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A characterization of the Banach property for summability matrices

by

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Dedicated to Prof. Z. Ciesielski on his 50th birthday

Abstract. A doubly infinite matrix $A = \{a_{nk}: n, k = 1, 2, \ldots\}$ of real numbers is said to have the *Banach property* if for every orthonormal system $\{\varphi_k(x): k = 1, 2, \ldots\}$ in (0, 1) we have

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}\varphi_k(x) = 0 \quad \text{a.e.}$$

We define a norm $\|A\|$ in such a way that a matrix A has the Banach property if and only if $\|A\| < \infty$. Some consequences of this characterization are also included.

1. Introduction. Let $\varphi = \{\varphi_k(x): k = 1, 2, \ldots\}$ be an orthonormal system (in abbreviation: ONS) in the unit interval (0,1) and let $A = \{a_{nk}: n, k = 1, 2, \ldots\}$ be a doubly infinite matrix of real numbers. Following Banach (see e.g. [2]) we say that the matrix A has the *Banach property* (shortly, $A \in (\mathrm{BP})$) if for every ONS φ in (0,1) we have

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}\varphi_k(x) = 0 \quad \text{a.e.} \tag{1.1}$$

Taking φ to be the Rademacher ONS $r = \{r_k(x) = \operatorname{sign}\sin 2^k\pi x: k = 1, 2, \ldots\}$ (see e.g. [5, p. 212]), one can easily deduce that if $A \in (\mathrm{BP})$, then

$$\lim_{n\to\infty} a_{nk} = 0 \quad (k = 1, 2, \ldots). \tag{1.2}$$

In fact, since

$$r_1(x+\tfrac{1}{2}) = -r_1(x), \qquad r_k(x+\tfrac{1}{2}) = r_k(x) \quad (k = 2, 3, \ldots),$$

one can write

$$\sum_{k=1}^{\infty} a_{nk} r_k(x+\tfrac{1}{2}) = -2a_{n1} r_1(x) + \sum_{k=1}^{\infty} a_{nk} r_k(x).$$

If (1.1) with $\varphi = r$ holds for both x and $x+1/2$ (which happens for almost every x in (0,1)), then letting $n \to \infty$ in the last equality yields (1.2) for $k = 1$. The proof for $k = 2, 3, \ldots$ is quite similar.

In this paper we assume that

(1.3) $$\sum_{k=1}^{\infty} |a_{nk}| < \infty \quad (n = 1, 2, \ldots).$$

This condition is satisfied if the matrix A is row finite, i.e. for each n there exists k_n such that

$$a_{nk} = 0 \quad \text{for } k > k_n \quad (n = 1, 2, \ldots).$$

Under condition (1.3) the infinite sums $\sum_{k=1}^{\infty} a_{nk}\varphi_k(x)$ in (1.1) exist a.e. for every ONS since

$$\int_0^1 |\varphi_k(x)|\, dx \leqslant \{\int_0^1 \varphi_k^2(x)\, dx\}^{1/2} = 1.$$

Remark 1. As is known (see e.g. [5, p. 74]), conditions (1.2) and

(1.4) $$\sum_{k=1}^{\infty} |a_{nk}| \leqslant C \quad (n = 1, 2, \ldots),$$

with a finite constant C, are necessary and sufficient in order that for every sequence $\{s_k\colon k = 1, 2, \ldots\}$ of real numbers tending to 0, the sequence $\{\sigma_n = \sum_{k=1}^{\infty} a_{nk} s_k\colon n = 1, 2, \ldots\}$ also tends to 0. Condition (1.4) is somewhat stronger than (1.3).

Remark 2. Hill [1] proved that if (1.1) holds for $\varphi = r$, then

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}^2 = 0,$$

which is weaker, in general, than (1.3).

2. Results. For $1 \leqslant M \leqslant N$ we introduce the quantity

$$\|A; M, N\| = \sup_{\varphi} \{\int_0^1 (\max_{M\leqslant n\leqslant N} |\sum_{k=1}^{\infty} a_{nk}\varphi_k(x)|)^2\, dx\}^{1/2},$$

where the supremum is taken over all ONS φ in (0,1), and define

(2.1) $$\|A\| = \lim_{N\to\infty} \|A; 1, N\| \quad (\leqslant \infty).$$

This limit exists since $\|A; M, N\|$ is nondecreasing in N for every fixed M. Clearly,

$$\|A\| = \sup_{\varphi} \{\int_0^1 (\sup_{n\geqslant 1} |\sum_{k=1}^{\infty} a_{nk}\varphi_k(x)|)^2\, dx\}^{1/2}.$$

Our main result is expressed in the following

THEOREM 1. (i) *If* $\|A\| < \infty$, *then* $A \in (\mathrm{BP})$.

(ii) *If* $\|A\| = \infty$, *then there exists an ONS* $\Phi = \{\Phi_k(x)\}$ *of step functions in* (0, 1) *such that*

(2.2) $$\limsup_{n\to\infty} |\sum_{k=1}^{\infty} a_{nk}\Phi_k(x)| = \infty \quad a.e.$$

In particular, $A \notin (\mathrm{BP})$.

A function $\Phi(x)$ is called a *step function* if there exists a partition of (0, 1) into a finite number of disjoint intervals such that $\Phi(x)$ is constant on each of them.

The proof of Theorem 1 is based on the following

THEOREM 2. (i) *If* $\|A\| < \infty$, *then*

(2.3) $$\lim_{M,N\to\infty} \|A; M, N\| = 0 \quad (M \leqslant N);$$

(ii) *If* $\|A\| = \infty$, *then for every* $M = 1, 2, \ldots$,

(2.4) $$\lim_{N\to\infty} \|A; M, N\| = \infty.$$

The next theorem is interesting in itself.

THEOREM 3. *If* $A \in (\mathrm{BP})$, *then there exists a double sequence*

$$\mu = \{\mu_{nk}\colon n, k = 1, 2, \ldots\}$$

of positive numbers such that

$$\lim \mu_{nk} = \infty \quad \text{as } \max(n, k) \to \infty$$

and $\mu A = \{\mu_{nk} a_{nk}\} \in (\mathrm{BP})$. *Conversely, if* $A \notin (\mathrm{BP})$, *then there exists a double sequence* $\mu = \{\mu_{nk}\}$ *of positive numbers such that*

$$\lim \mu_{nk} = 0 \quad \text{as } \max(n, k) \to \infty$$

and $\mu A \notin (\mathrm{BP})$.

Remark 3. Under the usual addition and multiplication of matrices by scalars, $\|A\|$ satisfies the three axioms of vector norm.

Remark 4. Theorems 1 and 2 remain valid if definition (2.1) is changed for

$$\|A\|_p = \lim_{N\to\infty} \|A; 1, N\|_p \quad (1 \leqslant p \leqslant 2),$$

where

$$\|A; M, N\|_p = \sup_{\varphi} \{\int_0^1 (\max_{M\leqslant n\leqslant N} |\sum_{k=1}^{\infty} a_{nk}\varphi_k(x)|)^p\, dx\}^{1/p}.$$

Remark 5. The special case $\{a_{nk} = a_k/\lambda_n\}$, where $\{a_k\colon k = 1, 2, \ldots\}$ is an arbitrary sequence of numbers and $\{\lambda_n\colon n = 1, 2, \ldots\}$ is a nondecreasing sequence of positive numbers tending to ∞, was extensively studied by the second author [4]. The method of proof in the present paper is partially based on that of [3] by the first author.

3. Auxiliary results.

LEMMA 1. *If* $1 < M \leqslant N$, *then*

$$\|A;\,1,\,N\| \leqslant \|A;\,1,\,M-1\| + \|A;\,M,\,N\|.$$

Proof. This inequality is a consequence of the Minkowski inequality and the fact that

$$\max_{1\leqslant n\leqslant N}\Big|\sum_{k=1}^{\infty} a_{nk}\,\varphi_k(x)\Big| \leqslant \Big(\max_{1\leqslant n\leqslant M-1} + \max_{M\leqslant n\leqslant N}\Big)\Big|\sum_{k=1}^{\infty} a_{nk}\,\varphi_k(x)\Big|.$$

LEMMA 2. *For every* $\varepsilon > 0$ *and* $M \geqslant 1$, *there exists* $N_0 \geqslant M$ *such that for all* $N_0 \leqslant N_1 \leqslant N_2$

$$\|A;\,1,\,M\|^2 + \|A;\,N_1,\,N_2\|^2 \leqslant \|A;\,1,\,N_2\|^2 + \varepsilon.$$

The next two lemmas will be used in the proof of Lemma 2.

LEMMA 3. *For every* $\varepsilon > 0$ *and* $1 \leqslant M \leqslant N$ *there exists* $K \geqslant 1$ *such that*

$$\sup_{\varphi}\Big\{\int_0^1\Big(\max_{M\leqslant n\leqslant N}\Big|\sum_{k=K+1}^{\infty} a_{nk}\,\varphi_k(x)\Big|\Big)^2 dx\Big\}^{1/2} < \varepsilon; \tag{3.1}$$

consequently, there exists an ONS φ *such that*

$$\Big\{\int_0^1\Big(\max_{M\leqslant n\leqslant N}\Big|\sum_{k=1}^{K} a_{nk}\,\varphi_k(x)\Big|\Big)^2 dx\Big\}^{1/2} \geqslant \|A;\,M,\,N\| - \varepsilon.$$

Proof. By the Minkowski inequality and (1.3),

$$\sup_{\varphi}\Big\{\int_0^1\Big(\max_{M\leqslant n\leqslant N}\Big|\sum_{k=K+1}^{\infty} a_{nk}\,\varphi_k(x)\Big|\Big)^2 dx\Big\}^{1/2} \leqslant \sum_{k=K+1}^{\infty}\Big(\max_{M\leqslant n\leqslant N}|a_{nk}|\Big) < \varepsilon$$

if K is sufficiently large.

LEMMA 4. *For every* $\varepsilon > 0$ *and* $K \geqslant 1$, *there exists* $N_0 \geqslant 1$ *such that for all* $N_0 \leqslant M \leqslant N$,

$$\sup_{\varphi}\Big\{\int_0^1\Big(\max_{M\leqslant n\leqslant N}\Big|\sum_{k=1}^{K} a_{nk}\,\varphi_k(x)\Big|\Big)^2 dx\Big\}^{1/2} < \varepsilon; \tag{3.2}$$

consequently, there exists an ONS φ *such that*

$$\Big\{\int_0^1\Big(\max_{M\leqslant n\leqslant N}\Big|\sum_{k=K+1}^{\infty} a_{nk}\,\varphi_k(x)\Big|\Big)^2 dx\Big\}^{1/2} \geqslant \|A;\,M,\,N\| - \varepsilon.$$

Proof. Again by the Minkowski inequality,

$$\sup_{\varphi}\Big\{\int_0^1\Big(\max_{M\leqslant n\leqslant N}\Big|\sum_{k=1}^{K} a_{nk}\,\varphi_k(x)\Big|\Big)^2 dx\Big\}^{1/2} \leqslant \sum_{k=1}^{K}\Big(\max_{M\leqslant n\leqslant N}|a_{nk}|\Big) < \varepsilon$$

if $M \geqslant N_0$ and N_0 is sufficiently large, due to (1.2).

Proof of Lemma 2. Given $\varepsilon > 0$, by Lemma 3 there exist an integer K and an ONS $\bar{\varphi} = \{\bar{\varphi}_k(x)\}$ such that

$$\int_0^1\Big(\max_{1\leqslant n\leqslant M}\Big|\sum_{k=1}^{K} a_{nk}\,\bar{\varphi}_k(x)\Big|\Big)^2 dx \geqslant \|A;\,1,\,M\|^2 - \varepsilon/2. \tag{3.3}$$

By Lemma 4, there exists an integer N_0 such that for all N_1 and N_2, $N_0 \leqslant N_1 \leqslant N_2$, there exists an ONS $\bar{\bar{\varphi}} = \{\bar{\bar{\varphi}}_k(x)\}$ such that

$$\int_0^1\Big(\max_{N_1\leqslant n\leqslant N_2}\Big|\sum_{k=K+1}^{\infty} a_{nk}\,\bar{\bar{\varphi}}_k(x)\Big|\Big)^2 dx \geqslant \|A;\,M,\,N\|^2 - \varepsilon/2. \tag{3.4}$$

We define for $k = 1, 2, \ldots, K$,

$$\psi_k(x) = \begin{cases}\sqrt{2}\,\bar{\varphi}_k(2x) & \text{for } x \in (0,\,1/2),\\ 0 & \text{otherwise;}\end{cases}$$

while for $k = K+1, K+2, \ldots$,

$$\psi_k(x) = \begin{cases}\sqrt{2}\,\bar{\bar{\varphi}}_k(2x-1) & \text{for } x \in (1/2,\,1),\\ 0 & \text{otherwise.}\end{cases}$$

Clearly, $\psi = \{\psi_k(x)\colon k = 1, 2, \ldots\}$ is an ONS in $(0, 1)$. From (3.3) and (3.4) it follows that

$$\begin{aligned}
&\|A;\,1,\,M\|^2 + \|A;\,N_1,\,N_2\|^2 - \varepsilon\\
&\qquad\leqslant 2\int_0^{1/2}\Big(\max_{1\leqslant n\leqslant M}\Big|\sum_{k=1}^{K} a_{nk}\,\bar{\varphi}_k(2x)\Big|\Big)^2 dx\\
&\qquad\quad + 2\int_{1/2}^1\Big(\max_{N_1\leqslant n\leqslant N_2}\Big|\sum_{k=K+1}^{\infty} a_{nk}\,\bar{\bar{\varphi}}_k(2x-1)\Big|\Big)^2 dx\\
&\qquad\leqslant \int_0^1\Big(\max_{1\leqslant n\leqslant N_2}\Big|\sum_{k=1}^{\infty} a_{nk}\,\psi_k(x)\Big|\Big)^2 dx\\
&\qquad\leqslant \|A;\,1,\,N_2\|^2.
\end{aligned}$$

LEMMA 5. *If for an ONS* $\varphi = \{\varphi_k(x)\colon K_1 \leqslant k \leqslant K_2\}$

$$S = \Big\{\int_0^1\Big(\max_{N_1\leqslant n\leqslant N_2}\Big|\sum_{k=K_1}^{K_2} a_{nk}\,\varphi_k(x)\Big|\Big)^2 dx\Big\}^{1/2} > 0,$$

then there exists another ONS $\psi = \{\psi_k(x)\colon K_1 \leqslant k \leqslant K_2\}$ *of step functions in*

(0, 1) *such that*

$$\max_{N_1\leqslant n\leqslant N_2}\Big|\sum_{k=K_1}^{K_2} a_{nk}\psi_k(x)\Big| \geqslant S/4 \qquad a.e.$$

A similar lemma was proved in [4] by the second author.

Proof. First we choose a (not necessarily orthogonal) system $\bar\varphi = \{\bar\varphi_k(x)\colon K_1 \leqslant k \leqslant K_2\}$ such that

$$\int_0^1 [\varphi_k(x)-\bar\varphi_k(x)]^2\,dx \leqslant \delta^2 \qquad (K_1 \leqslant k \leqslant K_2); \tag{3.5}$$

where δ is a sufficiently small positive number to be specified later. We set

$$\alpha_{kl} = \int_0^1 \bar\varphi_k(x)\,\bar\varphi_l(x)\,dx,$$

$$\beta_l = \Big\{\sum_{k=K_1}^{l-1} + \sum_{k=l+1}^{K_2}\Big\}|\alpha_{kl}| \qquad (K_1 \leqslant k,\ l \leqslant K_2).$$

Evidently, $\alpha_{kl}\approx 0$ for $k\neq l$ and $\alpha_{kk}\approx 1$ provided δ is small enough. More precisely, by (3.5) and the Minkowski inequality,

$$\Big\{\int_0^1 \bar\varphi_k^2(x)\,dx\Big\}^{1/2} \leqslant 1+\delta,$$

whence, by the Schwarz and Minkowski inequalities,

$$|\alpha_{kk}-1| = \Big|\int_0^1 [\bar\varphi_k(x)-\varphi_k(x)]\,[\bar\varphi_k(x)+\varphi_k(x)]\,dx\Big| \leqslant \delta(2+\delta), \tag{3.6}$$

while for $k\neq l$, again by the Schwarz inequality,

$$|\alpha_{kl}| \leqslant \delta(2+\delta),$$

whence

$$\beta_k \leqslant (K_2-K_1)\,\delta(2+\delta) \qquad (K_1 \leqslant k \leqslant K_2). \tag{3.7}$$

By (3.5),

$$\bar S = \Big\{\int_0^1\Big(\max_{N_1\leqslant n\leqslant N_2}\Big|\sum_{k=K_1}^{K_2} a_{nk}\,\bar\varphi_k(x)\Big|\Big)^2 dx\Big\}^{1/2} \tag{3.8}$$
$$\geqslant S-\delta\sum_{k=K_1}^{K_2}\big(\max_{N_1\leqslant n\leqslant N_2}|a_{nk}|\big) \geqslant \tfrac12 S$$

if δ is small enough. Obviously,

$$\Big\{\int_0^1\Big(\max_{N_1\leqslant n\leqslant N_2}\Big|\sum_{k=K_1}^{K_2} a_{nk}\frac{\bar\varphi_k(x)}{\sqrt{\alpha_{kk}+\beta_k}}\Big|\Big)^2 dx\Big\}^{1/2} \geqslant \bar S - S_1, \tag{3.9}$$

where

$$S_1 = \Big\{\int_0^1\Big(\max_{N_1\leqslant n\leqslant N_2}\Big|\sum_{k=K_1}^{K_2} a_{nk}\Big(1-\frac{1}{\sqrt{\alpha_{kk}+\beta_k}}\Big)\bar\varphi_k(x)\Big|\Big)^2 dx\Big\}^{1/2}.$$

Taking into account (3.6), (3.7) and the elementary inequality

$$\Big|1-\frac{1}{\sqrt{1+t}}\Big| \leqslant |t| \qquad \text{for } t \geqslant -\tfrac12,$$

we obtain, for sufficiently small δ,

$$S_1 \leqslant \sum_{k=K_1}^{K_2}\big(\max_{N_1\leqslant n\leqslant N_2}|a_{nk}|\big)\Big(1-\frac{1}{\sqrt{\alpha_{kk}+\beta_k}}\Big)(1+\delta)$$
$$\leqslant (1+\delta)\sum_{k=K_1}^{K_2}\big(|\alpha_{kk}-1|+\beta_k\big)\big(\max_{N_1\leqslant n\leqslant N_2}|a_{nk}|\big) \leqslant \tfrac12\bar S.$$

Combining this estimate with (3.9) yields

$$\Big\{\int_0^1\Big(\max_{N_1\leqslant n\leqslant N_2}\Big|\sum_{k=K_1}^{K_2} a_{nk}\frac{\bar\varphi_k(x)}{\sqrt{\alpha_{kk}+\beta_k}}\Big|\Big)^2 dx\Big\}^{1/2} \geqslant \frac14 S. \tag{3.10}$$

Now we extend the domain of the functions in $\bar\varphi$ from the interval (0, 1) to (0, 2) in such a way that $\bar\varphi$ will be an ONS of step functions in (0, 2). To this end, we divide the interval (1,2) into as many subintervals of equal length as the number of the ordered pairs of integers k, l with $K_1\leqslant k,\ l\leqslant K_2$, $k\neq l$. Denote the subintervals by I_{kl} and set

$$\bar\varphi_k(x) = \begin{cases} \{\tfrac12\gamma|\alpha_{kl}|\}^{1/2} & \text{for } x\in I_{kl},\\ -\{\tfrac12\gamma|\alpha_{kl}|\}^{1/2}\operatorname{sign}\alpha_{kl} & \text{for } x\in I_{lk},\\ 0 & \text{for } x\in(1,2)\setminus\bigcup_{l:l\neq k}(I_{kl}\cup I_{lk}), \end{cases}$$

where $\gamma = (K_2-K_1+1)(K_2-K_1)$. Then

$$\int_0^2 \bar\varphi_k(x)\,\bar\varphi_l(x)\,dx = \Big\{\int_0^1 + \int_{I_{kl}} + \int_{I_{kl}}\Big\}\bar\varphi_k(x)\,\bar\varphi_l(x)\,dx$$
$$= \alpha_{kl}-|\alpha_{kl}|\operatorname{sign}\alpha_{kl} = 0,$$

$$\int_0^2 \bar\varphi_k^2(x)\,dx = \Big\{\int_0^1 + 2\sum_{l\neq k}\int_{I_{kl}}\Big\}\bar\varphi_k^2(x)\,dx = \alpha_{kk}+\beta_k.$$

These equalities mean that the step functions

$$\bar\psi_k(x) = \frac{\bar\varphi_k(x)}{\sqrt{\alpha_{kk}+\beta_k}} \qquad (K_1 \leqslant k \leqslant K_2)$$

form an ONS in (0, 2). In addition, by (3.10),

$$\{\int_0^2 \big(\max_{N_1\le n\le N_2}\big|\sum_{k=K_1}^{K_2} a_{nk}\bar\psi_k(x)\big|\big)^2 dx\}^{1/2} \geqslant \tfrac14 S. \tag{3.11}$$

Since

$$\bar\Psi(x) = \max_{N_1\le n\le N_2}\big|\sum_{k=K_1}^{K_2} a_{nk}\bar\psi_k(x)\big|$$

is a step function, we can divide the interval (0, 2) into a finite number of subintervals (x_{r-1}, x_r), $0 = x_0 < x_1 < \ldots < x_\varrho = 2$, such that $\bar\Psi(x)$ takes a constant value, say y_r, on each subinterval (x_{r-1}, x_r), $r = 1, 2, \ldots, \varrho$. Set

$$u_0 = 0, \quad u_r = \frac{1}{T^2}\sum_{s=1}^{r} y_s^2(x_s - x_{s-1}),$$

where

$$T^2 = \sum_{s=1}^{\varrho} y_s^2(x_s-x_{s-1}) = \int_0^2 \bar\Psi^2(x)\,dx.$$

By (3.11), $T \geqslant S/4$. Furthermore, set

$$\psi_k(u) = \frac{T}{y_r}\bar\psi_k\Big(\frac{T^2}{y_r^2}(u-u_{r-1})+x_{r-1}\Big) \quad \text{for } u\in(u_{r-1}, u_r)$$

$$(r = 1, 2, \ldots, \varrho).$$

In this definition we have tacitly assumed $y_r \neq 0$. In case $y_r = 0$, we have $u_{r-1} = u_r$.

Now it is routine to check that $\{\psi_k(u)\colon K_1 \leqslant k \leqslant K_2\}$ is an ONS of step functions in (0, 1). Indeed, performing the substitution

$$x = \frac{T^2}{y_r^2}(u-u_{r-1})+x_{r-1}, \qquad dx = \frac{T^2}{y_r^2}du,$$

we get

$$\int_0^1 \psi_k(u)\psi_l(u)\,du = \sum_{r=1}^{\varrho}\int_{u_{r-1}}^{u_r}\psi_k(u)\psi_l(u)\,du$$

$$= \sum_{r=1}^{\varrho}\int_{x_{r-1}}^{x_r}\bar\psi_k(x)\bar\psi_l(x)\,dx = \int_0^2\bar\psi_k(x)\bar\psi_l(x)\,dx = \delta_{kl}.$$

Finally, by the definition of $\bar\Psi(x)$ and $\psi_k(u)$, we can see that for each $u\in(u_{r-1}, u_r)$ there exists an $x\in(x_{r-1}, x_r)$ such that

$$\max_{N_1\le n\le N_2}\big|\sum_{k=K_1}^{K_2} a_{nk}\psi_k(u)\big| = \frac{T}{y_r}\max_{N_1\le n\le N_2}\big|\sum_{k=K_1}^{K_2} a_{nk}\bar\psi_k(x)\big|$$

$$= \frac{T}{y_r}\bar\Psi(x) = T \geqslant S/4.$$

4. Proofs.

Proof of Theorem 2. (i) Assume $\|A\| < \infty$. By (2.1), for every $\varepsilon > 0$ there exists $M \geqslant 1$ such that

$$\|A; 1, M\|^2 \geqslant \|A\|^2 - \varepsilon.$$

By Lemma 2, there exists $N_0 \geqslant M$ such that for all $N_0 \leqslant N_1 \leqslant N_2$

$$\|A; N_1, N_2\|^2 \leqslant \|A; 1, N_2\|^2 + \varepsilon - \|A; 1, M\|^2 \leqslant 2\varepsilon.$$

(ii) In case $\|A\| = \infty$, by (2.1) we have (2.4) for $M = 1$. If $M > 1$, then Lemma 1 shows

$$\|A; M, N\| \geqslant \|A; 1, N\| - \|A; 1, M-1\|.$$

Since $\|A; 1, M-1\|$ is finite for every M, (2.4) follows.

Proof of Theorem 1. (i) If $\|A\| < \infty$, then by Theorem 2 (i) for $\varepsilon_r = 2^{-3r}$ there exists N_r such that

$$\|A; N_r, N\| \leqslant 2^{-3r} \quad \text{for } N \geqslant N_r \quad (r = 1, 2, \ldots).$$

We may assume $N_1 < N_2 < \ldots$ Then, given an ONS φ,

$$\int_0^1\big(\max_{N_r\le n\le N_{r+1}}\big|\sum_{k=1}^{\infty} a_{nk}\varphi_k(x)\big|\big)^2 dx \leqslant 2^{-3r}.$$

By setting

$$H_r = \{x\in(0, 1)\colon \max_{N_r\le n\le N_{r+1}}\big|\sum_{k=1}^{\infty} a_{nk}\varphi_k(x)\big| \geqslant 2^{-r}\},$$

this implies

$$\operatorname{mes} H_r \leqslant 2^{-r}, \quad \text{a fortiori} \quad \sum_{r=1}^{\infty}\operatorname{mes} H_r < \infty.$$

By mes H we denote the Lebesgue measure of the measurable set H. Thus, the Borel–Cantelli lemma yields that for almost every x there exists $r_0 = r_0(x)$ such that $x \notin H_r$ if $r \geqslant r_0$; i.e.

$$\max_{n\ge N_r}\big|\sum_{k=1}^{\infty} a_{nk}\varphi_k(x)\big| \leqslant 2^{-r+1} \quad (r \geqslant r_0).$$

This proves (1.1).

(ii) Since $\|A\| = \infty$, we can choose $M_1 = 1$ and N_1 such that

$$\|A; M_1, N_1\| \geqslant 5/4.$$

By Lemma 3, there exist $K_1 \geq 1$ and an ONS $\varphi^{(1)} = \{\varphi_k^{(1)}(x)\}$ such that

$$\sup_{\varphi} \Big\{\int_0^1 \Big(\max_{M_1 \leq n \leq N_1} \Big| \sum_{k=K_1+1}^{\infty} a_{nk}\varphi_k(x)\Big|\Big)^2 dx\Big\}^{1/2} \leq 1/4,$$

$$\Big\{\int_0^1 \Big(\max_{M_1 \leq n \leq N_1} \Big| \sum_{k=1}^{K_1} a_{nk}\varphi_k^{(1)}(x)\Big|\Big)^2 dx\Big\}^{1/2} \geq 1.$$

By Lemma 4, there exists $M_2 > N_1$ such that for $N \geq M_2$

$$\sup_{\varphi} \Big\{\int_0^1 \Big(\max_{M_2 \leq n \leq N} \Big| \sum_{k=1}^{K_1} a_{nk}\varphi_k(x)\Big|\Big)^2 dx\Big\}^{1/2} \leq 1/16.$$

Now we choose $N = N_2$ in such a way that

$$\|A;\, M_2,\, N_2\| \geq 17/8.$$

By applying Lemma 3 again, there exist $K_2 > K_1$ and an ONS $\varphi^{(2)} = \{\varphi_k^{(2)}(x)\}$ such that

$$\sup_{\varphi} \Big\{\int_0^1 \max_{M_2 \leq n \leq N_2} \Big| \sum_{k=K_2+1}^{\infty} a_{nk}\varphi_k(x)\Big|\Big)^2 dx\Big\}^{1/2} \leq 1/16,$$

$$\Big\{\int_0^1 \Big(\max_{M_2 \leq n \leq N_2} \Big| \sum_{k=K_1+1}^{K_2} a_{nk}\varphi_k^{(2)}(x)\Big|\Big)^2 dx\Big\}^{1/2} \geq 2.$$

Following this pattern, by induction we can define two sequences $1 = M_1 < N_1 < M_2 < N_2 < \ldots$ and $0 = K_0 < K_1 < \ldots$ of integers and ONS $\varphi^{(r)} = \{\varphi_k^{(r)}(x)\}$ $(r = 1, 2, \ldots)$ such that

$$\text{(4.1)} \qquad \sup_{\varphi} \Big\{\int_0^1 \Big(\max_{M_r \leq n \leq N_r} \Big| \sum_{k=1}^{K_{r-1}} a_{nk}\varphi_k(x)\Big|\Big)^2 dx\Big\}^{1/2} \leq 2^{-2r},$$

$$\text{(4.2)} \qquad \sup_{\varphi} \Big\{\int_0^1 \Big(\max_{M_r \leq n \leq N_r} \Big| \sum_{k=K_r+1}^{\infty} a_{nk}\varphi_k(x)\Big|\Big)^2 dx\Big\}^{1/2} \leq 2^{-2r},$$

$$\Big\{\int_0^1 \Big(\max_{M_r \leq n \leq N_r} \Big| \sum_{k=K_{r-1}+1}^{K_r} a_{nk}\varphi_k^{(r)}(x)\Big|\Big)^2 dx\Big\}^{1/2} \geq r$$

$$(r = 1, 2, \ldots).$$

By Lemma 5, there exist ONS $\psi^{(r)} = \{\psi_k^{(r)}(x)\}$ of step functions such that

$$\text{(4.3)} \qquad \max_{M_r \leq n \leq N_r} \Big| \sum_{k=K_{r-1}+1}^{K_r} a_{nk}\psi_k^{(r)}(x)\Big| \geq r/4 \quad \text{a.e.}$$

We are going to define a single ONS $\Phi = \{\Phi_k(x)\}$ of step functions such that for all r

$$\text{(4.4)} \qquad \max_{M_r \leq n \leq N_r} \Big| \sum_{k=K_{r-1}+1}^{K_r} a_{nk}\Phi_k(x)\Big| \geq r/4 \quad \text{a.e.}$$

We use an induction argument with respect to r. First we set

$$\Phi_k(x) = \psi_k^{(1)}(x) \qquad (k = 1, 2, \ldots, K_1).$$

Then (4.3) and (4.4) coincide for $r = 1$.

Now assume that the functions $\Phi_k(x)$ are defined for $k = 1, 2, \ldots, K_{r_0-1}$ so that (4.4) holds for $r = 1, 2, \ldots, r_0 - 1$. Then we can divide the interval $(0, 1)$ into a finite number of subintervals, say $I_1, I_2, \ldots, I_\sigma$, such that the functions $\Phi_k(x)$ for $k \leq K_{r_0-1}$ are constant in each I_s. We denote by I_s' and I_s'' the two halves of I_s and set

$$\Phi_k(x) = \sum_{s=1}^{\sigma} [\psi_k^{(r)}(I_s';\, x) - \psi_k^{(r)}(I_s'';\, x)] \qquad (K_{r_0-1} \leq k \leq K_{r_0}),$$

where $f(I;\, x)$ is defined for $I = (u, v)$ by

$$f(I;\, x) = \begin{cases} f\left(\dfrac{x-u}{v-u}\right) & \text{if } x \in (u, v), \\ 0 & \text{otherwise.} \end{cases}$$

It is not hard to check that the step functions $\Phi_k(x)$, $k = 1, 2, \ldots, K_{r_0}$, are orthonormal and (4.4) is also satisfied for $r = r_0$. This completes the proof of the induction step.

To see (2.2), we can estimate as follows: for $M_r \leq n \leq N_r$

$$\Big|\sum_{k=1}^{\infty} a_{nk}\Phi_k(x)\Big| \geq \Big|\sum_{k=K_{r-1}+1}^{K_r} a_{nk}\Phi_k(x)\Big| - \Big|\sum_{k=1}^{K_{r-1}} a_{nk}\Phi_k(x)\Big| - \Big|\sum_{k=K_r+1}^{\infty} a_{nk}\Phi_k(x)\Big|.$$

By using the same argument as in part (i), (4.1) and (4.2) imply that the inequalities

$$\max_{M_r \leq n \leq N_r} \Big|\sum_{k=1}^{K_{r-1}} a_{nk}\Phi_k(x)\Big| \leq 2^{-r},$$

$$\max_{M_r \leq n \leq N_r} \Big|\sum_{k=K_r+1}^{\infty} a_{nk}\Phi_k(x)\Big| \leq 2^{-r}$$

are satisfied except for a finite number of $r = 1, 2, \ldots$ for almost every x. Consequently, for almost every x

$$\limsup_{n\to\infty} \Big|\sum_{k=1}^{\infty} a_{nk}\Phi_k(x)\Big| \geq \limsup_{r\to\infty} \max_{M_r \leq n \leq N_r} \Big|\sum_{k=K_{r-1}+1}^{K_r} a_{nk}\Phi_k(x)\Big|$$

and the latter limit equals ∞, by (4.4).

P r o o f o f T h e o r e m 3. It runs along the same lines as that of Theorem 1. Therefore, we only sketch it.

(i) If $A \in (\mathrm{BP})$, then there exist two sequences $1 = N_0 < N_1 < \ldots$ and $0 = K_0 < K_1 < \ldots$ of integers such that

$$\|A; N_r, N_{r+1}\| \leqslant 2^{-r} \quad (r = 1, 2, \ldots),$$

$$\sup_{\varphi} \Big\{ \int_0^1 \Big(\max_{1 \leqslant n \leqslant N_r} \Big| \sum_{k=K_r+1}^{K_{r+1}} a_{nk} \varphi_k(x) \Big| \Big)^2 dx \Big\}^{1/2} \leqslant 2^{-r}. \tag{4.5}$$

Setting $\mu_{nk} = r+1$ for $N_r \leqslant n < N_{r+1}$ and $K_r < k \leqslant K_{r+1}$ $(r = 0, 1, \ldots)$, it is easy to check that $\|\mu A\| < \infty$.

(ii) Similarly, if $A \notin (\mathrm{BP})$, then there exist two sequences $1 = N_0 < N_1 < \ldots$ and $0 = K_0 < K_1 < \ldots$ of integers such that

$$\|A; N_r, N_{r+1}\| \geqslant 2^r \quad (r = 1, 2, \ldots)$$

and (4.5) is satisfied. Now we set $\mu_{nk} = (r+1)^{-1}$ for $N_r \leqslant n < N_{r+1}$ and $K_r < k \leqslant K_{r+1}$ $(r = 0, 1, \ldots)$ and conclude that $\|\mu A\| = \infty$.

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Analytic functions in non-locally convex spaces and applications

by

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Abstract. The aim of this paper is to determine, for a general p-normable space X, what can in general be said about X-valued analytic functions on the disc. The results obtained are used to solve a problem raised by Turpin [17] on tensor products of quasi-Banach spaces.

1. Summary of main results. Suppose Ω is an open subset of the complex plane C and X is a quasi-Banach space. A map $f\colon \Omega \to X$ is said to be *analytic* if for every $z_0 \in \Omega$ there exists $r > 0$ such that f can be expanded in a power series for $|z - z_0| < r$, i.e.

$$f(z) = \sum_{n=0}^{\infty} x_n (z - z_0)^n$$

for $|z - z_0| < r$. This definition of analyticity is forced on us by simple examples which demonstrate that complex differentiability of f does not suffice to produce reasonable properties (cf. Aleksandrov [3], p. 39 or Turpin [16], Chapitre IX).

A key property of analytic functions is ([16], p. 195) that if $f\colon \Omega \to X$ is analytic and $\Omega_0 \subset \Omega$ is open and relatively compact in Ω then there is a Banach space B, an analytic function $g\colon \Omega_0 \to B$ and a bounded linear operator $T\colon B \to X$ so that $f(z) = T(g(z))$, $z \in \Omega_0$. From this many of the standard properties of analytic functions in a Banach space can be lifted to quasi-Banach spaces.

In this paper we will primarily be concerned with the case $\Omega = \Delta$, the open unit disc. In this case one has, for example,

$$f(z) = \sum_{n=0}^{\infty} x_n z^n, \quad |z| < 1,$$

where $\limsup \|x_n\|^{1/n} \leqslant 1$.

It seems that the main obstacle to developing the theory of analytic functions for non-locally convex spaces is the failure of the Maximum Modulus Principle. It has been observed by several authors (Etter [7], Aleksandrov [3], Peetre [14], Davis–Garling–Tomczak [5]) that some standard spaces, e.g. L_p for $0 < p < 1$, have a plurisubharmonic quasi-norm and hence if $f\colon \bar{\Delta} \to L_p$ is analytic on Δ and

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