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Kernel estimates for fractional integrals with polynomial weights

by

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Abstract. A proof based on kernel estimates is given for a two weight function norm inequality for fractional integrals. The inequality was previously proved by the authors using a different technique. The present method can also be used to determine the form of the fractional integral operator for general functions for which the norm inequality is valid.

§ 1. Introduction. For $-\infty < x < +\infty$ and $\alpha > 0$, let $(I_\alpha f)(x)$ denote the fractional integral of order α of f defined by $(I_\alpha f)\hat{}(x) = |x|^{-\alpha}\hat{f}(x)$, where "$\hat{}$" denotes the Fourier transform. This definition makes sense if $\hat{f}(x)$ vanishes sufficiently rapidly as $|x| \to 0$ and ∞. For example, f could be any Schwartz function such that $\hat{f}(x) = 0$ near $x = 0$, or such that $(d^k/dx^k)\hat{f}(x) = 0$ at $x = 0$ for $k = 0, 1, \ldots, K$ with K sufficiently large. Also, f could be any (∞, K) atom, by which we mean any bounded f with compact support which satisfies the moment conditions

$$\int_{-\infty}^{\infty} x^k f(x)\,dx = 0, \qquad k = 0, 1, \ldots, K,$$

K sufficiently large.

The following weighted norm inequality for $I_\alpha f$ is a special case of a result proved in [11]. We use the usual notation A_p for the class of weight functions studied in [6], and we say that a weight function $w(x)$ satisfies the *reverse Hölder condition* of order r, $r > 1$, and write $w \in \mathrm{RH}_r$, if

$$\Big(\int_I w(x)^r\,dx/|I|\Big)^{1/r} \leqslant c\int_I w(x)\,dx/|I|$$

for all intervals I with c independent of I. Also, L^p_v denotes the class of f such that

$$\|f\|_{L^p_v} = \Big(\int_{-\infty}^{\infty} |f(x)|^p\, v(x)\ dx\Big)^{1/p} < \infty .$$

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THEOREM A. *Let* $\alpha > 0$, $1 < p < \infty$, $0 \leqslant 1/p - 1/q \leqslant \alpha$, $\beta = \alpha - (1/p - 1/q)$, *and* N *be a nonnegative integer with* $N \geqslant \beta$. *Let* $v(x) = |x|^{Np} w(x)$ *and* $u(x) = |x|^{(N-\beta)q} w(x)^{q/p}$ *where* $w \in A_p$, *and assume in addition when* $q > p$ *that* $w \in \mathrm{RH}_{q/p}$. *Then* $\|I_\alpha f\|_{L^q_u} \leqslant c \|f\|_{L^p_v}$ *for all* (∞, K) *atoms* f.

Here, K must be chosen sufficiently large depending on α and N. We note that the hypothesis $w \in A_p \cap \mathrm{RH}_{q/p}$ $(q \geqslant p)$ is equivalent to $w^{q/p} \in A_{(q/p')+1}$, $1/p + 1/p' = 1$ (see § 2).

The proof given in [11] of this result, as well as its more general versions, relies heavily on the results of [9] and [10]. One purpose of this paper is to give a different proof based on kernel estimates. Another purpose is to identify the form of $I_\alpha f$ for *general* $f \in L^p_v$. It is not hard to describe the form when f is an (∞, K) atom. In fact, if K is large enough, it then follows from [3], p. 194–195, that

$$(I_\alpha f)(x) = c_\alpha \int_{-\infty}^{\infty} f(y) |x-y|^{\alpha-1} dy \quad \text{for } \alpha \neq 1, 3, 5, \ldots,$$

and

$$(I_\alpha f)(x) = c_\alpha \int_{-\infty}^{\infty} f(y) |x-y|^{\alpha-1} \log|x-y| \, dy \quad \text{for } \alpha = 1, 3, 5, \ldots$$

Since the class of f's which are constant multiples of (∞, K) atoms is dense in L^p_v ([9], [10]), it follows from Theorem A that $I_\alpha f$ has an extension by continuity to all of L^p_v. This extension cannot have the form above since for $N \geqslant 1$ there are functions in L^p_v which are not locally integrable: e.g., take $w(x) \equiv 1$ and $f(x) = |x|^{-N} \chi(|x| < 1)$.

We shall consider these questions for weights of a more general form. To describe the results, we need some additional notation. Throughout the paper, Q will denote a polynomial,

$$Q(x) = \prod_{k=1}^{m} (x - a_k)^{\mu_k}, \quad \sum \mu_k = N,$$

with distinct real roots $\{a_k\}$ and order N. Given α, β, p and q as in Theorem A, let

$$\begin{aligned} &v(x) = |Q(x)|^p w(x), \quad \text{and} \\ (1.1) \qquad &u(x) = \left\{ |Q(x)| (1+|x|)^{-\beta} \prod \left(\frac{|x-a_k|}{1+|x-a_k|} \right)^{-\beta_k} \right\}^q w(x)^{q/p}, \\ &\qquad\qquad \beta_k = \min(\mu_k, \beta). \end{aligned}$$

The point to keep in mind is that u is formed by dividing $Q(x)$ by essentially $|x|^\beta$ when $|x|$ is large and by $|x-a_k|^{\beta_k}$ when x is near a_k. Note that for large $|x|$, $u(x) \approx |x|^{(N-\beta)q} w(x)^{q/p}$.

Associated with Q and any sufficiently smooth function $\varphi(x)$ is the interpolating polynomial $\mathscr{P}^Q_\varphi(x)$: i.e., the polynomial in x of degree $N-1$ whose first $\mu_k - 1$ derivatives at a_k coincide with the corresponding derivatives of φ at a_k, $k = 1, \ldots, m$. For more details about $\mathscr{P}$, see § 4 and [10]. For $x \neq a_k$, we will need to consider expressions like $\mathscr{P}^Q_{(x-\cdot)_+^{\alpha-1}}(y)$, where $x_+^{\alpha-1}$ is the function equal to $x^{\alpha-1}$ if $x > 0$ and to 0 if $x \leqslant 0$. If α is a positive integer, note that $\int_{-\infty}^{\infty} f(y)(x-y)_+^{\alpha-1} dy$ is the αth iterated indefinite integral of f. Note also that if the first N moments of f vanish, i.e., if $\int_{-\infty}^{\infty} f(y) y^j dy = 0$ for $j = 0, 1, \ldots, N-1$, then $\int_{-\infty}^{\infty} f(y) \mathscr{P}^Q_\varphi(y) dy = 0$.

We will prove the following result.

THEOREM 1. *Let* $\alpha > 0$, $1 < p < \infty$, $0 \leqslant 1/p - 1/q \leqslant \alpha$, $\beta = \alpha - (1/p - 1/q)$; *and* Q, u, v *and* $\mathscr{P}^Q$ *be as above. Let* $N \geqslant \beta$ *and* $w \in A_p$, *and assume in addition when* $q > p$ *that* $w \in \mathrm{RH}_{q/p}$. *Then if* $f \in L^p_v$, *the expression*

$$(I^+_\alpha f)(x) = \int_{-\infty}^{\infty} f(y) \{(x-y)_+^{\alpha-1} - \mathscr{P}^Q_{(x-\cdot)_+^{\alpha-1}}(y)\} dy$$

converges absolutely for a.e. $x \neq a_k$, *and* $\|I^+_\alpha f\|_{L^q_u} \leqslant c \|f\|_{L^p_v}$ *with* c *independent of* f.

In case $1/p \leqslant \alpha$, the restriction on q should be interpreted as $p \leqslant q < \infty$.

An analogous result holds for the function $I^-_\alpha f$ defined by replacing $(\ )_+^{\alpha-1}$ by $(\ )_-^{\alpha-1}$, where $x_-^{\alpha-1} = |x|^{\alpha-1}$ if $x < 0$ and $x_-^{\alpha-1} = 0$ otherwise. Thus, $(x-y)_+^{\alpha-1}$ and $(x-\cdot)_+^{\alpha-1}$ above can be replaced resp. by $|x-y|^{\alpha-1}$ and $|x-\cdot|^{\alpha-1}$. In the case $\alpha = 1, 3, 5, \ldots$, they can also be replaced by $|x-y|^{\alpha-1} \log|x-y|$ and $|x-\cdot|^{\alpha-1} \log|x-\cdot|$.

The proof of Theorem 1 will depend on making careful estimates on the size of the kernel $(x-y)_+^{\alpha-1} - \mathscr{P}^Q_{(x-\cdot)_+^{\alpha-1}}(y)$. These estimates are proved in § 4, and Theorem 1 itself is proved in § 5. One ingredient of the proof which may be of independent interest is as follows:

THEOREM 2. *If* $1 < p < \infty$, $\alpha > 0$, $0 \leqslant 1/p - 1/q \leqslant \alpha$, $\beta = \alpha - (1/p - 1/q)$ *and* $w \in A_p \cap \mathrm{RH}_{q/p}$, *then*

$$\Big\| \int_{|y| < 2|x|} f(y) |x-y|^{\alpha-1} dy \Big\|_{L^q_{|x|^{-\beta q} w^{q/p}}} \leqslant c \|f\|_{L^p_w}.$$

This result is proved in § 2. The purpose of § 3 is to list some technical lemmas which will be used to prove Theorem 1.

Let us now consider the extension results. We start with the assumption that

$$(1.2) \qquad \|I_\alpha f\|_{L^q_u} \leqslant c \|f\|_{L^p_v}$$

for all f's which are constant multiples of (∞, K) atoms, or with the analogous assumption for $f*\{x_+^{\alpha-1}\}$. Such an estimate was proved in our earlier paper [11] and also follows from Theorem 1. Alternatively, we could assume (1.2) holds for all f in the class $\mathscr{S}_{0,0}$ of Schwartz functions whose Fourier transform has compact support not containing 0. For v as in (1.1) with $w \in A_p$, each of these classes of f's is dense in L_v^p (see [9] and [10]). Thus, $I_\alpha f$ may be extended to all $f \in L_v^p$ by continuity. We denote the extension by $I_\alpha f$ again and refer to it as "the function obtained by extending (1.2)." It is a function in L_u^q. The next theorem, whose proof follows easily from Theorem 1, gives the form of this extension.

THEOREM 3. *With the same assumptions as in Theorem* 1, *if* $\alpha \neq 1, 3, 5, \ldots$ *and* $I_\alpha f$ *is the function obtained by extending* (1.2), *then*

$$I_\alpha f(x) = \int_{-\infty}^{\infty} f(y)\{|x-y|^{\alpha-1} - \mathscr{P}^Q_{|x-\cdot|^{\alpha-1}}(y)\}\,dy$$

for almost every x.

In case $\alpha = 1, 3, \ldots$, the conclusion holds if $|\cdot|^{\alpha-1}$ is replaced by $|\cdot|^{\alpha-1}\log|\cdot|$. Similarly, if $\alpha > 0$ and we assume that the analogue of (1.2) holds for the operator $f*\{x_+^{\alpha-1}\}$ and all f's in either dense subset, then the corresponding extension is the operator $I_\alpha^+ f$ of Theorem 1.

A result concerning the extension of $I_\alpha f$ to the Hardy space H_v^p, $1 < p < \infty$, is also discussed in § 6. Finally, in § 7, a theorem related to the Lusin area integral is given. This result is a simple application of the H_v^p results of [11].

§ 2. Theorem 2. As mentioned in the introduction, the condition that $w \in A_p \cap \mathrm{RH}_{q/p}$, $q \geq p$, is equivalent to $w^{q/p} \in A_{(q/p')+1}$, $1/p + 1/p' = 1$. To see this, recall that $w \in A_p$, $1 < p < \infty$, means

$$\Big(|I|^{-1}\int_I w\,dx\Big)\Big(|I|^{-1}\int_I w^{-1/(p-1)}\,dx\Big)^{p-1} \leq c.$$

Since the opposite inequality (with $c = 1$) is always true, it is easy to see that $w \in A_p \cap \mathrm{RH}_{q/p}$ iff

$$\Big(|I|^{-1}\int_I w^{q/p}\,dx\Big)^{p/q}\Big(|I|^{-1}\int_I w^{-1/(p-1)}\,dx\Big)^{p-1} \leq c.$$

By a simple calculation, this is in turn equivalent to $w^{q/p} \in A_{(q/p')+1}$.

The proof of Theorem 2 uses interpolation, and we will need the following two lemmas.

LEMMA (2.1). *Let* $1 \leq p_1, p_2, q_1, q_2 \leq \infty$ *and* T *be a linear operator such that*

$$\|Tf\cdot k_i\|_{q_i} \leq M_i\|f\cdot u_i\|_{p_i} \quad \textit{for } i = 1, 2.$$

If $0 \leq t \leq 1$ *and* p_t, q_t *are defined by* $1/p_t = (1-t)/p_1 + t/p_2$ *and* $1/q_t = (1-t)/q_1 + t/q_2$, *then*

$$\|Tf\cdot k_1^{1-t}k_2^t\|_{q_t} \leq M_1^{1-t}M_2^t\|f\cdot u_1^{1-t}u_2^t\|_{p_t}.$$

This is Theorem 2 of [8].

LEMMA (2.2). *If* $1 \leq r < p < s < \infty$ *and* $w \in A_p$, *there exist* $u \in A_r$ *and* $v \in A_s$ *such that* $w = u^{(s-p)/(s-r)}v^{(p-r)/(s-r)}$.

Proof. This follows easily from the factorization result of P. Jones [5]. In fact, by [5], we may express $w = w_1^{1-p}w_2$ with $w_1, w_2 \in A_1$. Rewrite this as

$$w = (w_1^{1-r}w_2)^{(s-p)/(s-r)}(w_1^{1-s}w_2)^{(p-r)/(s-r)}.$$

Now letting $u = w_1^{1-r}w_2$ and $v = w_1^{1-s}w_2$, and noting that $u \in A_r$ and $v \in A_s$, we obtain the lemma.

Proof of Theorem 2. The proof is divided into the cases $\alpha < 1/p$ and $\alpha \geq 1/p$. If $\alpha < 1/p$, fix α and p, and let $1/p^* = (1/p) - \alpha$. The restriction on q amounts to $p \leq q \leq p^*$, and the corresponding range of β is $\alpha \geq \beta \geq 0$. We will prove the theorem in this case by interpolating between the extreme values $q = p$ $(\beta = \alpha)$ and $q = p^*$ $(\beta = 0)$. Starting with $q = p$, we claim that

$$\Big\|\int_{|y|<2|x|} f(y)|x-y|^{\alpha-1}\,dy\Big\|_{L^p_{|x|^{-\alpha p}w_2}} \leq c\|f\|_{L^p_{w_2}} \quad \text{if } w_2 \in A_p. \tag{2.3}$$

In fact, the left side of (2.3) is the $L^p_{w_2}$ norm of

$$|x|^{-\alpha}\int_{|y|<2|x|} f(y)|x-y|^{\alpha-1}\,dy,$$

which is majorized by a constant times the Hardy–Littlewood maximal function of f. Thus, (2.3) follows from [6]. On the other hand, for $q = p^*$, we claim that [7] implies

$$\Big\|\int_{|y|<2|x|} f(y)|x-y|^{\alpha-1}\,dy\Big\|_{L^{p^*}_{w_1^{p^*/p}}} \leq c\|f\|_{L^p_{w_1}} \quad \text{if} \quad w_1 \in A_p \cap \mathrm{RH}_{p^*/p}. \tag{2.4}$$

In fact, the same is true even if the integral on the left is extended over all of $(-\infty, \infty)$ since $\alpha < 1/p$ and the condition $w_1 \in A_p \cap \mathrm{RH}_{p^*/p}$ is equivalent to the condition required for this conclusion in [7], namely,

$$\Big(|I|^{-1}\int_I w_1(x)^{p^*/p}\,dx\Big)^{p/p^*}\Big(|I|^{-1}\int_I w_1(x)^{-1/(p-1)}\,dx\Big)^{p-1} \leq c.$$

More generally, as noted above, the condition $w \in A_p \cap \mathrm{RH}_{q/p}$ is the same as $w^{q/p} \in A_{(q/p')+1}$. To apply the interpolation, note that if $p < q < p^*$ then

$$p = (p/p')+1 < (q/p')+1 < (p^*/p')+1.$$

Thus, if $w \in A_{(q/p')+1}$ we may use Lemma (2.2) with r, p and s there taken to be $(p/p')+1$, $(q/p')+1$ and $(p^*/p')+1$, respectively, to write

$$w^{q/p} = u^{(p^*/p' - q/p')/(p^*/p' - p/p')}v^{(q/p' - p/p')/(p^*/p' - p/p')}$$

with $u \in A_p$ and $v \in A_{(p^*/p')+1}$. This is the same as

$$w = u^{(p^*-q)p/(p^*-p)q}v^{(q-p)p/(p^*-p)q}, \quad u \in A_p,\ v \in A_{(p^*/p)+1}.$$

Now pick $t=(p^*-q)p/(p^*-p)q$ and define $w_2=u$ and $w_1=v^{p/p^*}$. A simple computation shows that $w=w_1^{1-t}w_2^t$. Moreover, $0<t<1$, and in fact by using the formulas $\beta=\alpha-(1/p-1/q)$ and $\alpha=1/p-1/p^*$ we have $t=\beta/\alpha$. Also, $w_2(=u)\in A_p$ and $w_1^{p/p^*}(=v)\in A_{(p^*/p)+1}$, so that the inequalities in both (2.3) and (2.4) hold. Now apply Lemma (2.1) with $q_1=p^*$, $q_2=p_1=p_2=p$; $k_1^p=w_1$, $k_2^p=|x|^{-\alpha p}w_2$, $u_1^p=w_1$, $u_2^p=w_2$ and t as above. Then $p_t=p$ and

$$\begin{aligned}1/q_t&=(1-t)/p^*+t/p=1/p^*+t(1/p-1/p^*)\\&=(1/p-\alpha)+(\beta/\alpha)(\alpha)=1/p-\alpha+\beta=1/q.\end{aligned}$$

Thus, $q_t=q$. Also, $(k_1^{1-t}k_2^t)^q=(w_1^{(1-t)/p}|x|^{-\alpha t}w_2^{t/p})^q=|x|^{-\beta q}w^{q/p}$ and $(u_1^{1-t}u_2^t)^p=(w_1^{(1-t)/p}w_2^{t/p})^q=w$. This completes the proof of the case $\alpha<1/p$.

It remains to consider the case $\alpha\geqslant 1/p$, in which the q range should be interpreted as $p\leqslant q<\infty$. Fix p, q and w and pick $\alpha_0<1/p$ so close to $1/p$ that the index p^* defined by $1/p^*=1/p-\alpha_0$ satisfies $p\leqslant q\leqslant p^*$. Thus, $0\leqslant 1/p-1/q\leqslant\alpha_0$, and by the case already considered we have

$$\Big\|\int_{|y|<2|x|}f(y)|x-y|^{\alpha_0-1}\,dy\Big\|_{L^q_{|x|^{-\beta_0 q}w^{q/p}}}\leqslant c\|f\|_{L^p_w},\quad \beta_0=\alpha_0-(1/p-1/q). \tag{2.5}$$

Since $\alpha>\alpha_0$ and $|y|<2|x|$,

$$|x-y|^{\alpha-1}=|x-y|^{\alpha-\alpha_0}|x-y|^{\alpha_0-1}\leqslant c|x|^{\alpha-\alpha_0}|x-y|^{\alpha_0-1}.$$

Therefore

$$\int_{|y|<2|x|}|f(y)||x-y|^{\alpha-1}\,dy\leqslant c|x|^{\alpha-\alpha_0}\int_{|y|<2|x|}|f(y)||x-y|^{\alpha_0-1}\,dy,$$

and

$$\Big\|\int_{|y|<2|x|}f(y)|x-y|^{\alpha-1}\,dy\Big\|_{L^q_{|x|^{-\beta q}w^{q/p}}}\leqslant c\Big\|\int_{|y|<2|x|}f(y)|x-y|^{\alpha_0-1}\,dy\Big\|_{L^q_{|x|^{(\alpha-\alpha_0)q-\beta q}w^{q/p}}}.$$

Noting that $(\alpha-\alpha_0)q-\beta q=-\beta_0 q$ and using (2.5), we obtain the desired estimate. This completes the proof of Theorem 2.

§ 3. **Lemmas for Theorem 1.** In this section we list some technical lemmas which will be used in § 5 to prove Theorem 1. The first two are versions of Hardy's inequality. Their proofs follow easily from the results of [1].

LEMMA (3.1). *If $1<p\leqslant q<\infty$, then*

$$\Big\|\int_{|y|>|x|}f(y)\,dy\Big\|_{L^q_u}\leqslant c\|f\|_{L^p_v}$$

for all $f\geqslant 0$ if

$$\Big(\int_{|x|<r}u(x)\,dx\Big)^{1/q}\Big(\int_{|x|>r}v(x)^{-p'/p}\,dx\Big)^{1/p'}\leqslant c,\qquad r>0.$$

We will also use the compact version of this lemma: if $1<p\leqslant q<\infty$ and $0<R<\infty$, then

$$\Big(\int_{|x|<R}\Big[\int_{|x|<|y|<R}f(y)\,dy\Big]^q u(x)\,dx\Big)^{1/q}\leqslant c\Big(\int_{|x|<R}f(x)^p v(x)\,dx\Big)^{1/p}$$

for all $f\geqslant 0$ if

$$\Big(\int_{|x|<r}u(x)\,dx\Big)^{1/q}\Big(\int_{r<|x|<R}v(x)^{-p'/p}\,dx\Big)^{1/p'}\leqslant c,\qquad 0<r<R. \tag{3.2}$$

This follows from Lemma (3.1) by letting $u=0$, $f=0$ and $v=\infty$ for $|x|>R$. (In Lemma (3.1), $0\cdot\infty$ should be interpreted as 0.)

LEMMA (3.3). *If $1<p\leqslant q<\infty$, then*

$$\Big\|\int_{|y|<|x|}f(y)\,dy\Big\|_{L^q_u}\leqslant c\|f\|_{L^p_v}$$

for $f\geqslant 0$ if

$$\Big(\int_{|x|>r}u(x)\,dx\Big)^{1/q}\Big(\int_{|x|<r}v(x)^{-p'/p}\,dx\Big)^{1/p'}\leqslant c,\qquad r>0.$$

We shall also use the following lemma.

LEMMA (3.4). *If $1<p<\infty$, $p\leqslant q<\infty$ and $w\in A_p\cap \mathrm{RH}_{q/p}$, then for $\delta\geqslant -(1/p-1/q)$,*

$$\int_{|x|>r}\frac{w(x)^{-1/(p-1)}}{|x|^{(1+\delta)p'}}\,dx\leqslant cr^{-\delta p'}\Big(\int_{|x|<r}w(x)\,dx\Big)^{-1/(p-1)},$$

and for $\delta<-(1/p-1/q)$,

$$\int_{r<|x|<R}\frac{w(x)^{-1/(p-1)}}{|x|^{(1+\delta)p'}}\,dx\leqslant c_R r^{(1/p-1/q)p'}\Big(\int_{|x|<r}w(x)\,dx\Big)^{-1/(p-1)},$$

where $c_R=cR^{[-\delta-1/p+1/q]p'}$ and c is independent of r.

Proof. The hypothesis on w is that $w^{q/p}\in A_{(q/p')+1}$. Since for any W, the statements $W\in A_s$ and $W^{-1/(s-1)}\in A_{s'}$ are the same, it follows that $w^{-1/(p-1)}\in A_{(p'/q)+1}$. If we then apply the fact (see (2.3) of [4]) that $W\in A_s$ implies

$$\int_{|x|>r}\frac{W(x)}{|x|^s}\,dx\leqslant c\frac{1}{r^s}\Big(\int_{|x|<r}W(x)\,dx\Big),\qquad r>0, \tag{3.5}$$

we can prove the first part of the lemma. Instead of giving the details, however, we will give a direct proof of the first part of the lemma which can also be used to prove the second part.

We have

$$\int_{r<|x|<R} \frac{w(x)^{-1/(p-1)}}{|x|^{(1+\delta)p'}}\,dx \leqslant c \sum_{\substack{k\geqslant 0\\ 2^k r\leqslant R}} (2^k r)^{-(1+\delta)p'} \int_{|x|\approx 2^k r} w(x)^{-1/(p-1)}\,dx$$

$$\leqslant c \sum_{\substack{k\geqslant 0\\ 2^k r\leqslant R}} (2^k r)^{-\delta p'} \Big(\int_{|x|\approx 2^k r} w(x)\,dx\Big)^{-1/(p-1)}$$

since $w\in A_p$ implies

$$\int_{|x|\approx 2^k r} w(x)^{-1/(p-1)}\,dx \leqslant c(2^k r)^{p'} \Big(\int_{|x|\approx 2^k r} w(x)\,dx\Big)^{-1/(p-1)}.$$

We will show that $w\in \mathrm{RH}_{q/p}$, $q>p$, implies

$$\int_{tI} w\,dx \geqslant ct^{1-p/q+\varepsilon}\int_I w\,dx, \qquad t>1, \tag{3.6}$$

for some $\varepsilon>0$. To see this, start with the inequality

$$\Big(\frac{1}{|tI|}\int_{tI} w(x)^{q/p}\,dx\Big)^{p/q} \leqslant c\frac{1}{|tI|}\int_{tI} w(x)\,dx.$$

Restricting the integration on the left to I, we get

$$\frac{1}{t^{p/q}}\Big(\frac{1}{|I|}\int_I w(x)^{q/p}\,dx\Big)^{p/q} \leqslant c\frac{1}{t}\cdot\frac{1}{|I|}\int_{tI} w(x)\,dx,$$

so that by Hölder's inequality,

$$\frac{1}{t^{p/q}}\cdot\frac{1}{|I|}\int_I w(x)\,dx \leqslant c\frac{1}{t}\cdot\frac{1}{|I|}\int_{tI} w(x)\,dx.$$

This is equivalent to (3.6) with $\varepsilon=0$. The fact that (3.6) holds for some $\varepsilon>0$ now follows since $w\in \mathrm{RH}_{q/p}$ implies $w\in \mathrm{RH}_s$ for some $s>q/p$. In case $q=p$, we obtain (3.6) from the fact that $w\in A_p$ implies $w\in \mathrm{RH}_\eta$ for some $\eta>1$.

Thus, the last sum above is at most

$$c\sum_{\substack{k\geqslant 0\\ 2^k r\leqslant R}} (2^k r)^{-\delta p'}\Big(2^{k(1-p/q+\varepsilon)}\int_{|x|<r} w(x)\,dx\Big)^{-1/(p-1)}$$

$$= cr^{-\delta p'}\Big(\int_{|x|<r} w(x)\,dx\Big)^{-1/(p-1)} \sum_{\substack{k\geqslant 0\\ 2^k r\leqslant R}} (2^k)^{(-\delta-1/p+1/q-\varepsilon/p)p'}.$$

If $-\delta-1/p+1/q\leqslant 0$, this last sum converges even if $R=\infty$, and the first part of the lemma follows. If $-\delta-1/p+1/q>0$, we drop the ε above and obtain that the resulting sum $\approx (R/r)^{(-\delta-1/p+1/q)p'}$. The second part of the lemma follows easily.

§ 4. The kernel estimates. Let $Q(x)=\prod_{k=1}^{m}(x-a_k)^{\mu_k}$ where a_k is real, $a_1<a_2<\ldots<a_m$, μ_k is a positive integer and $\sum\mu_k=N$. Let $\mathscr{P}_\varphi^Q(x)$ be the interpolating polynomial for φ based on Q, i.e.

$$\mathscr{P}_\varphi^Q(x)=Q(x)\mathscr{D}_u^Q[\varphi(u)/(x-u)]$$

where $\mathscr{D}_u^Q$ is the distribution with $\mathscr{D}_u^Q[1/(x-u)]=1/Q(x)$. Specifically, if

$$\frac{1}{Q(x)}=\sum_{k=1}^{m}\sum_{l=1}^{\mu_k}\frac{A_{k,l}}{(x-a_k)^l}$$

is the partial fraction decomposition of $1/Q$, then

$$\mathscr{D}^Q=\sum_{k=1}^{m}\sum_{l=1}^{\mu_k}\frac{A_{k,l}}{(l-1)!}\delta_{a_k}^{(l-1)},$$

where $\delta_{a_k}^{(l-1)}$ is the $(l-1)$th derivative of the δ-function at a_k (see [10]).

For $k_0=1,\ldots,m$, let

$$E_{k_0}=\{y\colon |y-a_{k_0}|=\min_k|y-a_k|\}.$$

Thus, $y\in E_{k_0}$ means that y is closer to a_{k_0} than to any other a_k. For $y\in E_{k_0}$, we will consider the regions

(1) $|x-a_{k_0}|\leqslant\frac{3}{4}|y-a_{k_0}|$,
(2) $\frac{3}{4}|y-a_{k_0}|\leqslant|x-a_{k_0}|\leqslant\frac{4}{3}|y-a_{k_0}|$,
(3) $\frac{4}{3}|y-a_{k_0}|\leqslant|x-a_{k_0}|$,

intersected with

(a) $x\leqslant -2R$,
(b) $-2R\leqslant x\leqslant 2R$,
(c) $x\geqslant 2R$,

where R is chosen so that $-R<a_1<\ldots<a_m<R$. In this way, we obtain nine regions denoted (1a), (1b), (1c), (2a), (2b), (2c), (3a), (3b), (3c).

LEMMA (4.1). *Let Q, $\mathscr{P}_\varphi=\mathscr{P}_\varphi^Q$ and E_{k_0} be as defined above. If $y\in E_{k_0}$ and x, $y\notin\{a_k\}$ then $|(x-y)_+^{\alpha-1}-\mathscr{P}_{(x-\cdot)_+^{\alpha-1}}(y)|$ is bounded by a constant times the following functions in the indicated regions:*

(1a) $|y-a_{k_0}|^{\alpha-1}$,

(1b) $|y-a_{k_0}|^{\alpha-1}+\dfrac{|Q(y)|}{1+|y-a_{k_0}|}+\dfrac{|Q(y)|}{|Q(x)|}\cdot\dfrac{|x-a_{k_0}|^\alpha}{|y-a_{k_0}|}$,

(1c) $|y-a_{k_0}|^{\alpha-1}+\dfrac{|Q(y)|}{|Q(x)|}\cdot\dfrac{|x-a_{k_0}|^\alpha}{|y-a_{k_0}|}$,

(2a) $|x-y|^{\alpha-1}$,

(2b) $|x-y|^{\alpha-1}+|Q(y)|+\dfrac{|Q(y)|}{|Q(x)|}\cdot\dfrac{|x-a_{k_0}|^{\alpha}}{|y-a_{k_0}|}$,

(2c) $|x-y|^{\alpha-1}+\dfrac{|Q(y)|}{|Q(x)|}\dfrac{|x-a_{k_0}|^{\alpha}}{|y-a_{k_0}|}$,

(3a) 0,

(3b) $\dfrac{|Q(y)|}{|Q(x)|}\{|x-a_{k_0}|^{\alpha-1}\chi_{E_{k_0}}(x)+\sum_k(|Q(x)|+|x-a_k|^{\alpha})\chi_{E_k}(x)\}$

(3c) $\dfrac{|Q(y)|}{|Q(x)|}|x-a_{k_0}|^{\alpha-1}$.

These estimates can be rewritten in different but equivalent forms. For example, in (1b), $|Q(x)|\approx|x-a_{k_0}|^{\mu_{k_0}}$ since x is bounded and $x\in E_{k_0}$. Similar reasoning shows that $|Q(x)|\approx|x-a_{k_0}|^{\mu_{k_0}}$ in (2b), and that in (3b) we have $|Q(x)|\approx|x-a_k|^{\mu_k}$ if $x\in E_k$. Furthermore, for any case of type (2), $|Q(y)|\approx|Q(x)|$ as can be seen by considering the cases when $|x|$ and $|y|$ are both large or both bounded. Note also that if $|x|$ is large then $|Q(x)|\approx|x|^N$; similarly, $|Q(y)|\approx|y|^N$ for large $|y|$.

Proof. We may assume without loss of generality that $a_{k_0}=0$ since $\mathscr{P}$ has the translation property $\mathscr{P}_{\varphi}^{Q}(y-a)=\mathscr{P}_{\varphi(\cdot-a)}^{Q(\cdot-a)}(y)$. In all cases except (3b) and (3c), we will estimate $(x-y)_+^{\alpha-1}$ and $|\mathscr{P}_{(x-\cdot)_+^{\alpha-1}}(y)|$ separately.

In case (a), $(x-\cdot)_+^{\alpha-1}$ is zero near every a_k, so that $\mathscr{P}_{(x-\cdot)_+^{\alpha-1}}\equiv 0$. Hence, we have only to estimate $(x-y)_+^{\alpha-1}$ for (1a), (2a) and (3a). For (1a), $(x-y)_+^{\alpha-1}\leqslant|x-y|^{\alpha-1}\leqslant c|y|^{\alpha-1}$ since $|y|\geqslant\frac{4}{3}|x|$. For (2a), we have simply $(x-y)_+^{\alpha-1}\leqslant|x-y|^{\alpha-1}$. For (3a), since $x\leqslant-2R$ and $|y|\leqslant\frac{3}{4}|x|$, we have $x-y<0$, so that $(x-y)_+^{\alpha-1}=0$. This completes all cases of type (a).

We now consider cases of type (b). For (1b), $|y|\geqslant\frac{4}{3}|x|$, so that $(x-y)_+^{\alpha-1}\leqslant c|y|^{\alpha-1}$ as above. For (2b), $(x-y)_+^{\alpha-1}\leqslant|x-y|^{\alpha-1}$. Now consider (1b) and (2b) for $\mathscr{P}_{(x-\cdot)_+^{\alpha-1}}(y)$. Write

$$\mathscr{P}_{(x-\cdot)_+^{\alpha-1}}(y)=Q(y)\,\mathscr{D}_u\left(\frac{(x-u)_+^{\alpha-1}}{y-u}\right),$$

and $\mathscr{D}_u=\mathscr{D}_u^0+\mathscr{D}_u^1$ where $\mathscr{D}_u^0$ is supported at $0\ (=a_{k_0})$ and $\mathscr{D}_u^1$ is supported at the other zeros of Q. In (1b) and (2b), x is bounded and $|x|\leqslant\frac{4}{3}|y|$. Since y is closer to 0 than to any other a_k, it follows from $|x|\leqslant\frac{4}{3}|y|$ that $|x-a_k|\geqslant c>0$ for $k\neq k_0$. Thus, since x is also bounded, $c\leqslant|x-a_k|\leqslant c^{-1}$ for $k\neq k_0$, and we obtain

$$\left|\mathscr{D}_u^1\left(\frac{(x-u)_+^{\alpha-1}}{y-u}\right)\right|\leqslant c\max_{\substack{j,k\\0\leqslant j\leqslant\mu_k-1\\k\neq k_0}}\frac{1}{|y-a_k|^{1+j}}.$$

The expression on the right is at most $c/(1+|y|)$ since $y\in E_{k_0}$, and therefore,

$$\left|Q(y)\,\mathscr{D}_u^1\left(\frac{(x-u)_+^{\alpha-1}}{y-u}\right)\right|\leqslant c\frac{|Q(y)|}{1+|y|}.$$

For $\mathscr{D}_u^0$, we have

$$\left|\mathscr{D}_u^0\left(\frac{(x-u)_+^{\alpha-1}}{y-u}\right)\right|\leqslant c\max_{\substack{i,j\\0\leqslant i+j\leqslant\mu_{k_0}-1}}|x|^{\alpha-i-1}|y|^{-j-1}.$$

Since $|y|>\frac{3}{4}|x|$, $|x|^{\alpha-i-1}|y|^{-j-1}\leqslant c|x|^{\alpha-i-j-1}|y|^{-1}$, which is bounded by $c|x|^{\alpha-\mu_{k_0}}|y|^{-1}$ since x is bounded and $i+j\leqslant\mu_{k_0}-1$. Finally, $|x|^{-\mu_{k_0}}\approx|Q(x)|^{-1}$ since x is bounded, $y\in E_{k_0}$ and $|x|<\frac{4}{3}|y|$, and we obtain

$$\left|Q(y)\,\mathscr{D}_u^0\left(\frac{(x-u)_+^{\alpha-1}}{y-u}\right)\right|\leqslant c\frac{|Q(y)|}{|Q(x)|}\cdot\frac{|x|^{\alpha}}{|y|}.$$

This completes the estimation of (1b) and (2b).

For (3b), write

$$\begin{aligned}(x-y)_+^{\alpha-1}-\mathscr{P}_{(x-\cdot)_+^{\alpha-1}}(y)&=Q(y)\,\mathscr{D}_u\left[\frac{(x-y)_+^{\alpha-1}-(x-u)_+^{\alpha-1}}{y-u}\right]\\&=(x-y)_+^{\alpha-1}\,Q(y)\,\mathscr{D}_u^1\left(\frac{1}{y-u}\right)\\&\quad-Q(y)\,\mathscr{D}_u^1\left(\frac{(x-u)_+^{\alpha-1}}{y-u}\right)\\&\quad+Q(y)\,\mathscr{D}_u^0\left[\frac{(x-y)_+^{\alpha-1}-(x-u)_+^{\alpha-1}}{y-u}\right]\end{aligned}$$

where $\mathscr{D}_u^0$ and $\mathscr{D}_u^1$ are defined as before. We now have $|x|\geqslant\frac{4}{3}|y|$ and x is bounded. Since $|x|\geqslant\frac{4}{3}|y|$, $(x-y)_+^{\alpha-1}\leqslant c|x|^{\alpha-1}$, and therefore since $|\mathscr{D}_u^1[1/(y-u)]|\leqslant c$, the first term on the right above is at most $c|x|^{\alpha-1}|Q(y)|$ in absolute value. This is certainly bounded by $c|x|^{\alpha-1}|Q(y)|/|Q(x)|$ since x is bounded, and if $x\in E_k$, $k\neq k_0$, it is bounded by $c|Q(y)|$ since $|x|$ is bounded away from 0 and ∞. Thus the first term on the right above satisfies the desired estimate for (3b).

For the second term, if $x\in E_k$,

$$\begin{aligned}\left|\mathscr{D}_u^1\left(\frac{(x-u)_+^{\alpha-1}}{y-u}\right)\right|&\leqslant c[1+\max_{\substack{j\\0\leqslant j\leqslant\mu_k-1}}|x-a_k|^{\alpha-1-j}]\\&\leqslant c(1+|x-a_k|^{\alpha-\mu_k})\leqslant c\left(1+\frac{|x-a_k|^{\alpha}}{|Q(x)|}\right)\end{aligned}$$

since x is bounded and $|Q(x)|\approx|x-a_k|^{\mu_k}$ for bounded x in E_k. Thus,

$$\left|Q(y)\mathscr{D}_u^1\left(\frac{(x-u)_+^{\alpha-1}}{y-u}\right)\right| \leqslant c\frac{|Q(y)|}{|Q(x)|}(|Q(x)|+|x-a_k|^\alpha)$$

for bounded $x\in E_k$.

The remaining part of (3b) is

$$Q(y)\mathscr{D}_u^0\left[\frac{(x-y)_+^{\alpha-1}-(x-u)_+^{\alpha-1}}{y-u}\right].$$

If $x<0$, then for u near 0 we have $(x-u)_+^{\alpha-1}=0$. Also, since $|x|\geqslant\frac{4}{3}|y|$, we have $x-y<0$ if $x<0$, so that $(x-y)_+^{\alpha-1}=0$. Thus,

$$\frac{(x-y)_+^{\alpha-1}-(x-u)_+^{\alpha-1}}{y-u}=0$$

for u near 0 if $x<0$, and therefore the expression to be estimated is 0 if $x<0$. If instead $x>0$, then $x-y>0$ if $|x|\geqslant\frac{4}{3}|y|$, and $x-u>0$ for u near 0. Thus,

$$\frac{(x-y)_+^{\alpha-1}-(x-u)_+^{\alpha-1}}{y-u}=\frac{(x-y)^{\alpha-1}-(x-u)^{\alpha-1}}{y-u}$$

$$=-\frac{1}{y-u}\int_0^1\frac{d}{ds}[\{x-y-s(u-y)\}^{\alpha-1}]\,ds=-(\alpha-1)\int_0^1\{x-y-s(u-y)\}^{\alpha-2}\,ds.$$

Note that when $u=0$, $|x-y-s(u-y)|=|x-(1-s)y|\approx|x|$ since $|x|\geqslant\frac{4}{3}|y|$. Thus

$$\left|\frac{d^j}{du^j}\left[\frac{(x-y)_+^{\alpha-1}-(x-u)_+^{\alpha-1}}{y-u}\right]_{u=0}\right|\leqslant c|x|^{\alpha-2-j}.$$

Since x is bounded, it follows that

$$\left|Q(y)\mathscr{D}_u^0\left[\frac{(x-y)_+^{\alpha-1}-(x-u)_+^{\alpha-1}}{y-u}\right]\right|\leqslant c|Q(y)|\,|x|^{\alpha-1-\mu_{k_0}}.$$

The right side is at most $c|Q(y)|\,|x|^{\alpha-1}/|Q(x)|$ if $x\in E_{k_0}$ since then $|Q(x)|\approx|x|^{\mu_{k_0}}$ (x is also bounded), and is at most $c|Q(y)|$ otherwise since $|x|$ is then bounded away from 0 and ∞. This completes the estimation for (3b).

For the (c) cases, we have $x>2R$. For (1c), $|x|\leqslant\frac{3}{4}|y|$ and consequently $(x-y)_+^{\alpha-1}\leqslant|x-y|^{\alpha-1}\approx|y|^{\alpha-1}$, while for (2c), we simply have $(x-y)_+^{\alpha-1}\leqslant|x-y|^{\alpha-1}$. To estimate $\mathscr{P}_{(x-\cdot)_+^{\alpha-1}}(y)$ for (1c) and (2c), we will use the fact that $\mathscr{P}_{u^j}(y)=y^j$ if $0\leqslant j\leqslant N-1$. For $x>2R$ and $|u|<R$, we apply the Maclaurin expansion of $(x-u)^{\alpha-1}$ as a function of u to obtain

$$(x-u)_+^{\alpha-1}=(x-u)^{\alpha-1}=\sum_{j=0}^{N-1}c_jx^{\alpha-1-j}u^j+R(x,u),$$

where

$$R(x,u)=\frac{1}{(N-1)!}\int_0^1(1-s)^{N-1}\frac{d^N}{ds^N}[(x-su)^{\alpha-1}]\,ds$$
$$=c_{N,\alpha}\int_0^1(1-s)^{N-1}u^N(x-su)^{\alpha-1-N}\,ds.$$

We will first show that for x large compared to $|u|$,

$$\left|\frac{\partial^i}{\partial u^i}R(x,u)\right|\leqslant c|u|^{N-i}|x|^{\alpha-1-N}.$$

To see this, apply Leibniz's formula and note that if $i=i'+i''$, then

$$\int_0^1(1-s)^{N-1}|u|^{N-i'}s^{i''}|x-su|^{\alpha-1-N-i''}\,ds\leqslant c|u|^{N-i'}|x|^{\alpha-1-N-i''}$$
$$\leqslant c|u|^{N-i}|x|^{\alpha-1-N}$$

since $|x|>2|u|$.

It follows that since all the a_k's satisfy $|a_k|<R$,

$$\mathscr{P}_{(x-\cdot)_+^{\alpha-1}}(y)=\sum_{j=0}^{N-1}c_jx^{\alpha-1-j}y^j+\mathscr{P}_{R(x,\cdot)}(y)$$

with

$$|\mathscr{P}_{R(x,\cdot)}(y)|=|Q(y)|\left|\mathscr{D}_u\left(\frac{R(x,u)}{y-u}\right)\right|\leqslant c|Q(y)|\,|x|^{\alpha-1-N}|y|^{-1},$$

since in (1c) and (2c), the fact that $|x|$ is large implies that $|y|$ is also large. Hence, due to $|Q(x)|\approx|x|^N$,

$$|\mathscr{P}_{R(x,\cdot)}(y)|\leqslant c\frac{|Q(y)|}{|Q(x)|}\cdot\frac{|x|^\alpha}{|y|}\cdot\frac{1}{|x|}\leqslant c\frac{|Q(y)|}{|Q(x)|}\cdot\frac{|x|^\alpha}{|y|}.$$

Moreover,

$$\left|\sum_{j=0}^{N-1}c_jx^{\alpha-1-j}y^j\right|\leqslant\sum_{j=0}^{N-1}|c_j|\left(\frac{|y|}{|x|}\right)^j|x|^{\alpha-1}\leqslant c\left(\frac{|y|}{|x|}\right)^{N-1}|x|^{\alpha-1}$$

because $|y|/|x|\geqslant 3/4$ in (1c) and (2c). Finally, since $|Q(x)|\approx|x|^N$ and $|Q(y)|\approx|y|^N$,

$$(|y|/|x|)^{N-1}|x|^{\alpha-1}\leqslant c|Q(y)|\,|x|^\alpha/|Q(x)|\,|y|.$$

This completes the estimation of (1c) and (2c).

Only case (3c) remains to be considered. In this case, $x > 2R$ and $x > \frac{4}{3}|y|$. The expansion used just above holds with u replaced by y:

$$(x-y)_+^{\alpha-1} = (x-y)^{\alpha-1} = \sum_{j=0}^{N-1} c_j x^{\alpha-1-j} y^j + R(x, y),$$

$$|R(x, y)| \leqslant c|x|^{\alpha-1-N}|y|^N.$$

We claim that $|R(x, y)| \leqslant c|Q(y)|\,|x|^{\alpha-1}/|Q(x)|$. Since $|x|$ is large, $|Q(x)| \approx |x|^N$, and so the claim will be established if $|y|^N \leqslant c|Q(y)|$. This is obvious if $|y|$ is large, while if $|y|$ is bounded, it follows from $|y|^N \leqslant c|y|^{\mu_{k_0}} \leqslant c|Q(y)|$, the last inequality being due to $|y-a_k| \geqslant c > 0$ if $k \neq k_0$. Thus, $R(x, y)$ satisfies the desired estimate, and since

$$(x-y)_+^{\alpha-1} - \mathscr{P}_{(x-\cdot)_+^{\alpha-1}}(y) = R(x, y) - \mathscr{P}_{R(x,\cdot)}(y),$$

it follows that we only need to estimate $\mathscr{P}_{R(x,\cdot)}(y)$. Write

$$|\mathscr{P}_{R(x,\cdot)}(y)| \leqslant |Q(y)| \left|\mathscr{D}_u^1\left(\frac{R(x, u)}{y-u}\right)\right| + |Q(y)| \left|\mathscr{D}_u^0\left(\frac{R(x, u)}{y-u}\right)\right|.$$

We have

$$\left|\mathscr{D}_u^1\left(\frac{R(x, u)}{y-u}\right)\right| \leqslant c \max_{\substack{i,k \\ 0 \leqslant i \leqslant \mu_k - 1 \\ k \neq k_0}} \left|\left(\frac{\partial^i}{\partial u^i} R(x, u)\Big|_{u=a_k}\right)\right|$$

since $|y-a_k| > c > 0$ if $k \neq k_0$. By our earlier estimate, this is at most $c|x|^{\alpha-1-N} \approx c|x|^{\alpha-1}/|Q(x)|$. Finally, $\mathscr{D}_u^0(R(x, u)/(y-u)) = 0$ since $(\partial^i/\partial u^i) R(x, u)|_{u=0} = 0$ for $i = 0, 1, \ldots, N-1$. Thus,

$$|\mathscr{P}_{R(x,\cdot)}(y)| \leqslant c|Q(y)|\,|x|^{\alpha-1}/|Q(x)|,$$

and the lemma follows.

§ 5. Proof of Theorem 1. Under the hypotheses of Theorem 1, it is clear from definition (see (1.1)) that v is locally integrable. Moreover, as we will show, $u(x) \leqslant cw(x)^{q/p}$ for bounded x, and therefore u is also locally integrable. In fact, from the definition of u, for bounded x,

$$u(x) \leqslant c|Q(x)|^q \prod |x-a_k|^{-q\min(\mu_k,\beta)} w(x)^{q/p}$$
$$\leqslant cw(x)^{q/p},$$

since $|Q(x)|^q$ contains the factor $|x-a_k|^{q\mu_k}$ for each k.

Also, both u and v belong to the class $A_\infty = \bigcup_{p>1} A_p$, as we now show. By considering the cases when $|x|$ is large or small and using $N-\beta \geqslant 0$, we see

$$u(x) \approx \prod \left(\frac{|x-a_k|}{1+|x-a_k|}\right)^{(\mu_k-\beta_k)q} (1+|x|^{(N-\beta)q})\, w(x)^{q/p}.$$

Thus, $u \in A_\infty$ by Lemma (6.5) of [11] since $w^{q/p} \in A_\infty$, $(\mu_k - \beta_k)q \geqslant 0$ and $(N-\beta)q \geqslant 0$. The fact that $v \in A_\infty$ can be deduced immediately from the same lemma.

We will estimate the norm of $I_\alpha^+ f(x)$ by considering individually the nine expressions obtained by decomposing the domain of integration into the regions (1a), (1b), (1c),... Writing

$$I_\alpha^+ f(x) = \int_{-\infty}^{\infty} f(y)\{(x-y)_+^{\alpha-1} - \mathscr{P}_{(x-\cdot)_+^{\alpha-1}}(y)\}\, dy$$

where $\mathscr{P} = \mathscr{P}^Q$, we may assume without loss of generality that $y \in E_{k_0}$ by splitting $f = \sum_{k=1}^{m} f_k$, $f_k = f\chi_{E_k}$, and considering each f_k separately. We may also assume as in the proof of Lemma (4.1) that $a_{k_0} = 0$.

We first consider regions of type (1). For (1a), the expression to be estimated is

$$\Big(\int_{x<-2R} \Big\{\int_{|y|>(4/3)|x|} |f(y)|\,|y|^{\alpha-1}\, dy\Big\}^q u(x)\, dx\Big)^{1/q}.$$

Here,

$$u(x) = |Q(x)|^q (1+|x|)^{-\beta q} \prod \left(\frac{|x-a_k|}{1+|x-a_k|}\right)^{-q\beta_k} w(x)^{q/p},$$

and since $|x|$ and $|y|$ are both large in (1a), we have $u(x) \approx |x|^{(N-\beta)q} w(x)^{q/p}$ and $|Q(y)| \approx |y|^N$. Thus, we must estimate

$$\Big(\int_{x<-2R} \Big\{\int_{|y|>(4/3)|x|} |f(y)Q(y)|\,|y|^{\alpha-N-1}\, dy\Big\}^q |x|^{(N-\beta)q} w(x)^{q/p}\, dx\Big)^{1/q}.$$

Enlarging the x-domain to $(-\infty, \infty)$ and applying Lemma (3.1), we will obtain the desired bound $c\big(\int_{-\infty}^{\infty} |fQ|^p w\, dx\big)^{1/p} = c\|f\|_{L_v^p}$ provided we show that

$$(5.1)\qquad \Big(\int_{|x|<r} |x|^{(N-\beta)q} w(x)^{q/p}\, dx\Big)^{1/q} \left(\int_{|x|>r} \left[\frac{w(x)^{1/p}}{|x|^{\alpha-N-1}}\right]^{-p'} dx\right)^{1/p'} \leqslant c$$

for $r > 0$. To estimate the second factor in this product, apply the first part of Lemma (3.4) with $\delta = N-\alpha$. The requirement that $\delta \geqslant -(1/p-1/q)$ amounts to $N \geqslant \alpha-(1/p-1/q)$, that is, to $N \geqslant \beta$, which is true by hypothesis. Thus,

$$\int_{|x|>r} \left[\frac{w(x)^{1/p}}{|x|^{\alpha-N-1}}\right]^{-p'} dx \leqslant cr^{(\alpha-N)p'} \Big(\int_{|x|<r} w\, dx\Big)^{-1/(p-1)}.$$

Furthermore, since $N-\beta \geqslant 0$, we have for the first factor in (5.1)

$$\int_{|x|<r} |x|^{(N-\beta)q} w(x)^{q/p}\, dx \leqslant r^{(N-\beta)q} \int_{|x|<r} w^{q/p}\, dx.$$

Hence, (5.1) will follow if

$$r^{N-\beta}\Big(\int_{|x|<r} w^{q/p}\,dx\Big)^{1/q} r^{\alpha-N}\Big(\int_{|x|<r} w\,dx\Big)^{-1/p} \leqslant c.$$

Since $N-\beta+\alpha-N=\alpha-\beta=1/p-1/q$, this is the same as

$$\left(\frac{1}{r}\int_{|x|<r} w\,dx\right)^{q/p} \leqslant c\frac{1}{r}\int_{|x|<r} w\,dx,$$

which is true since $w\in \mathrm{RH}_{q/p}$.

For (1b), there are three terms in the kernel estimate, and correspondingly we consider

$$(5.2)\qquad \Big(\int_{|x|<2R}\Big\{\int_{|y|>(4/3)|x|}|f(y)|\,|y|^{\alpha-1}\,dy\Big\}^q u(x)\,dx\Big)^{1/q},$$

$$(5.3)\qquad \left(\int_{|x|<2R}\left\{\int_{|y|>(4/3)|x|}|f(y)\,Q(y)|\frac{dy}{1+|y|}\right\}^q u(x)\,dx\right)^{1/q},$$

$$(5.4)\qquad \left(\int_{|x|<2R}\left\{\int_{|y|>(4/3)|x|}|f(y)\,Q(y)|\frac{dy}{|y|}\right\}^q \frac{|x|^{\alpha q}u(x)}{|Q(x)|^q}\,dx\right)^{1/q}.$$

For the first of these, we consider separately the parts when $|y|>2R$ and $|y|<2R$. The part of the inner integral in (5.2) with $|y|>2R$ is at most a constant times

$$\int_{|y|>2R}|f(y)\,Q(y)|\,|y|^{\alpha-N-1}\,dy$$
$$\leqslant\Big(\int_{-\infty}^{+\infty}|f|^p|Q|^p w\,dy\Big)^{1/p}\left(\int_{|y|>2R}\frac{w(x)^{-1/(p-1)}}{|y|^{(1+N-\alpha)p'}}\,dy\right)^{1/p'} \leqslant c\,\|f\|_{L^p_v}$$

by Hölder's inequality and Lemma (3.4). Hence, the corresponding part of (5.2) is bounded by

$$c\,\|f\|_{L^p_v}\Big(\int_{|x|<2R} u\,dx\Big)^{1/q} = c\,\|f\|_{L^p_v}$$

since u is locally integrable. Thus, the estimation of this part is complete.

Consider next the part of (5.2) with $|y|<2R$. Note that x is bounded, and $x\in E_{k_0}$ since $y\in E_{k_0}$. Thus, $|Q(y)|\approx|y|^{\mu_{k_0}}$, $|Q(x)|\approx|x|^{\mu_{k_0}}$, and the part of (5.2) in question is at most

$$c\Big(\int_{|x|<2R}\Big\{\int_{2R>|y|>(4/3)|x|}|f(y)\,Q(y)|\,|y|^{\alpha-\mu_{k_0}-1}\,dy\Big\}^q |x|^{q\mu_{k_0}-q\min(\mu_{k_0},\beta)}\,w(x)^{q/p}\,dx\Big)^{1/q}.$$

To this we apply Hardy's inequality in the compact form (see (3.2)), obtaining the bound

$$c\Big(\int_{-\infty}^{\infty}|fQ|^p w\,dx\Big)^{1/p} = c\,\|f\|_{L^p_v}$$

provided that for $0<r\leqslant 2R$ we show

$$(5.5)\qquad \Big(\int_{|x|<R}|x|^{q\mu_{k_0}-q\beta_{k_0}}\,w(x)^{q/p}\,dx\Big)^{1/q}$$
$$\times\Big(\int_{2R>|x|>r}\big[|x|^{1+\mu_{k_0}-\alpha}\,w(x)^{1/p}\big]^{-p'}\,dx\Big)^{1/p'}\leqslant c.$$

By Lemma (3.4), the second factor in (5.5) is at most

$$c_R\big(r^{\alpha-\mu_{k_0}}+r^{1/p-1/q}\big)\Big(\int_{|x|<r} w\,dx\Big)^{-1/p},$$

and the first is clearly bounded by $cr^{\mu_{k_0}-\beta_{k_0}}\Big(\int_{|x|<r} w^{q/p}\,dx\Big)^{1/q}$. Thus, (5.5) is less than

$$c_R\big\{r^{\alpha-\beta_{k_0}}+r^{\mu_{k_0}-\beta_{k_0}+1/p-1/q}\big\}\Big(\int_{|x|<r} w^{q/p}\,dx\Big)^{1/q}\Big(\int_{|x|<r} w\,dx\Big)^{-1/p}.$$

Since r is bounded, the term in curly brackets is less than $c(r^{\alpha-\beta}+r^{1/p-1/q}) = cr^{1/p-1/q}$, and (5.5) follows from $w\in\mathrm{RH}_{q/p}$. This completes the estimation of (5.2).

For (5.3) and (5.4), we again have $u(x)\approx|x|^{\mu_{k_0}q-\beta_{k_0}q}\,w(x)^{q/p}$ since x is bounded and in E_{k_0}. Applying Hölder's inequality to the inner integral in (5.3) shows that (5.3) is at most

$$\left[\int_{|x|<2R}\Big(\int_{-\infty}^{\infty}|f|^p|Q|^p w\,dy\Big)^{q/p}\left(\int_{-\infty}^{\infty}\frac{w^{-1/(p-1)}}{(1+|y|)^{p'}}\,dy\right)^{q/p'}u(x)\,dx\right]^{1/q}.$$

Each of the two inner integrals here is independent of x, and since u is locally integrable, we obtain the bound $c\,\|f\|_{L^p_v}$ by performing the x-integration.

Since $|Q(x)|\approx|x|^{\mu_{k_0}}$ in (5.4), (5.4) is less than

$$c\left(\int_{|x|<2R}\left\{\int_{|y|>(4/3)|x|}|f(y)\,Q(y)|\frac{dy}{|y|}\right\}^q|x|^{(\alpha-\beta_{k_0})q}\,w(x)^{q/p}\,dx\right)^{1/q}.$$

The facts that $|x|$ is bounded and $\beta_{k_0}\leqslant\beta$ give $|x|^{(\alpha-\beta_{k_0})q}\leqslant c\,|x|^{(\alpha-\beta)q}$, and consequently it is enough to estimate

$$(5.6)\qquad \left(\int_{-\infty}^{\infty}\left\{\int_{|y|>(4/3)|x|}|f(y)\,Q(y)|\frac{dy}{|y|}\right\}^q|x|^{(\alpha-\beta)q}\,w(x)^{q/p}\,dx\right)^{1/q}.$$

By Lemma (3.1), this is bounded by $c\,\|fQ\|_{L^p_w}=c\,\|f\|_{L^p_v}$ provided that for $r>0$,

$$\Big(\int_{|x|<r}|x|^{(\alpha-\beta)q}\,w(x)^{q/p}\,dx\Big)^{1/q}\Big(\int_{|x|>r}\big[|x|\,w(x)^{1/p}\big]^{-p'}\,dx\Big)^{1/p'}\leqslant c.$$

Since $w \in A_p$, this is the same as

$$r^{\alpha-\beta}\Big(\int_{|x|<r} w^{q/p}\,dx\Big)^{1/q}\Big(\int_{|x|<r} w\,dx\Big)^{-1/p} \leqslant c,$$

which follows from $w \in \mathrm{RH}_{q/p}$ since $\alpha-\beta = 1/p-1/q$. This completes case (1b).

For (1c), the expression to estimate is

$$\left(\int_{x>2R}\left\{\int_{|y|>(3/4)|x|} |f(y)|\left[|y|^{\alpha-1}+\frac{|Q(y)|}{|Q(x)|}\cdot\frac{|x|^{\alpha}}{|y|}\right]dy\right\}^{q} u(x)\,dx\right)^{1/q}$$

The part arising from $|y|^{\alpha-1}$ can be estimated exactly as for (1a). The remaining part, since $|x|$ is now large, is bounded by

$$\left(\int_{x>2R}\left\{\int_{|y|>(3/4)|x|} |f(y)Q(y)|\frac{dy}{|y|}\right\}^{q} |x|^{(\alpha-\beta)q} w(x)^{q/p}\,dx\right)^{1/q}.$$

This is less than (5.6), which was treated above. We have now completed all cases of type (1).

Next, we shall consider cases of type (3). The kernel estimate for (3a) is 0 and so there is nothing to do in this case. For (3b), there are three parts:

$$\left(\int_{\substack{|x|<2R\\ x\in E_{k_0}}}\left\{\int_{|y|<(3/4)|x|} |f(y)Q(y)|\frac{|x|^{\alpha-1}}{|Q(x)|}dy\right\}^{q} u(x)\,dx\right)^{1/q}, \tag{5.7}$$

$$\Big(\int_{\substack{|x|<2R\\ x\in E_{k}}}\Big\{\int_{|y|<(3/4)|x|} |f(y)Q(y)|\,dy\Big\}^{q} u(x)\,dx\Big)^{1/q}, \tag{5.8}$$

$$\left(\int_{\substack{|x|<2R\\ x\in E_{k}}}\left\{\int_{|y|<(3/4)|x|} |f(y)Q(y)|\frac{|x-a_k|^{\alpha}}{|Q(x)|}dy\right\}^{q} u(x)\,dx\right)^{1/q}. \tag{5.9}$$

For (5.7), since x is bounded and in E_{k_0},

$$u(x) \approx |Q(x)|^{q}|x|^{-q\min(\mu_{k_0},\beta)} w(x)^{q/p} \leqslant c\,|Q(x)|^{q}|x|^{-\beta q} w(x)^{q/p}.$$

Thus, (5.7) is less than

$$c\Big(\int_{-\infty}^{\infty}\Big\{\int_{|y|<(3/4)|x|} |f(y)Q(y)|\,dy\Big\}^{q} |x|^{(\alpha-\beta-1)q} w(x)^{q/p}\,dx\Big)^{1/q}.$$

By Hardy's inequality, Lemma (3.3), this satisfies the desired estimate provided that

$$\Big(\int_{|x|>r} |x|^{(\alpha-\beta-1)q} w(x)^{q/p}\,dx\Big)^{1/q}\Big(\int_{|x|<r} w^{-1/(p-1)}\,dx\Big)^{1/p'} \leqslant c, \qquad r>0. \tag{5.10}$$

To obtain a bound for the first factor in (5.10), note that $(\alpha-\beta-1)q = -(1/p'+1/q)q = -(q/p')-1$, and apply (3.5) with $W = w^{q/p}$ and $s = (q/p')+1$, using the fact that $w^{q/p} \in A_{(q/p')+1}$. Thus,

$$\Big(\int_{|x|>r} |x|^{(\alpha-\beta-1)q} w(x)^{q/p}\,dx\Big)^{1/q} \leqslant \frac{c}{r^{1/p'+1/q}}\Big(\int_{|x|<r} w^{q/p}\,dx\Big)^{1/q},$$

and (5.10) follows immediately from $w \in A_p \cap \mathrm{RH}_{q/p}$. This completes the estimation of (5.7).

For (5.8) and (5.9), since x is bounded and in E_k, we have $|Q(x)| \approx |x-a_k|^{\mu_k}$ and $u(x) \approx |x-a_k|^{(\mu_k-\beta_k)q} w(x)^{q/p}$. Thus, the sum of (5.8) and (5.9) is at most

$$c\Big[\int_{|x|<2R}\Big(\int_{|y|<2R} |f(y)Q(y)|\,dy\Big)^{q}\{|x-a_k|^{(\mu_k-\beta_k)q}+|x-a_k|^{(\alpha-\beta_k)q}\}\,w(x)^{q/p}\,dx\Big]^{1/q}.$$

Since $\mu_k-\beta_k \geqslant 0$ and $\alpha-\beta_k \geqslant 0$, and $|x-a_k|$ is bounded, the part of the integrand in curly brackets is bounded, and by Hölder's inequality it is enough to consider

$$c\Big(\int_{|x|<2R} w(x)^{q/p}\,dx\Big)^{1/q}\Big(\int_{|y|<2R} |f(y)|^{p}|Q(y)|^{p} w(y)\,dy\Big)^{1/p} \times\Big(\int_{|y|<2R} w(y)^{-1/(p-1)}\,dy\Big)^{1/p'}.$$

This, however, is clearly at most a constant depending on R times $\|f\|_{L^p_v}$. The estimation of all parts of (3b) is now complete.

To handle (3c), note that since $|x|$ is large, $u(x) \approx |Q(x)|^{q}|x|^{-\beta q} w(x)^{q/p}$, and the term to be estimated is

$$\left(\int_{|x|>2R}\left\{\int_{|y|<(3/4)|x|} |f(y)Q(y)|\frac{|x|^{\alpha-1}}{|Q(x)|}dy\right\}^{q} |Q(x)|^{q}|x|^{-\beta q} w(x)^{q/p}\,dx\right)^{1/q}$$
$$\leqslant \Big(\int_{-\infty}^{\infty}\Big\{\int_{|y|<(3/4)|x|} |f(y)Q(y)|\,dy\Big\}^{q} |x|^{(\alpha-\beta-1)q} w(x)^{q/p}\,dx\Big)^{1/q}.$$

This same expression was already treated in the argument for (5.7). This completes all parts of type (1) and (3).

Finally, consider the parts of type (2). The kernel estimates for these involve $|x-y|^{\alpha-1}$ plus some extra terms, and we will consider these extra terms first. For (2a), there is no extra term. For (2b), the extra terms are $|Q(y)|$ and $|Q(y)|\,|x|^{\alpha}/|Q(x)|\,|y|$, and the corresponding integrals can be treated by some of the methods used for (1b): specifically, the same arguments used for (5.3) and (5.4) apply. The only difference is that the previous restriction $|x| < \frac{3}{4}|y|$ is now replaced by $\frac{3}{4}|y| < |x| < \frac{4}{3}|y|$. This together with $y \in E_{k_0}$ and $|x| < 2R$ implies $|y|$ is bounded and

$$u(x) \approx |x|^{\mu_{k_0}q-\beta_{k_0}q} w(x)^{q/p},$$

and the arguments can be easily adapted. For (2c), the extra term is $|Q(y)|\,|x|^{\alpha}/|Q(x)|\,|y|$. Since $|x|$ is large,

$$u(x) \approx |Q(x)|^q\,|x|^{-\beta q}\,w(x)^{q/p},$$

and consequently we must estimate

$$\Big(\int_{-\infty}^{\infty}\Big\{\int_{|y|\approx|x|}|f(y)\,Q(y)|\frac{dy}{|y|}\Big\}^q |x|^{(\alpha-\beta)q}\,w(x)^{q/p}\,dx\Big)^{1/q}.$$

This can be handled in the same way as (5.6).

It remains only to consider the parts of (2a), (2b) and (2c) which arise from the term $|x-y|^{\alpha-1}$ of the kernel estimate. Parts (2a) and (2c) lead to

$$\Big(\int_{|x|>2R}\Big\{\int_{(3/4)|x|\leqslant|y|\leqslant(4/3)|x|}|f(y)|\,|x-y|^{\alpha-1}\,dy\Big\}^q|Q(x)|^q\,|x|^{-\beta q}\,w(x)\,dx\Big)^{1/q},$$

since $|x|$ is large, and (2b) leads to

$$\Big(\int_{|x|<2R}\Big\{\int_{(3/4)|x|\leqslant|y|\leqslant(4/3)|x|}|f(y)|\,|x-y|^{\alpha-1}\,dy\Big\}^q \times|Q(x)|^q\prod|x-a_k|^{-q\beta_k}\,w(x)^{q/p}\,dx\Big)^{1/q}.$$

In any case, $|Q(x)| \approx |Q(y)|$ since $y\in E_{k_0}$ and $\frac{3}{4}|x| \leqslant |y| \leqslant \frac{4}{3}|x|$. Moreover, for (2b),

$$\prod|x-a_k|^{-q\beta_k} \approx |x|^{-q\beta_{k_0}} \leqslant c\,|x|^{-q\beta}$$

since x is also bounded. Thus, both expressions above are less than a constant times

$$\Big(\int_{-\infty}^{\infty}\Big\{\int_{|y|\approx|x|}|f(y)\,Q(y)|\,|x-y|^{\alpha-1}\,dy\Big\}^q|x|^{-\beta q}\,w(x)^{q/p}\,dx\Big)^{1/q}.$$

By Theorem 2 with f there replaced by fQ, this is bounded by $c\,\|fQ\|_{L^p_w} = c\,\|f\|_{L^p_v}$. The proof of the norm inequality in Theorem 1 is now complete. The fact that $I^+_\alpha f$ converges absolutely a.e. follows by noting that the arguments above involve the absolute value of the integrand of $I^+_\alpha f$. Thus, the proof of Theorem 1 is complete.

§ 6. Extensions by continuity. The first part of this section contains the proof of Theorem 3. The second part concerns the extension problem in case the norm on the left is a Hardy norm. We will concentrate on the case when α is not an integer.

Proof of Theorem 3. We start with the assumption that

$$\|I_\alpha f\|_{L^q_u} \leqslant c\,\|f\|_{L^p_v} \tag{6.1}$$

for a suitably restricted class of functions which is dense in L^p_v. The class of f's is either the set $\mathcal{S}_{0,0}$ of Schwartz functions whose Fourier transform has compact support not containing the origin, or the set of constant multiples of (∞, K) atoms with K sufficiently large and fixed. We always choose $K \geqslant N-1$ where N is the degree of Q. The definition of $I_\alpha f$ for f in either class is $I_\alpha f = f * |x|^{\alpha-1}$ ($\alpha \neq 1, 3, 5, \ldots$), and its extension to L^p_v is also denoted $I_\alpha f$. Fix $f\in L^p_v$ and let $\{f_j\}$ be a sequence in the dense subset such that $f_j \to f$ in L^p_v. Thus, by definition, $I_\alpha f_j \to I_\alpha f$ in L^p_u. We have $I_\alpha f_j = f_j * |x|^{\alpha-1}$ and, consequently, since $\mathcal{P}^Q_{|x-\cdot|^{\alpha-1}}(y)$ is a polynomial of degree at most $N-1$ and $\int_{-\infty}^{\infty} f_j(y)\,y^k\,dy = 0$ for $k = 0, 1, \ldots, N-1$,

$$(I_\alpha f_j)(x) = \int_{-\infty}^{\infty} f_j(y)\,\{|x-y|^{\alpha-1} - \mathcal{P}^Q_{|x-\cdot|^{\alpha-1}}(y)\}\,dy.$$

By Theorem 1, this integral converges in L^q_u as $j\to\infty$ to the analogous integral with f_j replaced by f, and Theorem 3 follows.

Extensions involving Hardy spaces. If f belongs to either of the dense subsets above, then $I_\alpha f$ defines a tempered distribution by

$$\langle I_\alpha f, \varphi\rangle = \int_{-\infty}^{\infty}(I_\alpha f)(x)\,\varphi(x)\,dx, \qquad \varphi\in\mathcal{S}.$$

This is clear if the dense subset is $\mathcal{S}_{0,0}$ since then $I_\alpha f\in\mathcal{S}_{0,0}$ too. On the other hand, if f is any bounded function with compact support in $|y| < R$, then

$$|(I_\alpha f)(x)| \leqslant \|f\|_\infty \int_{|y|<R}|x-y|^{\alpha-1}\,dy \leqslant c_R\,\|f\|_\infty\,(1+|x|)^{\alpha-1}.$$

Thus, $I_\alpha f$ has at most polynomial growth and so defines a distribution as above. This applies to any atom. We denote the "grand" maximal function (see [2] for the definition) of $I_\alpha f$ by $(I_\alpha f)^*$ and use the standard notation

$$\|I_\alpha f\|_{H^q_u} = \|(I_\alpha f)^*\|_{L^q_u}.$$

Instead of beginning with assumption (6.1), let us now assume that

$$\|I_\alpha f\|_{H^q_u} \leqslant c\,\|f\|_{L^p_v} \tag{6.2}$$

for all f in either dense class, and ask about the form of the extension. The extension is now a distribution in H^q_u which we denote by $\mathcal{I}_\alpha f$ and refer to as "the distribution obtained by extending (6.2)."

Before proceeding, we note several facts about (6.2). First, it implies (6.1) since $(I_\alpha f)^* \geqslant I_\alpha f$. Second, it does not matter whether we use $\|\cdot\|_{L_v^p}$ or $\|\cdot\|_{H_v^p}$ on the right because of the form of v. In fact, the main result of [10] is that L_v^p and H_v^p can be identified if $v=|Q|^p w$ with $w \in A_p$ and Q as usual, $1<p<\infty$. Such an identification cannot generally be made for the spaces on the left, however. Finally, inequalities of type (6.2) are derived in [11].

THEOREM (6.3). *With the same assumptions as in Theorem* 3, *if* $\mathscr{I}_\alpha f$ *is the distribution obtained by extending* (6.2), *then*

$$\langle \mathscr{I}_\alpha f, \varphi\rangle = \int_{-\infty}^{\infty} f(x)\{\varphi_\alpha(x) - \mathscr{P}^Q_{\varphi_\alpha}(x)\}\,dx, \qquad \varphi \in \mathscr{S},$$

where $\varphi_\alpha = \varphi * |x|^{\alpha-1}$.

Proof. We will first show that $\varphi_\alpha - \mathscr{P}^Q_{\varphi_\alpha}$ belongs to the dual space of L_v^p, i.e., $L^{p'}_{v^{-1/(p-1)}}$. For this, we use the simple known fact (see Lemma (3.1) of [10]) that if a function ψ has N bounded derivatives and $c_\psi = \max_{j=0,\ldots,N} \|\psi^{(j)}\|_\infty$, then

$$|\psi(x) - \mathscr{P}^Q_\psi(x)| \leqslant c_\psi |Q(x)|/(1+|x|).$$

If $\varphi \in \mathscr{S}$, then the function $\varphi_\alpha = |x|^{\alpha-1} * \varphi$ is infinitely differentiable and bounded by $c(1+|x|)^{\alpha-1}$. Let $\theta(x)$ be a smooth compactly supported function equal to 1 on the support of $\mathscr{D}^Q$, and write $\varphi_\alpha = \theta\varphi_\alpha + (1-\theta)\varphi_\alpha$. Thus

$$\mathscr{P}_{\varphi_\alpha} = \mathscr{P}_{\theta\varphi_\alpha} \quad \text{and} \quad \varphi_\alpha - \mathscr{P}_{\varphi_\alpha} = (\theta\varphi_\alpha - \mathscr{P}_{\theta\varphi_\alpha}) + (1-\theta)\varphi_\alpha.$$

Since $\theta\varphi_\alpha$ has N bounded derivatives, $|\theta\varphi_\alpha - \mathscr{P}_{\theta\varphi_\alpha}|$ is at most $c|Q(x)|/(1+|x|)$, and so by Lemma (3.4) (in the simplest form with $q=p$) belongs to $L^{p'}_{v^{-1/(p-1)}}$. For $(1-\theta)\varphi_\alpha$, use the bound $(1+|x|)^{\alpha-1}\chi(|x|>2R)$ and the fact that

$$\int_{|x|>2R} (1+|x|)^{(\alpha-1)p'} |Q(x)|^{-p'} w(x)^{-1/(p-1)}\,dx \leqslant c \int_{|x|>2R} |x|^{(\alpha-1-N)p'} w(x)^{-1/(p-1)}\,dx < \infty$$

for $N \geqslant \beta$ by Lemma (3.4).

To prove the theorem, let $f \in L_v^p$ and $\{f_j\}$ be a sequence in the dense subset such that $f_j \to f$ in L_v^p. Then $I_\alpha f_j \to \mathscr{I}_\alpha f$ in H_u^q and, consequently, in $\mathscr{S}'$. Thus, $\langle I_\alpha f_j, \varphi\rangle \to \langle \mathscr{I}_\alpha f, \varphi\rangle$ for $\varphi \in \mathscr{S}$. Here, $I_\alpha f_j$ is of polynomial growth at most, and

$$\langle I_\alpha f_j, \varphi\rangle = \int (I_\alpha f_j)\varphi\,dx = \int f_j \varphi_\alpha\,dx$$

by Fubini's theorem. Since the first N moments of f_j vanish, we obtain

$$\langle I_\alpha f_j, \varphi\rangle = \int f_j \{\varphi_\alpha - \mathscr{P}^Q_{\varphi_\alpha}\}\,dx,$$

and the theorem follows by passing to the limit.

Theorem (6.3) can be used to derive a relation between $\mathscr{I}_\alpha f$ and $I_\alpha f$. Before stating this relationship, we must introduce some notation. The integral part of x will be denoted by $\operatorname{int}(x)$, and x_+ is defined as usual by $x_+ = \max\{x, 0\}$. Given α and a polynomial $Q(x) = \prod (x-a_k)^{\mu_k}$, let

$$Q^*(x) = \prod (x-a_k)^{[\operatorname{int}(\mu_k-\alpha)]_+}, \tag{6.4}$$

$$Q^{**}(x) = \prod (x-a_k)^{[\mu_k - \operatorname{int}\alpha]_+}. \tag{6.5}$$

If α is an integer, $Q^* = Q^{**}$. In general, Q^* and Q^{**} are polynomials with $\deg Q^* \leqslant \deg Q^{**}$; in fact, Q^* is a factor of Q^{**}. This implies, by repeated use of the translation property $\mathscr{P}^Q_\varphi(y-a) = \mathscr{P}^{Q(\cdot-a)}_{\varphi(\cdot-a)}(y)$ and the formula $\mathscr{P}^{xQ}_\varphi = \mathscr{P}^Q_\varphi + \mathscr{D}^{xQ}(\varphi)\cdot Q$ (see Lemma (2.7) of [10]), that

$$\mathscr{P}^{Q^{**}}_\varphi = \mathscr{P}^{Q^*}_\varphi + PQ^* \quad \text{with } \deg(PQ^*) \leqslant \deg Q^{**} - 1. \tag{6.6}$$

THEOREM (6.7). *Let p, q, v and N be as in Theorem* 3, *and let Q^* and Q^{**} be the polynomials defined by* (6.4) *and* (6.5) *which are associated with Q. Let $\mathscr{I}_\alpha f$ be the distribution obtained by extending* (6.2), *and let $I_\alpha f$ be the function obtained by extending* (6.1). *Then for $\varphi \in \mathscr{S}$ and $f \in L_v^p$,*

$$\langle \mathscr{I}_\alpha f, \varphi\rangle = \int_{-\infty}^{\infty} I_\alpha(\chi f)\{\varphi - \mathscr{P}^{Q^{**}}_\varphi\}\,dx + \int_{-\infty}^{\infty} I_\alpha((1-\chi)f)\{\varphi - \mathscr{P}^{Q^*}_\varphi\}\,dx,$$

where χ is a smooth, compactly supported function equal to 1 *on an interval containing the zeros of Q.*

We shall not give the proof, which is technical and fairly long. The idea is to first obtain both of the following representations for the term $\varphi_\alpha - \mathscr{P}^Q_{\varphi_\alpha}$ in Theorem (6.3):

$$\begin{aligned}\varphi_\alpha(x) - \mathscr{P}^Q_{\varphi_\alpha}(x) &= \int_{-\infty}^{\infty} \{|x-y|^{\alpha-1} - \mathscr{P}^Q_{|\cdot-y|^{\alpha-1}}(x)\}\{\varphi(y) - \mathscr{P}^{Q^*}_\varphi(y)\}\,dy \\ &= \int_{-\infty}^{\infty} \{|x-y|^{\alpha-1} - \mathscr{P}^Q_{|\cdot-y|^{\alpha-1}}(x)\}\{\varphi(y) - \mathscr{P}^{Q^{**}}_\varphi(y)\}\,dy.\end{aligned}$$

Once this is done, the remainder of the proof consists of using the equation $1 = \chi + (1-\chi)$ to divide the integral in Theorem (6.3) into two pieces, substituting the representations above into the respective pieces, and verifying that Fubini's theorem can be used to interchange the order of integration. The result then follows immediately from Theorem 3.

§ 7. **Area-type integrals.** Let $\Gamma(x)$ denote the "cone" $\{(y,t)\colon |x-y|<at\}$ in $\boldsymbol{R}^2$ with aperture a. Fix a function $\psi \in \mathscr{S}$, let $\psi_t(x) = t^{-1}\psi(x/t)$, and define

$$S_\alpha(f)(x) = \Big(\iint_{\Gamma(x)} t^{\alpha-2} |(f*\psi_t)(y)|^2\,dy\,dt\Big)^{1/2}.$$

We will prove the following result about $S_\alpha(f)$.

THEOREM (7.1). *Let α, p, q, u and v be as in Theorem* 1. *Then*

$$\|S_\alpha(f)\|_{L_u^q} \leqslant c\|f\|_{L_v^p}$$

for any (∞, K) *atom with* K *sufficiently large.*

Proof. The idea is to show that $S_\alpha(f)$ is an ordinary area integral of $I_\alpha f$. Define $\varphi(x)$ by $\hat{\varphi}(x) = |x|^\alpha \hat{\psi}(x)$, and note that $t^\alpha(f*\psi_t)(x) = c((I_\alpha f)*\varphi_t)(x)$ if f is an (∞, K) atom, by checking Fourier transforms. Thus, $S_\alpha(f)$ is a multiple of $S(I_\alpha f)$, where

$$S(g)(x) = \Big(\iint_{\Gamma(x)} t^{-2}|(g*\varphi_t)(y)|^2\,dy\,dt\Big)^{1/2}.$$

Since $\hat{\varphi}(0) = 0$, S is an ordinary area integral, and therefore

$$\|S(g)\|_{L_u^q} \leqslant c\|g\|_{H_u^q}$$

by [9]. Here, we use the fact that $u \in A_\infty$, as was noted near the beginning of § 5. Therefore,

$$\|S_\alpha(f)\|_{L_u^q} \leqslant c\|I_\alpha f\|_{H_u^q}.$$

The theorem now follows immediately since $\|I_\alpha f\|_{H_u^q} \leqslant c\|f\|_{L_v^p}$ by Theorem (1.9) of [11].

In a similar way and with the same hypotheses, one obtains

$$\Big\|\sup_{(y,t)\in\Gamma(x)} t^\alpha|(f*\psi_t)(y)|\Big\|_{L_u^q} \leqslant c\|f\|_{L_v^p}. \tag{7.2}$$

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