, t \log\log(1/t)} = -1 \quad P\text{-a.s.} \tag{12}$$

Proof. We only have to prove (11); the second assertion follows from the first one and Proposition 2.1 (2). The proof of (11) is divided into two parts, the "upper class result" (lim sup $\geqslant 1$) and the "lower class result" (lim sup $\leqslant 1$). We prove first the "upper class result". Define $h(t) = \log\log(1/t)$, $\varphi(t) = (2/\pi)\, th(t)$ and choose $0 < \theta < 1$, $t = \theta^{n-1}$, $0 < \delta < 1$ and $c = (\pi/2t)(1+\delta)^{-1/2}$. By the submartingale inequality and Proposition 2.1 (5), we have

$$\begin{aligned} P[\max_{0 \leqslant s \leqslant t} \{cL_s\} > \sqrt{1+\delta}\, h(t)] &= P[\max_{0 \leqslant s \leqslant t} \{\exp(cL_s)\} > \exp(\sqrt{1+\delta}\, h(t))] \\ &\leqslant \exp(-\sqrt{1+\delta}\, h(t))\, E[\exp(cL_t)] \\ &= \text{const} \cdot (n-1)^{-\sqrt{1+\delta}}, \end{aligned}$$

which is the general term of a convergent sum. So by the first Borel–Cantelli lemma,

$$P\Big[\max_{0 \leqslant s \leqslant \theta^{n-1}} \{L_s\} \leqslant (1+\delta)\frac{2}{\pi}\theta^{n-1} h(\theta^{n-1}),\ n \nearrow \infty\Big] = 1.$$

Further, for sufficiently large $n$ and $\theta^n < t \leqslant \theta^{n-1}$,

$$L_t \leqslant \max_{\theta^n \leqslant s \leqslant \theta^{n-1}} \{L_s\} \leqslant (1+\delta)\, \varphi(\theta^{n-1}) \leqslant (1+\delta)\frac{1}{\theta}\varphi(t)\frac{h(\theta^{n-1})}{h(\theta^n)}$$

since $h$ is monotone increasing for small $t$. Letting $\theta \nearrow 1$ and $\delta \searrow 0$ completes the proof of the "upper class result".

To prove the lower class result set

$$A_n := \{[L(\theta^n) - L(\theta^{n+1})] \geqslant (1-\theta)^2 \varphi(\theta^n)\}, \quad 0 < \theta < 1,\ n \geqslant 1,$$

and choose the $\sigma$-algebras $\mathfrak{F}_{n+1} = \mathcal{M}_{\theta^{n+1}}$. By Lemma 2.6,

$$P[A_n | \mathfrak{F}_{n+1}] \geqslant \text{const} \cdot n^{-(1-\theta)},$$

which is the general term of a divergent series. Thus, by Lemma 2.2,

$$L(\theta^n) > (1-\theta)^2 \varphi(\theta^n) + L(\theta^{n+1}) \quad \text{infinitely often (i.o.).}$$

Also

$$L(\theta^{n+1}) < (1+\theta)\, \varphi(\theta^{n+1})$$

as $n \to \infty$ by the first part of the proof. So

$$L(\theta^n) > (1-\theta)^2 \varphi(\theta^n) - (1+\theta)\, \varphi(\theta^{n+1}) \geqslant (1-4\theta)\, \varphi(\theta^n) \quad \text{i.o.}$$

as $n \to \infty$, i.e.

$$\limsup_{t \searrow 0} L_t/\varphi(t) > 1 - 4\theta,$$

and the proof is completed by letting $\theta \searrow 0$. ∎

Remark 3.2. For a 1-dimensional Brownian motion $(b_t)$ the sample path behaviour "*at infinity*" can be read from the behaviour of the paths "*at zero*" by using the fact that the time-scaled process $(\hat{b}_t) = (tb_{1/t})$ is again a

Brownian motion. This method does not seem to work for $(L_t)$. Therefore, in order to prove the "lil" at infinity one has to repeat the essential steps of Theorem 3.1 (cf. also [2] or [6]).

**4. Related processes.**

($\mathscr{A}$) We now extend the result obtained so far for $(L_t)$ to processes $(L_t^A)$ defined by

(13) $$L_t^A := \int_0^t \langle AW_s, dW_s\rangle,$$

$A$ a $d\times d$ skew symmetric matrix and $(W_t)$ a $d$-dimensional, $d \geqslant 2$, Wiener process. It turns out that the asymptotic behaviour of the sample path of the 1-dimensional process $(L_t^A)$ only differs by a constant $a$ from the behaviour of $(L_t)$ and that

(14) $$a := \max\{|a_k| \mid \pm ia_k \text{ eigenvalues of } A\}.$$

THEOREM 4.1.

(15) $$\limsup_{t\searrow 0} \frac{L_t^A}{(2/\pi)\, at \log\log(1/t)} = 1 \quad P\text{-a.s.},$$

(16) $$\liminf_{t\searrow 0} \frac{L_t^A}{(2/\pi)\, at \log\log(1/t)} = -1 \quad P\text{-a.s.}$$

Proof. The proof of Theorem 4.1 is very much the same as that of Theorem 3.1. The only difference is that instead of Proposition 2.1 (5) and (7) one uses the formulae (cf. [7])

(17) $$E[\exp(i\lambda L_t^A)] = \prod_{k=1}^{[d/2]} \operatorname{ch}^{-1}(\lambda t a_k),$$

$[d/2]$ the integer part of $d/2$ and $\{\pm ia_k\}$ the eigenvalues of $A$, and

(18) $$E[\exp(iz(L_t^A + \langle\gamma, W_t\rangle))] = \prod_{k=1}^{[d/2]} \operatorname{ch}^{-1}(zta_k) \times\exp[-\tfrac{1}{2}((O\gamma)_{2k-1}^2 + (O\gamma)_{2k}^2) z \operatorname{th}(zTa_k)],$$

where $O$ is an orthogonal matrix which reduces $A$ to its "standard form".

The "upper class result" is then derived exactly as in the previous case, putting

$$\varphi(t) = \frac{2}{\pi} at \log\log(1/t) \quad \text{and} \quad c = \frac{\pi}{2at\sqrt{1+\delta}}.$$

To prove the "lower class result", we again need a lower bound for the tail



probabilities of the conditional increments of the process $(L_t^A)$. By Lemma 2.4,

(19) $$\chi(z) = \prod_{k=1}^{[d/2]} \exp[-\tfrac{1}{2}((O\gamma)_{2k-1}^2 + (O\gamma)_{2k}^2) z \operatorname{th}(zTa_k)]$$

is a characteristic function associated with a symmetric distribution function and thus

(20) $$\chi(z) \prod_{k=1}^{[d/2]}{}' \operatorname{ch}^{-1}(zTa_k)$$

is too, the prime indicating that the product is taken over all $a_k$'s except one whose modulus equals $a$. Therefore, as in the proof of Lemma 2.6 one gets, for $\xi \geqslant 0$, $T = t-s > 0$,

(21) $$P[L_t^A - L_s^A \geqslant \xi \mid \mathscr{M}_s] \geqslant (2\pi)^{-1} \exp(-\tfrac{1}{2}\pi\xi/(aT)).$$

The proof can then be completed in the same manner as the previous one. ∎

($\mathscr{B}$) Let $(L_t^{(i)})_{1\leqslant i\leqslant m}$ be $m$ independent area processes, i.e. there are $m$ independent 2-dimensional Brownian motions $(W_t^{(i)})$, each of them defining the associated area process

$$L_t^{(i)} = \int_0^t \langle JW_s^{(i)}, dW_s^{(i)}\rangle, \qquad 1 \leqslant i \leqslant m.$$

We examine here the asymptotic behaviour of $|\vec{L}_t|_1 := \sum_{i=1}^m |L_t^{(i)}|$.

THEOREM 4.2.

(22) $$\limsup_{t\searrow 0} \frac{|\vec{L}_t|_1}{(2/\pi)\, t \log\log(1/t)} = 1 \quad P\text{-a.s.}$$

Proof. We shall reduce the problem so that we can apply Theorem 4.1. Note that

(23) $$|\vec{L}_t|_1 = \sup_{|e|_\infty = 1} \{|\langle e, \vec{L}_t\rangle|\} = \sup_{|e|_\infty = 1} \{|\sum_{i=1}^m e_i L_t^{(i)}|\},$$

where $|e|_\infty := \max_{1\leqslant i\leqslant m} \{|e_i|\}$, $e \in \mathbf{R}^m$. If $E$ denotes the $d\times d$, $d = 2m$, skew symmetric matrix

$$E = \operatorname{diag}\{e_i J\}_{1\leqslant i\leqslant m},$$

we can express $\langle e, \vec{L}_t\rangle$ as $L_t^E = \int_0^t \langle EW_s, dW_s\rangle$. Hence, if $|e|_\infty = 1$ it follows from Theorem 4.1 that the "upper class result" holds. To show the converse inequality we assume the left-hand side of (22) to be larger than $c$, $c > 1$ (Assumption (*)). Now, there are $N = 2^m$ vertices $e^{(1)}, \ldots, e^{(N)}$ of the $m$-dimensional cube $\mathcal{K}_\infty = \{e \in \boldsymbol{R}^m |\ |e|_\infty \leqslant 1\}$. So, for $\varepsilon > 0$, let

$$K_\varepsilon^{(i)} := \{x \in \boldsymbol{R}^m |\ |\langle x, e^{(i)}\rangle| > 1-\varepsilon\}, \qquad 1 \leqslant i \leqslant N,$$

and observe that

$$\partial \mathcal{K}_1 := \{x \in \boldsymbol{R}^m |\ |x|_1 = \sum_{i=1}^{m} |x_i| = 1\} \subset \bigcup_{i=1}^{N} K_\varepsilon^{(i)}.$$

Then, for some $i_0 \in \{1, \ldots, N\}$, $\delta$ sufficiently small and

$$\varphi(t) = (2/\pi)\, t \log \log (1/t)$$

the assumption (*) implies

$$P\left[\limsup_{t \searrow 0} \frac{|\vec{L}_t|_1}{\varphi(t)} \geqslant 1+\delta,\ \frac{\vec{L}_t}{|\vec{L}_t|_1} \in K_\varepsilon^{(i_0)}\right] > 0.$$

Thence,

$$(24)\quad P\left[\limsup_{t \searrow 0} \left|\frac{\langle e^{(i_0)}, \vec{L}_t\rangle}{\varphi(t)}\right| = \limsup_{t \searrow 0} \left|\frac{\langle e^{(i_0)}, \vec{L}_t\rangle}{|\vec{L}_t|_1}\right| \left|\frac{\vec{L}_t}{\varphi(t)}\right|_1 > (1+\delta)(1-\varepsilon)\right] > 0$$

and, since $\varepsilon$ can be chosen so that $(1+\delta)(1-\varepsilon) > 1$, (24) contradicts Theorem 4.1. ∎

Remark 4.3. There are equivalent results to Theorems 3.1, 4.1 and 4.2 which describe the asymptotic behaviour of the sample paths at infinity. The proofs of these results run along the same lines as the given proofs.

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