Condition $\mathcal{E}$:

$$(\mathcal{E}1)\qquad \bar{\alpha}_{i,m} = \left(\alpha_{i,m} - (2q(i)+3X(i))\left|\frac{d\varphi_i'}{dy}\right|\right)\cdot\left|\frac{d\varphi_i'}{dx}\right|^{-1} > 0,$$

$$(\mathcal{E}2)\qquad \bar{\beta}_{j,n} = \left(\beta_{j,n} - (2p(j)+3Y(j))\left|\frac{d\psi_j'}{dx}\right|\right)\cdot\left|\frac{d\psi_j'}{dy}\right|^{-1} > 0,$$

$$(\mathcal{E}3)\qquad \bar{\alpha}_{i,m}\cdot\bar{\beta}_{j,n} > \max(X(i),\ 1+\mu_i)\max(Y(j),\ 1+\mu^j)$$

for all m, n, $i\in I(m)$, $j\in J(n)$ such that $\mathring{e}_i(P)\cap\mathring{E}_j(Q)\neq\emptyset$. Here $X(i)$ and $Y(j)$ are, respectively, the largest solutions of the equations

$$1 = \frac{2q(i)}{X} + \frac{q(i)}{X-3q(i)} + \frac{2}{X-\mu_i},$$

$$1 = \frac{2p(j)}{Y} + \frac{p(j)}{Y-3p(j)} + \frac{2}{Y-\mu^j},$$

where $q(i)$ is the number of components of $\mathscr{C}_i$, and $p(j)$ is the number of components of $\mathscr{C}^j$.

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Nilpotent groups with T_1 primitive ideal spaces

by

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Abstract. We prove that second countable locally compact nilpotent groups containing a compactly generated normal open subgroup have T_1 primitive ideal spaces.

§ 1. Introduction. We say that a locally compact group G has T_1 *primitive ideal space* if the group C^*-algebra, $C^*(G)$, has the property that every primitive ideal (i.e. kernel of an irreducible representation) is closed in the hull-kernel topology on the space of primitive ideals of $C^*(G)$, denoted $\operatorname{Prim} G$. Long ago Dixmier proved [5] that every connected nilpotent Lie group has T_1 primitive ideal space (in fact, such groups, being type I, are therefore CCR). More recently Poguntke showed [11] that discrete nilpotent groups have T_1 primitive ideal space. This then suggests the obvious conjecture that all locally compact nilpotent groups have T_1 primitive ideal space.

This note proves the following result in that direction.

THEOREM. *If G is a second countable locally compact nilpotent group with a compactly generated open normal subgroup then G has T_1 primitive ideal space.*

The notation of this paper is the same as that of [2], to which we also refer the reader for a more leisurely account of some of the techniques and ideas used here.

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§ 2. Preliminary arguments. We note firstly the following structure theorem for compactly generated nilpotent locally compact groups ([8], p. 104). Namely if G' is such a group then there exists a maximal compact normal subgroup K consisting of all elements whose powers form a relatively compact set such that the quotient $\bar{G}' = G'/K$ is a Lie group. By this last statement we mean that the connected component of the identity $\bar{G}_0$ of $\bar{G}'$ is a (connected) Lie group and $\bar{G}'/\bar{G}_0$ is discrete (possibly infinite). If G is as in the Theorem of the Introduction with G' normal and open in G, then K is normal in G so that $\bar{G} = G/K$ is a Lie group. Then $\bar{G}_0$ is the connected component of the identity of $\bar{G}$ and we note that $\bar{G}_0$ is simply connected by Lemma 4 of Howe [9].

Let G_0 be the pullback to G of $\bar{G}_0$ and note the following elementary result:

LEMMA 2.1. *The group G_0 is type* I.

Proof. As the dual $\hat{K}$ of K is discrete and G_0/K is connected G_0 acts trivially on $\hat{K}$. Thus the Mackey normal subgroup analysis [10] shows that G_0 is type I.

Our strategy is to exploit the following result systematically.

THEOREM 2.2. *If G is a second countable locally compact group and N is a normal subgroup such that G/N is discrete abelian then G has T_1 primitive ideal space if and only if G-quasiorbits in* Prim N *are closed.*

This theorem is the main result of [2]. We recall for the reader's convenience that a G-quasiorbit in Prim N is an equivalence class under the relation: $I_1 \sim I_2$, $I_j \in \operatorname{Prim} N$, $j = 1, 2$, whenever I_1 lies in the closure of the G-orbit through I_2 and vice versa. In order to use Theorem 2.2 we need to show that G-quasiorbits in $\hat{G}_0$ are closed and then (by an induction on the length of an ascending central series from G_0 to G) deduce that G-quasiorbits are closed "one level down" from G in the series and hence by Theorem 2.2 that G has T_1 primitive ideal space.

The first step in the argument is to note that, as a corollary of the proof of Lemma 2.1, G_0 lies in the stabilizer $S_\varkappa \subseteq G$ of each $\varkappa \in \hat{K}$. Then we have the following result.

LEMMA 2.3. *With the notation as in the first paragraph of this section, suppose that $\theta \subseteq \hat{G}_0$ is a G-quasiorbit and, for each $\pi \in \theta$, let π restrict on K to a multiple of $\varkappa_\pi \in \hat{K}$. Then θ is closed whenever, for each $\pi \in \theta$, the quasiorbit containing π of the stabilizer $S_{\varkappa_\pi}$ of $\varkappa_\pi$ is closed.*

Proof. Let $g_n \cdot \pi \to \pi_1$ with π_1 in the closure $\bar{\theta}$ of θ. Then $g_n \cdot \varkappa_\pi \to \varkappa_{\pi_1}$ but as $\hat{K}$ is discrete this says that $g_n \cdot \varkappa_\pi = \varkappa_\pi$ for all $n \geqslant N$ for some N. But then $g_n g_N^{-1} \cdot \varkappa_{\pi_1} = \varkappa_{\pi_1}$ for all $n \geqslant N$ so that $g_n g_N^{-1} \in S_{\varkappa_{\pi_1}}$. Now $(g_n g_N^{-1}) g_N \cdot \pi \to \pi_1$ as $n \to \infty$ and so as, by hypothesis, the $S_{\varkappa_{\pi_1}}$ quasiorbit in $\hat{G}_0$ containing $g_N \cdot \pi$ is closed, we conclude that there exists $\{h_n\}$ in $S_{\varkappa_{\pi_1}}$ such that $h_n \cdot \pi_1 \to g_N \cdot \pi$. But then $g_N^{-1} h_n \cdot \pi_1 \to \pi$ proving that θ is closed.

Now notice that if θ_1 denotes the $S_{\varkappa_{\pi_1}}$ quasiorbit in $\hat{G}_0$ containing π_1 and $\pi_2 \in \bar{\theta}_1$ then $\varkappa_{\pi_2} = \varkappa_{\pi_1}$ because there exists $\{h_n\}_{n=1}^{\infty} \subseteq S_{\varkappa_{\pi_1}}$ with $h_n \cdot \pi_1 \to \pi_2$ so $\varkappa_{\pi_1} = h_n \cdot \varkappa_{\pi_1} = \varkappa_{\pi_2}$ for sufficiently large n.

Thus the stabilizer $S_{\varkappa_{\pi_1}}$ is constant on the closure of the quasiorbit θ_1 since every representation in $\bar{\theta}_1$ restricts on K to $\varkappa_{\pi_1}$. So we write S_{θ_1} for this common stabilizer and note by the preceding lemma that to prove that the G-quasiorbit containing π is closed, we need only show that θ_1 is closed. Now let σ_{θ_1} denote the Mackey cocycle on S_{θ_1} which measures the obstruction to extending $\varkappa_{\pi_1}$ to G_0. Then we may identify $\bar{\theta}_1$ homeomorphically with a subset of the σ_{θ_1}-dual, $(G_0, \sigma_{\theta_1})\hat{}$ of G_0. (This identification is defined in [10] and proved continuous for example in [7] by noting that any irreducible representation of K is finite-dimensional so that the hypotheses of Theorem 18 of [7] are satisfied.) Denote by $(S_{\theta_1}/K)^{\sim}$ the central extension of S_{θ_1}/K determined by σ_{θ_1} and by $(G_0/K)^{\sim}$ the corresponding extension of G_0/K. Then to show that $\theta_1 = \bar{\theta}_1$ it is sufficient to show that $(S_{\theta_1}/K)^{\sim}$ quasiorbits in the dual of $(G_0/K)^{\sim}$ are closed. To do this we establish first the following result.

PROPOSITION 2.4. *If H is a second countable locally compact nilpotent group whose connected component G_0 is a Lie group then the H-quasiorbits in $\hat{G}_0$ are closed.*

Proof. We begin by noting that G_0 has the same Lie algebra $\mathfrak{g}_0$ as its simply connected covering group and moreover its dual $\hat{G}_0$ is a closed subset of the dual of this covering group. It follows then from the Kirillov conjecture as proved by Brown [1] that $\hat{G}_0$ is homeomorphic to a closed subset of the space of coadjoint orbits of the covering group in the dual $\mathfrak{g}_0^*$ of $\mathfrak{g}_0$. In fact, the coadjoint orbits of the covering group which lie in this subset are just those that are integral orbits for the coadjoint G_0 action (cf. [3] for this terminology and result). Now H acts by conjugation as automorphisms of G_0 and hence, by differentiating, as automorphisms of $\mathfrak{g}_0$. Write $\operatorname{Ad} h$ for this action of $h \in H$ on $\mathfrak{g}_0$ and $\operatorname{Ad}^* h$ for the dual action on $\mathfrak{g}_0^*$. Using the Kirillov conjecture we need only show that the $\operatorname{Ad}^* H$-quasiorbits in $\mathfrak{g}_0^*$ are closed. Note that $(\operatorname{Ad} h - 1)^n = 0$ for sufficiently large n by differentiating

$$1 = [g, [g, [g, \ldots [g, \exp tX] \ldots]$$

(n commutators) with respect to t at $t = 0$ for each $X \in \mathfrak{g}_0$. Consequently there is a basis of $\mathfrak{g}_0^*$ such that $\operatorname{Ad}^* H$ is represented by upper triangular matrices with one's on the diagonal (Kolchin's theorem [12]). The proposition is now a corollary of Theorem 1 of Abels [0] once we remark that an $\operatorname{Ad}^* H$-quasiorbit in $\mathfrak{g}_0^*$ being closed is equivalent to it being a minimal $\operatorname{Ad}^* H$ space.

Returning to the discussion preceding Proposition 2.4 we see that we may now conclude that the $(S_{\theta_1}/K)^{\sim}$ quasiorbits in the dual of $(G_0/K)^{\sim}$ are closed. This, together with the preceding results, may be combined into the main result of this section.

Proposition 2.5. *If G is as in the Theorem of the Introduction then G-quasiorbits in $\hat{G}_0$ are closed.*

§ 3. **Proof of the Theorem.** This section makes considerable use of arguments from [2] to which the reader may refer for more details. Let G be as in the statement of the theorem and adopt the notation of the first paragraph of Section 2. Let $\{H_i\}_{i=1}^n$ be a sequence of subgroups with $H_n = G$, $H_0 = G_0$, $[G, H_i] \subseteq H_{i-1}$ and each H_i of lower nilpotence class than its predecessor H_{i+1}.

Lemma 3.1. *If G-quasiorbits in $\operatorname{Prim} H_{n-1}$ are closed then G has T_1 primitive ideal space.*

Proof. This is immediate from Theorem 2.2.

Lemma 3.2. *The H_i-quasiorbits in $\hat{G}_0$ are closed for all $i = 1, \ldots, n$.*

Proof. Each group H_i satisfies the hypotheses of Proposition 2.5, hence the result.

Proposition 3.3. *The G-quasiorbits in $\operatorname{Prim} H_i$ are closed for all $i = 1, \ldots, n-1$.*

Proof. We induct on the length n of the central series $\{H_i\}_{i=0}^n$. The induction starts at $n = 1$ by Theorem 2.2. We may assume that the result is true for all groups for which the sequence $\{H_i\}$ has length less than n and hence, by Lemma 3.1, that such groups have T_1 primitive ideal space. Secondly we induct on the label i of the ascending central series $\{H_i\}$ noting that the induction starts at $i = 0$ by Proposition 2.5. Thus we may assume that we have a group G with ascending sequence $\{H_i\}_{i=1}^n$ of length n, each H_i having T_1 primitive ideal space and closed quasiorbits in $\operatorname{Prim} H_k$ for $k < i < n$. Moreover, we assume that the G-quasiorbits in $\operatorname{Prim} H_k$ for $k < i$ are closed and now prove as a consequence that G-quasiorbits in $\operatorname{Prim} H_i$ are closed.

With this in mind, let $\theta \subseteq \operatorname{Prim} H_i$ be a G-quasiorbit, $x \in \theta$ and $y \in \bar{\theta}$ and let $\{g_n\} \subseteq G$ be such that $g_n \cdot x \to y$. Both x and y being primitive ideals, define H_i-quasiorbits x_0 and y_0 respectively in $\operatorname{Prim} H_{i-1}$ by restriction (cf. Green [7] or Gootman–Rosenberg [6]). Thus $g_n \cdot x_0 \to y_0$ and by closure of G-quasiorbits in $\operatorname{Prim} H_{i-1}$ there exists a sequence $\{g'_n\}$ in G with $g'_n \cdot y_0 \to x_0$. Let $\Gamma = (H_i/H_{i-1})\hat{}$. Now the ideal induced from a primitive ideal in x_0, or equivalently all of x_0, has as its hull the closed subset $\Gamma \cdot x$ of $\operatorname{Prim} H_i$ (cf. Section 2 of [2] for the definition of the Γ action on $\operatorname{Prim} H_i$ and a similar argument). So by continuity of induction $g'_n \cdot \Gamma \cdot y \to \Gamma \cdot x$. Thus we can find a sequence $\{\gamma_n\}_{n=1}^\infty \subseteq \Gamma$ such that

$$g'_n \cdot \gamma_n \cdot y \to \gamma \cdot x$$

for some $\gamma \in \Gamma$. Observe that as H_{i-1}, H_i are successive terms in a central series the Γ and G actions commute. Therefore there is a sequence $\{\gamma'_n\} \subseteq \Gamma$ with

$$g'_n \cdot \gamma'_n \cdot y \to x.$$

But Γ is compact and so a subsequence of $\{\gamma'_n\}$ converges to some $\gamma_0^{-1} \in \Gamma$. So we have $g_n \cdot y \to \gamma_0 \cdot x$ and hence $\gamma_0 \cdot x \in \overline{\theta(y)}$ where $\theta(y)$ denotes the quasiorbit containing y.

Before continuing we need a result.

Lemma 3.4. *Let G be a second countable locally compact group and N a closed normal subgroup. Then every G-quasiorbit in $\operatorname{Prim} N$ is contained in the closure of a maximal G-quasiorbit.*

Proof. We will state the following argument in slightly greater generality by considering (following Green [7]) the situation where G acts on a second countable totally Baire space X (cf. [7], p. 222 for this notion and the fact that $\operatorname{Prim} N$ is totally Baire). We will show that every G-quasiorbit in X is contained in the closure of a maximal one. Taking the quasiorbit space $(X/G)^{\sim}$ it is enough to show that the partial ordering defined by: $x < y$ if and only if x lies in the closure of y, satisfies the hypotheses of Zorn's lemma. Let A be a chain in X and let B be the closure of A. It is enough to show that B is the closure of a single point. Since B is closed it is a Baire space [7]. Let $\{U_n\colon n \in N\}$ be a basis for the open sets of B. If $x \in U_n$ and $x < y$ then $y \in U_n$. It follows that U_n contains a cofinal subset of B and so is dense in B. Thus $\bigcap_n U_n$ is dense in B. Now if y is any element of $\bigcap_n U_n$, y is dense in B and the result follows.

Remark 3.5. This argument is just a variation on the proof in Dixmier ([4], Corollaire 2) of the existence of minimal primitive ideals.

We return to the proof of the proposition noting that as a consequence of Lemma 3.4 we may as well assume that θ is maximal, for if maximal quasiorbits are closed then all quasiorbits are closed. The Γ action on $\operatorname{Prim} H_i$ clearly takes maximal quasiorbits to maximal quasiorbits. Hence we have $\gamma_0 \cdot \theta$ maximal and $\gamma_0 \cdot \theta \subseteq \overline{\theta(y)}$. But this means that $\gamma_0 \cdot \theta = \theta(y)$ so $\theta(y)$ is maximal. But by the original hypothesis

$$\theta(y) \subseteq \bar{\theta} = \gamma_0^{-1}\theta(y),$$

so that $\theta(y) = \theta$ as required.

Combining the results of this section completes the proof of the theorem.

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Egorov's type convergence in the Dedekind completion of a C^*-algebra

by

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Abstract. The concept of convergence in Egorov's sense for nets in an abelian AW^*-algebra is introduced. We say that an abelian AW^*-algebra A has property E if every order convergent net in A also converges in Egorov's sense to the same limit. It is shown that the Dedekind completion of the hermitian part B_h of a given separable unital C^*-algebra B (regarded as an order unit vector space) satisfies property E if and only if B is abelian and its spectrum contains a dense subset of isolated points.

C^*-algebras have very nice properties as ordered vector spaces, they have not, however, the order completeness property in general. Since the hermitian part of a C^*-algebra is an Archimedean partially ordered vector space, it can be embedded, with preservation of suprema and infima, in a bounded complete vector lattice (the hermitian part of an abelian AW^*-algebra) called the *Dedekind completion* (of the C^*-algebra) (see, for example, [7] and [13]).

Our claim in this note is that this completion is very badly behaved for almost all separable C^*-algebras as far as the order convergence is concerned, in the following sense:

THEOREM. *Let A be a separable C^*-algebra and let $\mathscr{G}$ be the Dedekind completion of the hermitian part of the C^*-algebra A_1 obtained from adjunction of a unit to A, regarded as an order unit vector space. Then any bounded net $\{a_\lambda\}$ in $\mathscr{G}$ which converges to a in the order sense also converges in Egorov's sense (see below) to the same limit a if and only if A is an abelian C^*-algebra whose spectrum contains a dense set of isolated points.*

This is, however, an easy consequence of the following

PROPOSITION. *Keeping the above notations and definitions in mind, $\mathscr{G}$ is atomic (in the sense that it has sufficiently many minimal projections) if and only if A is abelian and its spectrum has a dense set of isolated points.*

Let Z be an abelian AW^*-algebra and Ω the spectrum of Z. If we denote by Z_h the set of all hermitian elements in Z, then Z_h is *-isomorphic to the set $C_r(\Omega)$ of all continuous real-valued functions on Ω. In its natural ordering, $C_r(\Omega)$ is a boundedly complete vector lattice ([2]).