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Isomorphism of regular Morse dynamical systems induced by arbitrary blocks

by

J. KWIATKOWSKI (Toruń)

Abstract. The problem of metric isomorphism in the class of Morse dynamical systems is investigated. The main theorem of this paper is the following:

Let $x = b^0 \times b^1 \times \ldots$, $y = \beta^0 \times \beta^1 \times \ldots$ be Morse sequences such that $|b^t| = |\beta^t| = \lambda_t$, $t \geq 0$, the sequences of blocks $\{b^0, b^1, \ldots\}$, $\{\beta^0, \beta^1, \ldots\}$ satisfy a condition (R) (the relative frequencies of the pairs 00, 11 and 01, 10 in each of the blocks b^t and β^t, $t \geq 0$, are greater than a positive number η) and $\lambda_t > 8/\eta$ for sufficiently large t. The shift dynamical systems $\theta(x)$ and $\theta(y)$ induced by x and y are metrically isomorphic iff there exist numbers q_t, $0 \leq q_t \leq \lambda_t - 1$, and a sequence of pairs $\{(r_t, s_t)\}$, $r_t, s_t = 0, 1$, $t \geq 0$, such that

(A)
$$\sum_{t=0}^{\infty} d\left(\beta^t, (b^t)^{r_t} (b^t)^{s_t} [q_t, q_t + \lambda_t - 1]\right) < \infty,$$

(B)
$$\sum_{t=0}^{\infty} \min(l_t/n_t, 1 - l_t/n_t) < \infty,$$

where $(b^t)^0 = b^t$ and $(b^t)^1 = \tilde{b}^t$, $n_t = \lambda_0 \ldots \lambda_t$, $l_0 = q_0$, $l_t = q'_0 + q'_1 n_0 + \ldots + q'_t n_{t-1}$, $q'_0 = q_0$, $q'_t = q_t$ if $l_{t-1} \leq n_{t-1} - l_{t-1} - 1$, $q'_t = q_t - 1 \pmod{\lambda_t}$ if $l_{t-1} > n_{t-1} - l_{t-1} - 1$, $t \geq 1$.

Introduction. Generalized Morse sequences were introduced by Keane in [4]. The construction of Morse dynamical systems gave the positive answer to Jacobs' question whether any infinite group of roots of unity can occur as the eigenvalue group of a strictly ergodic system having a continuous part of the spectrum. Moreover, this class enables one to construct some counter-examples in the general ergodic theory (see [1], [3], [7], [8]). It follows from [4] that all Morse dynamical systems are $\boldsymbol{Z}_2$-extensions of ergodic dynamical systems with discrete spectra for which the eigenvalues form a group of roots of unity. In [5] the detailed form of such systems was described and the problem of isomorphism was investigated. It was proved there that if $x = b^0 \times b^1 \times \ldots$, $y = \beta^0 \times \beta^1 \times \ldots$ are regular Morse sequences such that $|b^t| = |\beta^t| = \lambda_t$ and $\{\lambda_t\}$ is a bounded sequence then the shift dynamical systems $\theta(x)$ and $\theta(y)$ induced by x and y are metrically isomorphic iff $b^t = \beta^t$ for all sufficiently large t. If x is a given Morse sequence for which the sequence

$\{|b^t|\}$ is bounded then by the above result we can obtain a countable number of Morse sequences y such that $\theta(x)$ and $\theta(y)$ are metrically isomorphic.

In this paper we study the problem of metric isomorphism in the class of all regular Morse systems over 0 and 1 without the assumption that the sequence of lengths is bounded. The main result of the paper is formulated in the abstract. This theorem enables us to construct a class $\mathcal{M}^*(x)$ of Morse sequences y such that $\theta(y)$ is metrically isomorphic to $\theta(x)$. $\mathcal{M}^*(x)$ consists of a continuum of Morse shift systems $\theta(y)$ such that the measures m_y induced by y are pairwise orthogonal. The procedure of constructing $\mathcal{M}^*(x)$ is the following. Suppose that x is a regular Morse sequence of the form

$$x = b^0 \times b^1 \times \ldots$$

Depending upon this representation we can construct a set of Morse sequences having the form $y = \beta^0 \times \beta^1 \times \ldots$, where the blocks $\{\beta^t\}$ satisfy the conditions of the theorem in the abstract. Now we can represent x in another form by grouping the blocks $\{b^t\}$, and next find a new set of Morse sequences as above. In this way we obtain the class $\mathcal{M}^*(x)$.

Rojek [10] has shown that if $\theta(x)$ and $\theta(y)$ are metrically isomorphic then there exists a Morse sequence z such that x and y can be obtained from z as above. This enables one to construct the class $\mathcal{M}(x)$ of all Morse sequences y such that $\theta(x)$ and $\theta(y)$ are metrically isomorphic. Namely, the above procedure of obtaining the class $\mathcal{M}^*(x)$ must be completed. Suppose $y = \beta^0 \times \beta^1 \times \ldots$ is obtained as above. If some of the β^i can be represented as $\beta^i = a^i \times \bar{a}^i$ then we replace β^i by $a^i \times \bar{a}^i$ and obtain new forms of y, to which we apply the above procedure. In this way we obtain the class $\mathcal{M}(x)$. We remark that the assumptions $\lambda_t > 8/\eta$ from the abstract do not limit the possibility of constructing $\mathcal{M}(x)$ for a regular Morse sequence.

§ 1. Notation and definitions. First we introduce notions, definitions and notation used in the paper. Let $X = \prod_{-\infty}^{\infty} \{0, 1\}$. A finite sequence $B = (b_0 \ldots b_{n-1})$, $b_i = 0, 1$, $n \geqslant 1$, is called a *block*, the number n is called the *length* of B and denoted by $|B|$. If $x \in X$ and $B = (b_0 \ldots b_{n-1})$ is a block then $x[i, k]$, $B[i, k]$, $0 \leqslant i \leqslant k \leqslant n-1$, denote the blocks $(x_i \ldots x_k)$ and $(b_i \ldots b_k)$ respectively. We will write $B[i]$ and $x[i]$ instead of $B[i, i]$ and $x[i, i]$. Let us denote $\tilde{B} = (\tilde{b}_0 \ldots \tilde{b}_{n-1})$ where $\tilde{b} = 1-b$, $b = 0, 1$. If $C = (c_0 \ldots c_{m-1})$ is a block then we denote by BC the block $(b_0 \ldots b_{n-1} c_0 \ldots c_{m-1})$. Further we define

$$B \times C = B^{c_0} B^{c_1} \ldots B^{c_{m-1}},$$

where $B^0 = B$ and $B^1 = \tilde{B}$.

Assume $b^0, b^1, \ldots$ are finite blocks with $|b^t| \geqslant 2$, $t \geqslant 0$, starting with 0.

Then we may define a one-sided sequence x as follows:

$$x = b^0 \times b^1 \times \ldots \tag{1}$$

If B, C are blocks with $|B| \leqslant |C|$ then by $\mathrm{fr}(B, C)$ we denote the frequency of B in C, i.e.

$$\mathrm{fr}(B, C) = \mathrm{card}\{j\colon\ 0 \leqslant j \leqslant |C|-|B|,\ C[j, j+|B|-1] = B\}.$$

Next we set

$$\lambda_t = |b^t|, \quad r_t = \min\left\{\frac{1}{\lambda_t}\mathrm{fr}(0, b^t), \frac{1}{\lambda_t}\mathrm{fr}(1, b^t)\right\},$$

$$n_t = \lambda_0 \cdot \ldots \cdot \lambda_t, \quad c_t = b^0 \times \ldots \times b^t, \quad t = 0, 1, \ldots$$

We have $\lambda_t \geqslant 2$ and $n_t = |c_t|$.

Definition 1. A sequence x defined by (1) is called a *Morse sequence* if $\sum_{t=0}^{\infty} r_t = \infty$, infinitely many of the b^t are different from $00 \ldots 0$ and infinitely many are different from $01 \ldots 010$.

Definition 2. A Morse sequence x is called *continuous* if either infinitely many of the λ_t are even or λ_t are odd for $t \geqslant t_0$ and

$$\sum_{t=0}^{\infty} \min\{e_0(b^t)+o_1(b^t),\ o_0(b^t)+e_1(b^t)\} = \infty,$$

where

$$e_k(b^t) = \frac{1}{\lambda_t}\mathrm{card}\{j\colon\ 0 \leqslant j \leqslant \lambda_t-1,\ b^t[j] = k,\ j \text{ even}\},$$

$$o_k(b^t) = \frac{1}{\lambda_t}\mathrm{card}\{j\colon\ 0 \leqslant j \leqslant \lambda_t-1,\ b^t[j] = k,\ j \text{ odd}\},$$

$k = 0, 1$.

It is known [4] that if x is a continuous Morse sequence then each block B has the relative average frequency m_x in x. The function m_x is an ergodic measure on X invariant under the shift T on X. For each block B, $m_x(B) = m_x(\tilde{B})$ and in particular $m_x(0) = m_x(1) = \frac{1}{2}$. We will say that m_x is the *Morse measure* defined by x and $\theta(x) = (x, m_x, T)$ is the *Morse dynamical system* induced by x.

Following [6] we construct a special dynamical system metrically isomorphic to $\theta(x)$. Set

$$Z_{\lambda_t} = \{0, 1, \ldots, \lambda_t-1\}, \quad Z = \prod_{t=0}^{\infty} Z_{\lambda_t},$$

and let $\bar{p}$ be the product measure of the uniform measures on Z_{λ_t}, $t = 0, 1, \ldots$ Next let $\bar{Z}$ be the subset of Z consisting of all $\bar{z} = (z_0, z_1, \ldots)$ such that infinitely many of the z_t are different from 0 and infinitely many are different from $\lambda_t - 1$. Consider the transformation $\bar{S}$: $\bar{Z} \to \bar{Z}$ mapping $\bar{z}$ to $\bar{z}+1$, that is,

$$\bar{S}(z_0, z_1, \ldots) = (0, \ldots, 0, z_t+1, z_{t+1}, \ldots),$$

where $t = t(\bar{z})$ is the first number such that $z_t < \lambda_t - 1$.

The dynamical system $(\bar{Z}, \bar{p}, \bar{S})$ is an ergodic system with discrete spectrum. Indeed, the set Z may be identified with the dual group of the group Λ of all n_t-roots of unity, $n_t = \lambda_0 \cdot \ldots \cdot \lambda_t$, $t \geqslant 0$. The addition in Z is defined as in the set of p-adic numbers. It is easy to establish that the element $(1, 0, 0, \ldots) \in Z$ corresponds to the homomorphism from Λ into the circle group equal to the identity on Λ. (Z, p, S) is an ergodic dynamical system with discrete spectrum and Λ is its point spectrum.

It turns out [5], [6] that the dynamical system $\theta(x)$ is metrically isomorphic to a skew product of the automorphism $\bar{S}$ with a family $\{S(\bar{z})\}$, $z \in \bar{Z}$, of permutations of the two-point set $\{0, 1\}$, each with mass $\frac{1}{2}$. In order to describe the family $\{S(z)\}$, $z \in \bar{Z}$, we define a function p: $\bar{Z} \to \boldsymbol{Z}_2 = \{0, 1\}$. Take $z = (z_0, z_1, \ldots) \in \bar{Z}$ and put

$$p(\bar{z}) = b^t[z_t+1] - b^t[z_t] - b^{t-1}[\lambda_{t-1}-1] - \ldots - b^0[\lambda_0 - 1] \pmod 2,$$

where $t = t(\bar{z})$ is defined above.

If $(\bar{z}, i) \in \bar{Z} \times \boldsymbol{Z}_2$ then we define $S(\bar{z})(i) = i + p(\bar{z}) \pmod 2$. We obtain the $\boldsymbol{Z}_2$-extension

$$\theta^*(x) = \left(\bar{Z} \times \boldsymbol{Z}_2, \bar{p} \times \tfrac{1}{2}, S \times \{S(\bar{z})\}\right)$$

of the dynamical system $(\bar{Z}, \bar{p}, \bar{S})$. We will define a mapping f from $\bar{Z} \times \boldsymbol{Z}_2$ to X which is an isomorphism between the dynamical systems $\theta^*(x)$ and $\theta(x)$.

For $z = (z_0, z_1, \ldots) \in \bar{Z}$ we put

$$j_0 = z_0, \quad j_t = j_t(z) = z_0 + z_1 n_0 + \ldots + z_t n_{t-1}, \quad t > 0,$$
$$k_t = k_t(\bar{z}) = n_t - j_t - 1, \quad t \geqslant 0.$$

It is obvious that $\bar{z} \in \bar{Z}$ implies

$$j_{t+1} \geqslant j_t, \quad k_{t+1} \geqslant k_t, \quad t \geqslant 0,$$
$$j_t \to \infty, \quad k_t \to \infty \quad \text{as } t \to \infty.$$

Define a two-sided sequence $y = f(\bar{z}, i)$ by putting

(2) $$y[-j_t, k_t] = c_t^{q_t(\bar{z}, i)}$$

where

$$q_t(\bar{z}, i) = i - b^0[z_0] - \ldots - b^t[z_t] \pmod 2.$$

Let $X(x) = f(Z \times \boldsymbol{Z}_2)$. Then $m_x(X(x)) = 1$ and the mapping f is an isomorphism of the dynamical systems $\theta^*(x)$ and $\theta(x)$.

Denote by ξ_t, $t \geqslant 0$, the partition of $X(x)$ consisting of sets of the form

(3) $$\begin{aligned} C_0(k) = \{y \in X(x)\colon\ & y[-k, n_t-k-1] = c_t \text{ and each fragment} \\ & y[ln_t - k, (l+1)n_t - k - 1] \text{ of } y \text{ is } c_t \text{ or } \tilde{c}_t\}, \\ C_1(k) = \{y \in X(x)\colon\ & y[-k, n_t-k-1] = \tilde{c}_t \text{ and each fragment} \\ & y[ln_t - k, (l+1)n_t - k - 1] \text{ of } y \text{ is } c_t \text{ or } \tilde{c}_t\}, \end{aligned}$$

$k = 0, 1, \ldots, n_t - 1$. The sets $C_0(k)$ and $C_1(k)$, $0 \leqslant k \leqslant n_t - 1$, are pairwise disjoint and

(4) $$m_x(C_0(k)) = m_x(C_1(k)) = \frac{1}{2n_t}.$$

In [6] we have shown that $\xi_t \nearrow \varepsilon$.

It follows from the definition of the sets $C_0(k)$ and $C_1(k)$ that

$$C_0(0) \xrightarrow{T} C_0(1) \xrightarrow{T} \ldots \xrightarrow{T} C_0(n_t - 1)$$

and

$$C_1(0) \xrightarrow{T} C_1(1) \xrightarrow{T} \ldots \xrightarrow{T} C_1(n_t - 1).$$

Therefore the partitions ξ_t, $t \geqslant 0$, may be represented as towers of $X(x)$.

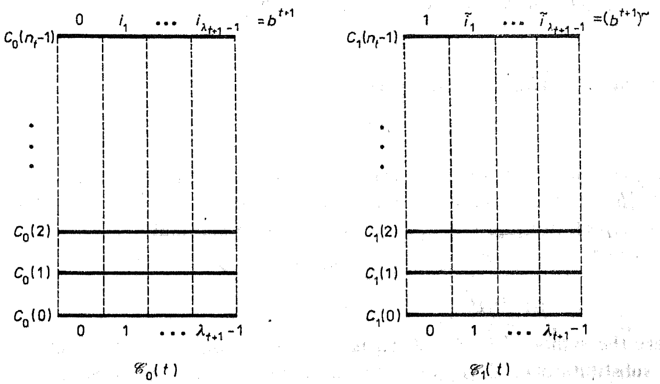

Fig. 1

Each tower ξ_t consists of two columns $\mathscr{C}_0(t)$ and $\mathscr{C}_1(t)$. The next tower ξ_{t+1} may be obtained from ξ_t as follows: we divide each of the two columns of ξ_t into λ_{t+1} equal subcolumns and mark them by $0, 1, \ldots, \lambda_{t+1}-1$. We write the block b^{t+1} over $\mathscr{C}_0(t)$ and $(b^{t+1})^{\sim}$ over $\mathscr{C}_1(t)$. Then $\mathscr{C}_0(t+1)$ consists of those subcolumns that have 0 over them and $\mathscr{C}_1(t+1)$ consists of the remaining successive subcolumns.

§ 2. Codes of partitions. Regularity of Morse sequences. In this section we describe all two-set partitions of $X(x)$ and introduce the notion of the regularity of a Morse sequence $x = b^0 \times b^1 \times \ldots$

Suppose $Q = (Q_0, Q_1)$ and $R = (R_0, R_1)$ are partitions of $X(x)$. The distance $|\cdot|$ between Q and R is defined by

$$|Q-R| = m_x(Q_0 \cap R_1) + m_x(Q_1 \cap R_0). \tag{5}$$

Fix $t \geqslant 0$ and assume that the partition Q satisfies $Q \prec \xi_t$. The partition Q permits to define two blocks A_t and B_t in the following way:

$$\begin{aligned} A_t[k] = i \quad &\text{iff} \quad C_0(k) \subset Q_i, \\ B_t[k] = i \quad &\text{iff} \quad C_1(k) \subset Q_i, \end{aligned} \tag{6}$$

$i = 0, 1$, $k = 0, 1, \ldots, n_t - 1$. The blocks A_t and B_t are called *codes* of Q with respect to ξ_t and we will denote them by $A_t(Q)$ and $B_t(Q)$.

Now, if $Q, R \prec \xi_t$ then by (4)–(6) we have

$$|Q-R| = \tfrac{1}{2} d\big(A_t(Q), A_t(R)\big) + \tfrac{1}{2} d\big(B_t(Q), B_t(R)\big), \tag{7}$$

where $d(A, B) = n^{-1}\,\mathrm{card}\{i\colon 0 \leqslant i \leqslant n-1, A[i] \neq B[i]\}$, and A, B are blocks with $|A| = |B| = n$.

Let $Q = (Q_0, Q_1)$ be a partition of $X(x)$. Since $\xi_t \nearrow \varepsilon$, there exists a sequence $\{Q^t\}$ of partitions of $X(x)$ such that $Q^t \prec \xi_t$ and

$$|Q - Q^t| \underset{t}{\to} 0. \tag{8}$$

The above condition is equivalent to

$$\sup_{k \geqslant 1} |Q^t - Q^{t+k}| \underset{t}{\to} 0. \tag{9}$$

The codes of the partitions Q^t with respect to ξ_t form the sequences of blocks $\{A_t\}$, $\{B_t\}$ such that $|A_t| = |B_t| = n_t$. Let us denote by $A_t^{(k)}, B_t^{(k)}$ the codes of Q^t with respect to ξ_{t+k}, $k = 1, 2, \ldots$ It is easy to see that

$$\begin{aligned} A_t^{(k)} &= \{A_t, B_t\} * (b^{t+1} \times \ldots \times b^{t+k}), \\ B_t^{(k)} &= \{B_t, A_t\} * (b^{t+1} \times \ldots \times b^{t+k}), \end{aligned}$$

where the symbol $\{A_t, B_t\} * E$ (E is a block) denotes the block obtained by the substitution of A_t, B_t in E, A_t in place of 0 and B_t in place of 1.

Using (7) and (9) we conclude that the sequences of blocks $\{A_t\}$, $\{B_t\}$

satisfy the conditions

$$|A_t| = |B_t| = n_t, \quad t \geqslant 0, \tag{10}$$

and

$$\begin{aligned} &\sup_{k \geqslant 1} d(A_{t+k}, A_t^{(k)}) \underset{t}{\to} 0, \\ &\sup_{k \geqslant 1} d(B_{t+k}, B_t^{(k)}) \underset{t}{\to} 0. \end{aligned} \tag{11}$$

Conversely, any sequences of blocks $\{A_t\}$, $\{B_t\}$ satisfying (10) and (11) define a partition $Q = (Q_0, Q_1)$ of $X(x)$. Other such sequences $\{\bar{A}_t\}$, $\{\bar{B}_t\}$ define the same partition Q iff

$$d(A_t, \bar{A}_t) \underset{t}{\to} 0 \quad \text{and} \quad d(B_t, \bar{B}_t) \underset{t}{\to} 0.$$

Any sequences of blocks $\{A_t\}$, $\{B_t\}$ satisfying (10) and (11) are called *codes* of Q with respect to $\{\xi_t\}$, $t \geqslant 0$.

Now, we define the notion of the regularity of a Morse sequence $x = b^0 \times b^1 \times \ldots$ For a fixed $t \geqslant 0$ let

$$x_t = b^t \times b^{t+1} \times \ldots \quad \text{and} \quad n_t^{(s)} = \lambda_t \cdot \ldots \cdot \lambda_{t+s}, \quad s \geqslant 0.$$

We have $|b^t \times \ldots \times b^{t+s}| = n_t^{(s)}$. If B is a block with $|B| = n_t^{(s)}$ then we put

$$m_s(x_t, B) = \frac{1}{n_t^{(s)}} \mathrm{fr}(B, b^t \times \ldots \times b^{t+s}).$$

Further let

$$\begin{aligned} h_t(00, 11) &= \mathrm{fr}(00, b^t) + \mathrm{fr}(11, b^t), \\ h_t(01, 10) &= \mathrm{fr}(01, b^t) + \mathrm{fr}(10, b^t), \\ m_s(x_t, 00 \vee 11) &= m_s(x_t, 00) + m_s(x_t, 11), \\ m_s(x_t, 01 \vee 10) &= m_s(x_t, 01) + m_s(x_t, 10), \end{aligned}$$

$t, s \geqslant 0$. It is easy to obtain

$$\begin{aligned} m_{s+1}(x_t, 00 \vee 11) &= m_s(x_t, 00 \vee 11) + \begin{cases} \dfrac{h_{t+s+1}(00, 11)}{n_t^{(s+1)}}, & s \in \bar{\mathrm{I}}_t, \\[2ex] \dfrac{h_{t+s+1}(01, 10)}{n_t^{(s+1)}}, & s \in \overline{\mathrm{II}}_t, \end{cases} \\ m_{s+1}(x_t, 01 \vee 10) &= m_s(x_t, 01 \vee 10) + \begin{cases} \dfrac{h_{t+s+1}(01, 10)}{n_t^{(s+1)}}, & s \in \bar{\mathrm{I}}_t, \\[2ex] \dfrac{h_{t+s+1}(00, 11)}{n_t^{(s+1)}}, & s \in \overline{\mathrm{II}}_t, \end{cases} \end{aligned} \tag{12}$$

where

$$\bar{\mathrm{I}}_t = \{s \geqslant 0\colon (b^t \times \ldots \times b^{t+s})[n_t^{(s)}] = 0\},$$

$$\bar{\mathrm{II}}_t = \{s \geqslant 0\colon (b^t \times \ldots \times b^{t+s})[n_t^{(s)}] = 1\}.$$

Thus (12) gives

$$\lim_s m_s(x_t, 00 \vee 11) = \frac{h_t(00, 11)}{\lambda_t} + \sum_{\substack{s=t+1 \\ s-1 \in \bar{\mathrm{I}}_t}}^{\infty} \frac{h_s(00, 11)}{n_t^{(s)}} + \sum_{\substack{s=t+1 \\ s-1 \in \bar{\mathrm{II}}_t}}^{\infty} \frac{h_s(01, 10)}{n_t^{(s)}},$$

(13)

$$\lim_s m_s(x_t, 01 \vee 10) = \frac{h_t(01, 10)}{\lambda_t} + \sum_{\substack{s=t+1 \\ s-1 \in \bar{\mathrm{I}}_t}}^{\infty} \frac{h_s(01, 10)}{n_t^{(s)}} + \sum_{\substack{s=t+1 \\ s-1 \in \bar{\mathrm{II}}_t}}^{\infty} \frac{h_s(00, 11)}{n_t^{(s)}}.$$

From (12) it follows that

$$m_{s+1}(x_t, 00 \vee 11) \geqslant m_s(x_t, 00 \vee 11),$$

$$m_{s+1}(x_t, 01 \vee 10) \geqslant m_s(x_t, 01 \vee 10),$$

$s, t \geqslant 0$. Further we put

$$\bar{p}_t = m_{x_t}(00) + m_{x_t}(11) = \lim_s m_s(x_t, 00 \vee 11),$$

(14)

$$\bar{q}_t = m_{x_t}(01) + m_{x_t}(10) = \lim_s m_s(x_t, 01 \vee 10).$$

Definition 3. We say that a sequence of blocks $\{b^0, b^1, \ldots\}$ satisfies *condition* (R) if there exists a positive number η such that

$$\frac{1}{\lambda_t} h_t(00, 11) \geqslant \eta, \quad \frac{1}{\lambda_t} h_t(01, 10) \geqslant \eta,$$

for $t = 0, 1, \ldots$

Definition 4. A sequence of blocks $\{b^0, b^1, \ldots\}$ is said to be *regular* if the numbers $\bar{p}_t, \bar{q}_t, t \geqslant 0$, satisfy the condition

(15) $$\overline{\lim_t} \min(\bar{p}_t, \bar{q}_t) > 0.$$

Remark 1. If a sequence of blocks $\{b^0, b^1, \ldots\}$ is regular then there exist numbers $0 \leqslant t_0 \leqslant t_1 \leqslant \ldots$ such that the sequence $\{g^0, g^1, \ldots\}$ of blocks satisfies condition (R), where

$$g^0 = b^0 \times \ldots \times b^{t_0}, \quad g^{j+1} = b^{t_j+1} \times \ldots \times b^{t_{j+1}}, \quad j \geqslant 0.$$

Proof. The assumption (15) implies that for infinitely many numbers $t'_0 < t'_1 < \ldots$ we have $\bar{p}_{t'_i} \geqslant 2\eta$, $\bar{q}_{t'_i} \geqslant 2\eta$ with $\eta > 0$, $i = 0, 1, \ldots$ It follows from (13) and (14) that for each $i = 0, 1, \ldots$ we can find a number $a(t'_i) = t'_j$, $t'_j > t'_i$, such that

$$m_s(x_{t'_i}, 00 \vee 11) \geqslant \eta \quad \text{and} \quad m_s(x_{t'_i}, 01 \vee 10) \geqslant \eta, \quad s = t_j - t_i - 1.$$

Now, we define $t_0 = a(t'_0)$ and $t_{j+1} = a(t_j)$ for $j = 0, 1, \ldots$ It is easy to see that the numbers $t_0, t_1, \ldots$ satisfy the assertion of Remark 1.

Definition 5. A Morse sequence x is said to be *regular* if there is a representation $x = b^0 \times b^1 \times \ldots$ by a regular sequence of blocks $\{b^0, b^1, \ldots\}$.

If x is a regular Morse sequence then it is continuous. Indeed, using Remark 1 we can represent x by blocks $\{b^0, b^1, \ldots\}$ satisfying condition (R). Then we have the inequalities

$$e_0 + o_1 \geqslant \frac{1}{\lambda_t} h_t(00, 11), \quad o_0 + e_1 \geqslant \frac{1}{\lambda_t} h_t(00, 11).$$

§ 3. Outline of the proof of the isomorphism theorem. Now, we are in a position to present the most important ideas of this paper. We use a coding technique and Ornstein's distance $\bar{d}$ between stationary processes as in [5]. However, the coding technique is improved in this paper.

First we formulate the main theorem and give an outline of the proof, which contains not difficult but laborious computations. The detailed proof of Theorem 1 is given in § 4.

In this section we will write b_i^t instead of $(b^t)^i$, $i = 0, 1$.

Theorem 1. *Let $x = b^0 \times b^1 \times \ldots$, $y = \beta^0 \times \beta^1 \times \ldots$ be Morse sequences such that $|b^t| = |\beta^t| = \lambda_t$, $t \geqslant 0$, the sequences of blocks $\{b^0, b^1, \ldots\}$, $\{\beta^0, \beta^1, \ldots\}$ satisfy condition* (R) *with a positive number η and $\lambda_t > 8/\eta$ for sufficiently large t.*

Then the dynamical systems $\theta(x)$ and $\theta(y)$ are metrically isomorphic iff there exist numbers q_t, $0 \leqslant q_t \leqslant \lambda_t - 1$, and a sequence of pairs $\{(r_t, s_t)\}$, $r_t, s_t = 0, 1$, $t \geqslant 0$, such that

(A) $$\sum_{t=0}^{\infty} d(\beta^t, b_{r_t}^t b_{s_t}^t [q_t, q_t + \lambda_t - 1]) < \infty,$$

(B) $$\sum_{t=0}^{\infty} \min(l_t/n_t, 1 - l_t/n_t) < \infty,$$

where $l_0 = q_0$, $l_t = q'_0 + q'_1 n_0 + \ldots + q'_t n_{t-1}$, $q'_0 = q_0$ and

$$q'_t = q_t \quad \text{if } l_{t-1} \leqslant n_{t-1} - l_{t-1} - 1,$$
$$q'_t = q_t - 1 \ (\mathrm{mod}\ \lambda_t) \quad \text{if } l_{t-1} > n_{t-1} - l_{t-1} - 1.$$

for $t \geqslant 1$.

Assume that $h\colon X(y) \to X(x)$ is an isomorphism between the dynamical systems $\theta(y)$ and $\theta(x)$. Denote by $P(x) = (P_0, P_1)$ the time zero partition of $X(x)$, i.e.

$$P_0 = \{v \in X(x)\colon v[0] = 0\}, \qquad P_1 = \{v \in X(x)\colon v[0] = 1\}.$$

Then $Q = h^{-1}(P(x))$ is a partition of $X(y)$. The partition Q defines the stationary process (T, Q, m_y) on $X(y)$. The process (T, Q, m_y) determines an invariant measure ν on the shift space $\prod_{-\infty}^{\infty}\{0, 1\}$ defined on cylinders by putting

$$(16) \qquad \nu(B) = m_y\Big(\bigcap_{j=0}^{n-1} T^{-1}(Q_{b_j})\Big).$$

where $B = (b_0 \ldots b_{n-1})$, $b_0, \ldots, b_{n-1} = 0, 1$, $n \geqslant 1$. We have

$$(17) \qquad m_y\Big(\bigcap_{j=0}^{n-1} T^{-1}(Q_{b_j})\Big) = m_x\Big(\bigcap_{j=0}^{n-1} T^{-j}(P_{b_j}(x))\Big)$$

because h is a measure-preserving transformation. At the same time, Q is a generator for T on $X(y)$. Conversely, if there is a generator $Q = (Q_0, Q_1)$ of $X(y)$ satisfying (17) then $\theta(x)$ and $\theta(y)$ are metrically isomorphic. In order to prove the theorem it suffices to show that the existence of such a generator implies conditions (A) and (B) and conversely that these conditions allow to construct Q.

Necessity. Suppose $Q = h^{-1}(P(x))$. Let ξ_t be the sequence of partitions of $X(y)$ defined in § 1. There exist sequences $\{A_t\}$, $\{B_t\}$ of codes of Q with respect to $\{\xi_t\}$ satisfying (11). We will prove that the codes $\{A_t\}$, $\{B_t\}$ may be chosen in the following form:

$$(18) \qquad \begin{aligned} &A_t = c_t^{r_t} c_t^{s_t}[l_t, l_t+n_t-1], \qquad 0 \leqslant l_t \leqslant l_t \leqslant n_t-1,\ r_t, s_t = 0, 1,\\ &l_{t+1} \equiv l_t \pmod{n_t},\\ &B_t = \tilde{A}_t, \qquad t \geqslant 0. \end{aligned}$$

The main tool in proving the above is the Ornstein distance $\bar{d}$ [9]. We use another definition of the Ornstein distance $\bar{d}$ [2] between invariant measures defined by stationary processes in the same way as in (16).

Take two invariant ergodic measures φ, ψ on $X = \prod_{-\infty}^{\infty}\{0, 1\}$ and denote by X_φ, X_ψ the sets of all generic sequences for φ and ψ respectively. Define

$$(19) \qquad \bar{d}(\varphi, \psi) = \inf_{\substack{y \in X_\varphi \\ \bar{y} \in X_\psi}} \overline{\lim}\, \frac{1}{n}\sum_{i=0}^{n-1} \varrho(y[i], \bar{y}[i]),$$

where

$$\varrho(u, v) = \begin{cases} 0 & \text{if } u = v,\\ 1 & \text{if } u \neq v, \end{cases}$$

$u, v = 0, 1$.

Let $\{Q^t\}$, $t \geqslant 0$, be the sequence of partitions of $X(y)$ corresponding to the codes $\{A_t, B_t\}$ (see § 2). Then $Q^t \prec \xi_t$ and

$$|Q - Q^t| \underset{t}{\to} 0.$$

As a simple property of the Ornstein distance $\bar{d}$ [9] we get the inequality

$$\bar{d}\big((T, Q^t, m_y), (T, Q, m_y)\big) \leqslant |Q - Q^t|.$$

Let ν_t be the invariant measures on X determined by the processes (T, Q^t, m_y), $t \geqslant 0$; then we have

$$(20) \qquad \bar{d}(\nu_t, m_x) \underset{t}{\to} 0,$$

because m_x is determined by the process (T, Q, m_y) (see (17)).

Now we describe the set of all generic sequences for ν_t. Observe that the dynamical systems (X, T, ν_t) are homomorphic images of $(X(y), T, m_y)$. Indeed, the partition Q^t (t fixed) defines a homomorphism g_t from $(X(y), T, m_y)$ to (X, T, ν_t) as follows:

$$(g_t(u))[j] = i \quad \text{iff} \quad T^j u \in Q_i^t,$$

where $u \in X(y)$, $j = 0, \pm 1, \ldots$, $i = 0$ or 1.

Denote by $\bar{f}$ the mapping from $Z \times Z_2$ to $X(y)$ defined in § 1 and let $u = \bar{f}(\bar{w}, i) \in X(y)$, $\bar{w} \in \bar{Z}$, $i = 0$ or 1. Put $j_t' = j_t(\bar{w})$, $k_t' = k_t(\bar{w})$. Then each fragment $u[-j_t'+kn_t, k_t'+kn_t]$, $k = 0, \pm 1, \ldots$, of u is either $\gamma_t = \beta^0 \times \beta^1 \times \ldots \times \beta^t$ or $\tilde{\gamma}_t$. It is not difficult to observe that $g_t(u)$ arises from u by substituting A_t in place of γ_t and B_t in place of $\tilde{\gamma}_t$. It is clear that $\nu_t\big(g_t(X(y))\big) = 1$.

Therefore ν_t is an ergodic measure on X as a homomorphic image of m_y. At the same time, each $u \in X(y)$ is a generic sequence for m_y [5] and it is easy to see that $g_t(u)$ is a generic sequence for ν_t.

Now, by using (20), for arbitrary small $\varepsilon > 0$ we can find t_ε such that

$$\bar{d}(\nu_t, m_x) < \varepsilon^2$$

whenever $t \geqslant t_\varepsilon$. It follows from (19) that there exist sequences $u \in X(y)$, $v \in X(x)$ such that

$$\overline{\lim}\, \frac{1}{n}\operatorname{card}\{i\colon 0 \leqslant i < n,\ v[i] \neq g_t(u)[i]\} < \varepsilon^2.$$

Each fragment $g_t(u)[j'_t+kn_t, j'_t+(k+1)n_t]$, $k = 0, \pm 1, \ldots$, of $g_t(u)$ is equal to A_t or B_t while each fragment $v[j'_t+kn_t, j'_t+(k+1)n_t]$, $k = 0, \pm 1, \ldots$, of v is equal to one of the four blocks

$$c_t^i c_t^j [j_t, j_t+n_t-1], \quad i, j = 0, 1,$$

where j_t is a number from $\{0, 1, \ldots, n_t\}$.

Evidently there are a lot of fragments $g_t(u)[j'_t+kn_t, j'_t+(k+1)n_t]$ of $g_t(u)$ such that the distance d between them and the corresponding fragments of v is smaller than ε. That is why the codes A_t and B_t have forms (18). The detailed considerations about them are given in Lemmas 1, 2 and 3.

As a consequence of the above we obtain codes satisfying (18) and

$$\sup_{k \geqslant 1} d(A_{t+k}, A_t^{(k)}) \underset{t}{\to} 0, \tag{21}$$

where $A_t^{(k)} = A_t \times \beta^{t+1} \times \ldots \times \beta^{t+k}$, $t, k \geqslant 0$.

The remaining part of the proof of the necessity may be obtained from (21) by computations. We present them in Lemmas 4, 5 and 6.

Sufficiency. We will construct a partition Q of $X(y)$ satisfying

(a) $$m_x\Big(\bigvee_{i=0}^{n} T^{-i} P(x)\Big) = m_y\Big(\bigvee_{i=0}^{n} T^{-i} Q\Big)$$

for infinitely many n,

(b) Q is a generator for T.

To do this we define q'_t and l_t in the same way as in (B) and we choose t_0 in such a way that

$$\min(l_t/n_t, 1-l_t/n_t) < \tfrac{1}{2} \quad \text{for } t \geqslant t_0.$$

Further we set

$$r'_{t_0} = r_{t_0}, \qquad s'_{t_0} = s_{t_0},$$

$$r'_t = r_t + r'_{t-1} \pmod 2, \quad s'_t = s_t + r'_{t-1} \pmod 2 \quad \text{if } \frac{l_{t-1}}{n_{t-1}} < 1 - \frac{l_{t-1}}{n_{t-1}},$$

$$r'_t = r_t + s'_{t-1} \pmod 2, \quad s'_t = s_t + s'_{t-1} \pmod 2 \quad \text{if } \frac{l_{t-1}}{n_{t-1}} > 1 - \frac{l_{t-1}}{n_{t-1}} \text{ and } q_t > 0.$$

If $q_t = 0$ and $l_{t-1}/n_{t-1} > 1 - l_{t-1}/n_{t-1}$ then we set $s'_t = r_t + s'_{t-1}$, r'_t arbitrary. In the above definitions $t \geqslant t_0$.

Next we define codes $\{A_t, B_t\}$, $t \geqslant t_0$, by putting

$$A_t = c_t^{r'_t} c_t^{s'_t} [l_t, l_t+n_t-1], \quad B_t = \tilde{A}_t.$$

We show (Lemma 7) that $\{A_t, B_t\}$ satisfy the conditions (11) so they define a partition Q of $X(y)$. Now we prove that Q satisfies (a) and (b).

Let Q^t, $t \geqslant 0$, be the partition of $X(y)$ determined by the codes $\{A_t, \tilde{A}_t\}$ and let ν_t be the invariant measure given by the process (T, Q^t, m_y). Then (20) implies

$$\lim_t \nu_t(B) = \nu(B) \tag{22}$$

for an arbitrary block B, where ν is the invariant measure determined by the process (T, Q, m_y). The numbers $\nu_t(B)$ are equal to the average relative frequency of B in the sequence $g_t(u)$. Since

$$A_t = c_t^{r'_t} c_t^{s'_t} [l_t, l_t+n_t-1]$$

it is easy to see that

$$\lim_t \nu_t(B) = m_x(B).$$

The last equality and (22) give (a).

It remains to show that Q is a generator of $X(y)$. Let S be the homomorphism from $X(y)$ to $X(x)$ defined by the partition Q and let S_t, $t \geqslant 0$, be the homomorphisms defined by Q^t. The properties $B_t = \tilde{A}_t$, $t \geqslant 0$, imply $S_t(\tilde{u}) = (S_t(u))^\sim$ for each $u \in X(y)$. For fixed n we have

$$\Big|\bigvee_{i=-n}^{n} T^{-i} Q^t - \bigvee_{i=-n}^{n} T^{-i} Q\Big| \underset{t}{\to} 0. \tag{23}$$

It follows that for a.e. $u \in X(y)$

$$S(\tilde{u}) = (S(u))^\sim. \tag{24}$$

Further let η_y be the partition of $X(y)$ into the sets of the form $\{u, \tilde{u}\}$ and let η_x be the partition of $X(x)$ into such sets. The property (24) implies $S(\eta_y) = \eta_x$, that is, S is a homomorphism from the factor dynamical system $(X(y)/\eta_y, T)$ to $(X(x)/\eta_x, T)$. It follows from the construction of the isomorphism f in § 1 that the dynamical systems $(X(x)/\eta_x, T)$ and $(X(y)/\eta_y, T)$ have discrete spectra and Λ is their eigenvalue group. This means that S is an isomorphism from $(X(y)/\eta_y, T)$ to $(X(x)/\eta_x, T)$, that is, S is one-to-one mod 0.

The last property and (24) imply that S is an a.e. one-to-one mapping from $X(y)$ to $X(x)$. Thus S is an isomorphism of the dynamical systems $(X(y), T, m_y)$ and $(X(x), T, m_x)$.

We finish the considerations of this section by the following theorem which is a consequence of Theorem 1.

THEOREM 2. *If x is a regular Morse sequence then there exists a continuum of Morse sequences y such that the measures m_y are pairwise* *orthogonal and the dynamical systems $\theta(y)$ are metrically isomorphic to $\theta(x)$.*

Proof. We can assume that x has the form

$$x = b^0 \times b^1 \times \ldots$$

and the sequence of blocks $\{b^0, b^1, \ldots\}$ satisfies condition (R) and $\sum_{t=0}^{\infty} 1/\lambda_t < \infty$, $\lambda_t = |b^t|$, $t = 0, 1, \ldots$ Each block b^t consists of finite sequences of zeros and unities. It follows from condition (R) that the length of the shortest sequences is at most $1/2\eta$. Now we define the block β^t by changing one shortest sequence in b^t into the mirror sequence. Then we have

$$d(b^t, \beta^t) \leqslant \frac{1}{2\eta} \sum_{t=0}^{\infty} \frac{1}{\lambda_t} < \infty,$$

and

$$|\mathrm{fr}(00, b^t) + \mathrm{fr}(11, b^t) - \mathrm{fr}(00, \beta^t) - \mathrm{fr}(11, \beta^t)| = 2,$$
$$|\mathrm{fr}(01, b^t) + \mathrm{fr}(10, b^t) - \mathrm{fr}(01, \beta^t) - \mathrm{fr}(10, \beta^t)| = 2.$$

We can construct Morse sequences y taking infinitely many blocks b^t and infinitely many blocks β^t. In this way we obtain a continuum of Morse sequences. Applying Theorem 1 and the formulas (13) we conclude that the dynamical systems $\theta(y)$ are metrically isomorphic to $\theta(x)$ and the measures m_y are pairwise orthogonal.

In [6] it was proved that there exist classes of spectrally isomorphic Morse dynamical systems consisting of a continuum of metrically nonisomorphic systems.

§ 4. Lemmas. We assume in Lemmas 1 to 6 that $h\colon X(y) \to X(x)$ is an isomorphism, $Q = h^{-1}(P(x))$ and the assumptions of Theorem 1 are satisfied.

Lemma 1. *The partition Q has codes $\{A_t', B_t'\}$ such that*

$$A_t' = c_t^{r_t'} c_t^{s_t'} [l_t, l_t + n_t - 1], \qquad B_t' = \tilde{A}_t',$$

where $r_t', s_t' = 0$ or 1, $0 \leqslant l_t \leqslant n_t - 1$, $t \geqslant 0$.

Proof. Let $\{\bar{A}_t, \bar{B}_t\}$ be codes of Q and let $\varepsilon_j \searrow 0$. Choose numbers $t_1 < t_2 < \ldots$ such that

(25) $$\bar{d}(v_t, m_x) < \varepsilon_j^2 \quad \text{for } t \geqslant t_j.$$

Let j_0 be a number such that $\varepsilon_j < \frac{1}{2}$ for $j \geqslant j_0$ and put $t_0 = t_{j_0}$. Fix $t \geqslant t_0$ and choose j satisfying the condition $t_j \leqslant t < t_{j+1}$. Then there exist sequences $u \in X(y)$, $v \in X(x)$ such that

(26) $$\overline{\lim_n} \frac{1}{n} \operatorname{card}\{i\colon 0 \leqslant i < n,\ v[i] \neq g_t(u)[i]\} < \varepsilon_j^2.$$

Next we define

$$Z_j = \{k \geqslant 0\colon d(g_t(u)[kn_t, (k+1)n_t - 1], v[kn_t, (k+1)n_t - 1]) < \varepsilon_j\}.$$

(25) and (26) imply

(27) $$\overline{\lim_n} \frac{1}{n} \operatorname{card}\{k\colon 0 \leqslant k < n,\ k \in Z_j\} > 1 - \varepsilon_j.$$

Suppose that $v = f(\bar{z}, i)$, $i = 0$ or 1, $\bar{z} = (z_0, z_1, \ldots) \in \bar{Z}$. Then the blocks $v[kn_t, (k+1)n_t - 1]$, $k = 0, \pm 1, \ldots$, have the following forms:

(28) $$\begin{aligned} \mathrm{I}_t &= c_t c_t [l_t, l_t + n_t - 1], & \mathrm{II}_t &= c_t \tilde{c}_t [l_t, l_t + n_t - 1],\\ \mathrm{III}_t &= \tilde{c}_t c_t [l_t, l_t + n_t - 1], & \mathrm{IV}_t &= \tilde{c}_t \tilde{c}_t [l_t, l_t + n_t - 1], \end{aligned}$$

where $l_t = j_t(\bar{z})$. As before (see § 3) let $u = \bar{f}(\bar{w}, i)$. We can assume that $j_t'(\bar{w}) = 0$, i.e. $g_t(u)[kn_t, (k+1)n_t - 1]$ is $\bar{A}_t$ or $\bar{B}_t$ for every $k = 0, \pm 1, \ldots$ Let

$$Z_j^{(0)} = \{k \in Z_j\colon g_t(u)[kn_t, (k+1)n_t - 1] = \bar{A}_t\},$$
$$Z_j^{(1)} = \{k \in Z_j\colon g_t(u)[kn_t, (k+1)n_t - 1] = \bar{B}_t\}.$$

Therefore the conditions

$$\lim_n \frac{1}{n} \operatorname{card}\{0 \leqslant k < n\colon g_t(u)[kn_t, (k+1)n_t - 1] = \bar{A}_t\} = m_{y_{t+1}}(0) = \tfrac{1}{2},$$

$$\lim_n \frac{1}{n} \operatorname{card}\{0 \leqslant k < n\colon g_t(u)[kn_t, (k+1)n_t - 1] = \bar{B}_t\} = m_{y_{t+1}}(1) = \tfrac{1}{2},$$

and (27) imply

$$\overline{\lim_n} \frac{1}{n} \operatorname{card}\{0 \leqslant k < n\colon k \in Z_j^{(0)}\} > \tfrac{1}{2} - \varepsilon_j,$$

$$\overline{\lim_n} \frac{1}{n} \operatorname{card}\{0 \leqslant k < n\colon k \in Z_j^{(1)}\} > \tfrac{1}{2} - \varepsilon_j.$$

Take an arbitrary $k \in Z_j^{(0)}$. Then there exist $r', s' \in \{0, 1\}$ depending on t such that

$$v[kn_t, (k+1)n_t - 1] = c_t^{r'} c_t^{s'} [l_t, l_t + n_t - 1].$$

Putting $A_t' = c_t^{r'} c_t^{s'} [l_t, l_t + n_t - 1]$ for $t \in [t_j, t_{j+1})$ we have

(29) $$d(A_t', \bar{A}_t) < \varepsilon_j.$$

Using the equalities

$$d(\mathrm{I}_t, \mathrm{IV}_t) = d(\mathrm{II}_t, \mathrm{III}_t) = 1$$

and

$$m_{x_{t+1}}(00) = m_{x_{t+1}}(11) = \tfrac{1}{2}\bar{p}_{t+1} > 0, \quad m_{x_{t+1}}(01) = m_{x_{t+1}}(10) = \tfrac{1}{2}\bar{q}_{t+1},$$
$$\bar{q}_{t+1} > 0, \quad \bar{p}_{t+1} + \bar{q}_{t+1} = 1,$$

we come to the conclusion that there exist $k_1 \in Z_j^{(0)}$, $k_2 \in Z_j^{(1)}$ satisfying

(30) $$d(v[k_1 n_t, (k_1+1)n_t - 1], v[k_2 n_t, (k_2+1)n_t - 1]) = 1.$$

Thus (29) and (30) imply

$$d(A_t, B_t) > 1 - 2\varepsilon_j.$$

If we put $B_t' = \tilde{A}_t'$ for $t \in [t_j, t_{j+1} - 1]$ then

(31) $$d(\bar{A}_t, A_t') \underset{t}{\to} 0, \quad d(\bar{B}_t, B_t') \underset{t}{\to} 0.$$

For $t < t_0$ we may take A_t' of the form $A_t' = c_t^{r_t'} c_t^{s_t'} [l_t, l_t + n_t - 1]$ in an arbitrary way requiring only that $l_{t+1} \equiv l_t \pmod{n_t}$. The sequences of blocks $\{A_t', B_t'\}$, $t \geq 0$, satisfy the assertion of the lemma.

LEMMA 2. *Let B, C be blocks of lengths n_{t+k+1} and n_t, respectively, $t \geq 0$, $k \geq 1$, of the form*

$$B = c_{t+k+1}^r c_{t+k+1}^s [l_{t+k+1}, l_{t+k+1} + n_{t+k+1} - 1],$$
$$C = c_t^p c_t^q [j_t, j_t + n_t - 1], \quad r, s, p, q \in \{0, 1\},$$

and let

$$C^{(i+1)} = C \times \beta^{t+1} \times \ldots \times \beta^{t+i+1}, \quad i = 0, 1, \ldots, k, \quad C^{(0)} = C.$$

If $d(B, C^{(k+1)}) \leq \eta$ then there exist blocks $A_t'', A_{t+1}'', \ldots, A_{t+k+1}''$ of the form

(32) $$A_{t+i}'' = c_{t+i}^{r_i'} c_{t+i}^{s_i'} [l_{t+i}, l_{t+i} + n_{t+i} - 1],$$

satisfying

$$d(A_{t+i}'', C^{(i)}) \leq \eta \quad \textit{and} \quad l_{t+i} \equiv l_{t+i+1} \pmod{n_{t+i}},$$

$i = 0, 1, \ldots, k$.

Proof. Put $A_{t+k+1}'' = B$. Suppose that for some i, $1 \leq i \leq k$, the blocks $A_{t+k+1}'', \ldots, A_{t+i+1}''$ are defined and have forms (32). Further we have

(33) $$d(A_{t+i+1}'', C^{(i+1)}) = \frac{1}{\lambda_{t+i+1}} \sum_{l=0}^{\lambda_{t+i+1}-1} d(A_{t+i+1}''[ln_{t+i}, (l+1)n_{t+i} - 1], C^{(i+1)}[ln_{t+i}, (l+1)n_{t+i} - 1]).$$

The inequality $d(A_{t+i+1}'', C^{(i+1)}) \leq \eta$ implies that at least one component of the sum (33) is $\leq \eta$. Suppose that this component corresponds to some number l, $0 \leq l \leq \lambda_{t+i+1} - 1$.

Then one of the two blocks

$$A_{t+i}'' = A_{t+i+1}''[ln_{t+i}, (l+1)n_{t+i} - 1] \quad \text{or} \quad A_{t+i}'' = A_{t+i+1}''[ln_{t+i}, (l+1)n_{t+i} - 1]$$

satisfies the condition (32). We can proceed in this way for each $i = k, k-1, \ldots, 1$ so the lemma is proved.

LEMMA 3. *Let Q be the partition of $X(y)$ as in Lemma 1. Then the codes $\{A_t, B_t\}$, $t \geq 0$, of Q can be improved in such a way that $l_{t+1} \equiv l_t \pmod{n_t}$, $t \geq 0$.*

Proof. Let $\{A_t'\}$ be a sequence of codes of Q satisfying the conclusion of Lemma 1. Take a sequence of positive numbers ε_j, $j \geq 1$, with $\sum_{j=1}^{\infty} \varepsilon_j < \infty$ and choose positive integers $1 \leq t_1 < t_2 < \ldots$ such that

(34) $$\sup_i d(A_{t_j+i}', A_{t_j}'^{(i)}) \leq \varepsilon_j, \quad j = 1, 2, \ldots$$

Now, for fixed $j \geq 1$ we will construct blocks $A_{t_1}(j), A_{t_1+1}(j), \ldots, A_{t_{j+1}-1}(j)$ satisfying the following conditions:

(35) $$\begin{gathered} A_{t_1+i}(j) = c_{t_1+i}^{r_i} c_{t_1+i}^{s_i} [l_{t_1+i}(j), l_{t_1+i}(j) + n_{t_1+i} - 1], \ 0 \leq i \leq t_{j+1} - t_1 - 1, \\ l_{t_1+i}(j) \equiv l_{t_1+i+1}(j) \pmod{n_{t_1+i}}, \quad i = 0, 1, \ldots, t_{j+1} - t_1 - 2, \\ d(A_{t_1+i}(j), A_{t_p}'^{(i-t_p+t_1)}) \leq \varepsilon_p + \ldots + \varepsilon_j, \\ t_p - t_1 \leq i \leq t_{p+1} - t_1 - 1, \quad p = 1, 2, \ldots, j. \end{gathered}$$

First we apply Lemma 2 with $t = t_j$, $B = A_{t_{j+1}-1}'$, $C = A_{t_j}'$, $k = t_{j+1} - t_j - 1$, $\eta = \varepsilon_j$. As a consequence we obtain blocks $A_{t_j+i}(j)$, $i = 0, 1, \ldots, t_{j+1} - t_j - 1$, satisfying (32). If $j = 1$ then the blocks $A_{t_1}(1), \ldots, A_{t_2-1}(1)$ satisfy (35). If $j > 1$ then using (32) and (34) we have

(36) $$d(A_{t_j}(j), A_{t_j}') \leq \varepsilon_j \quad \text{and} \quad d(A_{t_j}', A_{t_{j-1}}'^{(t_j - t_{j-1})}) \leq \varepsilon_{j-1}.$$

This gives

$$d(A_{t_j}(j), A_{t_{j-1}}'^{(t_j - t_{j-1})}) \leq \varepsilon_j + \varepsilon_{j-1}.$$

Applying Lemma 2 again with $B = A_{t_j}(j)$, $t = t_{j-1}$, $C = A_{t_{j-1}}'$, $k = t_j - t_{j-1} - 1$, $\eta = \varepsilon_j + \varepsilon_{j-1}$ we obtain blocks $A_{t_{j-1}+i}(j)$, $i = 0, 1, \ldots, t_j - t_{j-1} - 1$, satisfying (35). If $j > 2$ then (34) and (35) imply

$$d(A_{t_{j-1}}(j), A_{t_{j-2}}'^{(t_{j-1} - t_{j-2})}) \leq \varepsilon_j + \varepsilon_{j-1} + \varepsilon_{j-2}$$

and we can apply Lemma 2 with $B = A_{t_{j-1}}(j)$, $t = t_{j-2}$, $C = A_{t_{j-2}}'$, $k = t_{j-1} - t_{j-2} - 1$, $\eta = \varepsilon_j + \varepsilon_{j-1} + \varepsilon_{j-2}$. Proceeding in this manner we obtain blocks $A_{t_1}(j), \ldots, A_{t_{j+1}-1}(j)$ satisfying (35).

In order to prove the lemma we consider the sequences of blocks that we have obtained above:

$$\begin{aligned}
&A_{t_1}(1), \ldots, A_{t_2-1}(1),\\
&A_{t_1}(2), \ldots, A_{t_2-1}(2), A_{t_2}(2), \ldots, A_{t_3-1}(2),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&A_{t_1}(j), \ldots, A_{t_2-1}(j), \ldots, A_{t_3-1}(j), A_{t_3}(j), \ldots, A_{t_{j+1}-1}(j).
\end{aligned}$$

We can find a subsequence of numbers $\{j_s\}$, $s \geqslant 1$, such that

$$l_{t_1+i}(j_s) = l_{t_1+i} \quad \text{for } s \geqslant i+1,\ i = 0, 1, \ldots$$

Further we define

$$A_{t_1+i} = A_{t_1+i}(j_{i+1}), \quad i = 0, 1, \ldots$$

Then

$$l_{t_1+i+1} = l_{t_1+i+1}(j_{i+2}) \equiv l_{t_1+i}(j_{i+2}) \pmod{n_{t_1+i}}$$

and

$$l_{t_1+i}(j_{i+2}) = l_{t_1+i}, \quad i \geqslant 0.$$

In this way

$$l_{t_1+i+1} \equiv l_{t_1+i} \pmod{n_{t_1+i}}, \quad i \geqslant 0.$$

It follows by the construction of the blocks $A_{t_1+i}(j)$ that

$$A_{t_1+i} = c_{t_1+i}^{r_{t_1+i}} c_{t_1+i}^{s_{t_1+i}} [l_{t_1+i}, l_{t_1+i}+n_{t_1+i}-1], \quad i \geqslant 0.$$

Now we prove that $\{A_t\}$, $t \geqslant t_1$, are codes of Q. It remains to show that $d(A_{t_1+i}, A'_{t_1+i}) \to 0$ as i tends to ∞.

Suppose that $i \in \{t_p - t_1, t_p - t_1 + 1, \ldots, t_{p+1} - t_1 - 1\}$. Then (35) implies

$$d(A_{t_1+i}, A_{t_p}'^{(i-t_p+t_1)}) \leqslant \sum_{s=p}^{\infty} \varepsilon_s. \tag{37}$$

By (34) we have

$$d(A'_{t_1+i}, A_{t_p}'^{(i-t_p+t_1)}) \leqslant \varepsilon_p. \tag{38}$$

Now, (37) and (38) give

$$d(A'_{t_1+i}, A_{t_1+i}) \leqslant \sum_{s=p}^{\infty} \varepsilon_s + \varepsilon_p. \tag{39}$$

Thus we obtain

$$d(A'_{t_1+i}, A_{t_1+i}) \underset{i}{\to} 0.$$

For $i = 1, \ldots, t_1 - 1$ we take A_i of the form $c_i^{r_i} c_i^{s_i} [l_i, l_i + n_i - 1]$ where r_i, s_i are arbitrary from $\{0, 1\}$, and $l_1, \ldots, l_{t_1-1}$ satisfy the conditions $l_i \equiv l_{i+1} \pmod{n_i}$, $i = 0, 1, \ldots, t_1 - 1$. In this way we finish the proof of the lemma.

In the sequel we assume that the codes $\{A_t, \tilde{A}_t\}$ of Q have the form

$$A_t = c_t^{r'_t} c_t^{s'_t} [l_t, l_t + n_t - 1]$$

and

$$l_{t+1} \equiv l_t \pmod{n_t}, \quad t \geqslant 0.$$

Lemma 4. *The numbers l_t, $t \geqslant 0$, satisfy the condition*

$$\min(l_t/n_t, 1 - l_t/n_t) \underset{t}{\to} 0.$$

Proof. For convenience we put

$$\begin{aligned}
&\gamma_t^{(k)} = \beta^{t+1} \times \ldots \times \beta^{t+k},\\
&\delta_t^{(k)} = (b^{t+1} \times \ldots \times b^{t+k})^r (b^{t+1} \times \ldots \times b^{t+k})^s [l_t^{(k)}, l_t^{(k)} + n_t^{(k)} - 1],
\end{aligned}$$

where $r = r'_{t+k}$, $s = s'_{t+k}$, $n_t^{(k)} = \lambda_{t+1} \ldots \lambda_{t+k}$ and $l_t^{(k)}$ satisfy the formulas $l_{t+k} = l_t + n_t l_t^{(k)}$, $t, k \geqslant 0$. By the definition of the distance d we obtain the following formula:

$$d(A_{t+k}, A_t^{(k)}) = \frac{l_1(k)}{n_t^{(k)}} \cdot \frac{l_t}{n_t} + \frac{l_2(k)}{n_t^{(k)}} \left(1 - \frac{l_t}{n_t}\right) + \frac{l_3(k)}{n_t^{(k)}}, \tag{40}$$

where

$$\begin{aligned}
l_1(k) = \operatorname{card}\{i\colon 0 \leqslant i \leqslant n_t^{(k)} - 1,\ &\gamma_t^{(k)}[i] = 0 \text{ and } \delta_t^{(k)}[i, i+1] = r'_t \tilde{s}'_t\\
&\text{or } \gamma_t^{(k)}[i] = 1 \text{ and } \delta_t^{(k)}[i, i+1] = \tilde{r}'_t s'_t\}.\\
l_2(k) = \operatorname{card}\{i\colon 0 \leqslant i \leqslant n_t^{(k)} - 1,\ &\gamma_t^{(k)}[i] = 0 \text{ and } \delta_t^{(k)}[i, i+1] = \tilde{r}'_t s'_t\\
&\text{or } \gamma_t^{(k)}[i] = 1 \text{ and } \delta_t^{(k)}[i, i+1] = r'_t \tilde{s}'_t\},\\
l_3(k) = \operatorname{card}\{i\colon 0 \leqslant i \leqslant n_t^{(k)} - 1,\ &\gamma_t^{(k)}[i] = 0 \text{ and } \delta_t^{(k)}[i, i+1] = \tilde{r}'_t \tilde{s}'_t\\
&\text{or } \gamma_t^{(k)}[i] = 1 \text{ and } \delta_t^{(k)}[i, i+1] = r'_t s'_t\}.
\end{aligned}$$

Thus (40) implies

$$d(A_{t+k}, A_t^{(k)}) \geqslant \frac{l_2(k) + l_3(k)}{n_t^{(k)}} \left(1 - \frac{l_t}{n_t}\right) + \frac{l_1(k)}{n_t^{(k)}} \cdot \frac{l_t}{n_t}$$

and

$$d(A_{t+k}, A_t^{(k)}) \geqslant \frac{l_1(k) + l_3(k)}{n_t^{(k)}} \cdot \frac{l_t}{n_t} + \frac{l_2(k)}{n_t^{(k)}} \left(1 - \frac{l_t}{n_t}\right).$$

Next we obtain

(41) $$d(A_{t+k}, A_t^{(k)}) \geq \frac{l_1(k)+l_2(k)}{n_t^{(k)}} \min\left(\frac{l_t}{n_t}, 1-\frac{l_t}{n_t}\right).$$

The equality

$$\lim_k \frac{l_1(k)+l_2(k)}{n_t^{(k)}} = \begin{cases} \bar{q}_{t+1} & \text{if } r_t' = s_t', \\ \bar{p}_{t+1} & \text{if } r_t' = \tilde{s}_t', \end{cases}$$

condition (R), (11) and (41) imply the assertion of the lemma.

LEMMA 5. *There exist numbers q_t, $0 \leq q_t \leq \lambda_t - 1$, and a sequence of pairs $\{r_t, s_t\}$ such that*

$$\sum_{t=0}^{\infty} d(\beta^t, b^t_{r_t} b^t_{s_t}[q_t, q_t+\lambda_t-1]) < \infty.$$

Proof. There exist numbers q_t', $0 \leq q_t' \leq \lambda_t - 1$, $t \geq 0$, such that $l_0 = q_0'$, $l_t = q_0' + q_1' n_0 + \ldots + q_t' n_{t-1}$, $t \geq 1$, because the conditions $l_{t+1} \equiv l_t \pmod{n_t}$ hold. We define the numbers q_t in the same way as in Theorem 1.

Next we find r_t, s_t. Let $u_t(k)$ denote the minimum of four numbers

$$d(A_t^{(k)}, c^i_{t+k} c^j_{t+k}[l_{t+k}, l_{t+k}+n_{t+k}-1]), \quad i, j = 0, 1.$$

Thus

$$u_t(k) \leq d(A_{t+k}, A_t^{(k)})$$

which implies

(42) $$\sup_{k \geq 1} u_t(k) \underset{t}{\to} 0.$$

Further there exist $r = r_{t,k}$ and $s = s_{t,k}$ such that

$$u_t(k) = d(A_t^{(k)}, c^r_{t+k} c^s_{t+k}[l_{t+k}, l_{t+k}+n_{t+k}-1]).$$

Denote

$$v_t(k) = d(A_t^{(k)}, c^r_{t+k} c^{1-s}_{t+k}[l_{t+k}, l_{t+k}+n_{t+k}-1]),$$
$$\alpha_t^{(k)} = b_r^{t+k} b_s^{t+k}[q'_{t+k}, q'_{t+k}+\lambda_{t+k}].$$

Then we have

(43) $$u_t(k+1) = \frac{\bar{l}_1(k)}{\lambda_{t+k+1}} u_t(k) + (1-u_t(k))\frac{\bar{l}_2(k)}{\lambda_{t+k+1}} + v_t(k)\frac{\bar{l}_3(k)}{\lambda_{t+k+1}} + (1-v_t(k))\frac{\bar{l}_4(k)}{\lambda_{t+k+1}},$$

where

$$\bar{l}_1(k) = \operatorname{card}\{i\colon 0 \leq i \leq \lambda_{t+k+1}-1,\ \beta^{t+k+1}[i] = 0 \text{ and } \alpha_t^{(k+1)}[i, i+1] = rs \text{ or } \beta^{t+k+1}[i] = 1 \text{ and } \alpha_t^{(k+1)}[i, i+1] = \tilde{r}\tilde{s}\},$$

$$\bar{l}_2(k) = \operatorname{card}\{i\colon 0 \leq i \leq \lambda_{t+k+1}-1,\ \beta^{t+k+1}[i] = 0 \text{ and } \alpha_t^{(k+1)}[i, i+1] = \tilde{r}\tilde{s} \text{ or } \beta^{t+k+1}[i] = 1 \text{ and } \alpha_t^{(k+1)}[i, i+1] = rs\},$$

$$\bar{l}_3(k) = \operatorname{card}\{i\colon 0 \leq i \leq \lambda_{t+k+1}-1,\ \beta^{t+k+1}[i] = 0 \text{ and } \alpha_t^{(k+1)}[i, i+1] = r\tilde{s} \text{ or } \beta^{t+k+1}[i] = 1 \text{ and } \alpha_t^{(k+1)}[i, i+1] = \tilde{r}s\},$$

$$\bar{l}_4(k) = \operatorname{card}\{i\colon 0 \leq i \leq \lambda_{t+k+1}-1,\ \beta^{t+k+1}[i] = 0 \text{ and } \alpha_t^{(k+1)}[i, i+1] = \tilde{r}s \text{ or } \beta^{t+k+1}[i] = 1 \text{ and } \alpha_t^{(k+1)}[i, i+1] = r\tilde{s}\}.$$

We choose t_0 such that $u_{t_0}(k) < \frac{1}{4}$ for each $k \geq 0$. If $v_{t_0}(k) \leq \frac{1}{2}$ then (43) implies

$$\begin{aligned} u_{t_0}(k+1) &\geq u_{t_0}(k)\frac{\bar{l}_1(k)+\bar{l}_2(k)}{\lambda_{t_0+k+1}} + (1-2u_{t_0}(k))\frac{\bar{l}_2(k)}{\lambda_{t_0+k+1}} \\ &\quad + u_{t_0}(k)\frac{\bar{l}_3(k)}{\lambda_{t_0+k+1}} + (1-v_{t_0}(k))\frac{\bar{l}_4(k)}{\lambda_{t_0+k+1}} \\ &= u_{t_0}(k) + (1-2u_{t_0}(k))\frac{\bar{l}_2(k)}{\lambda_{t_0+k+1}} + (1-v_{t_0}(k)-u_{t_0}(k))\frac{\bar{l}_4(k)}{\lambda_{t_0+k+1}}. \end{aligned}$$

If $v_{t_0}(k) > \frac{1}{2}$ then (43) implies

$$u_{t_0}(k+1) \geq u_{t_0}(k) + (1-2u_{t_0}(k))\frac{\bar{l}_2(k)}{\lambda_{t_0+k+1}} + (v_{t_0}(k)-u_{t_0}(k))\frac{\bar{l}_3(k)}{\lambda_{t_0+k+1}}.$$

Both inequalities give

(44) $$u_{t_0}(k+1) \geq \begin{cases} u_{t_0}(k) + \dfrac{1}{4}\dfrac{\bar{l}_2(k)+\bar{l}_4(k)}{\lambda_{t_0+k+1}} & \text{if } v_{t_0}(k) \leq \frac{1}{2}, \\[2ex] u_{t_0}(k) + \dfrac{1}{4}\dfrac{\bar{l}_2(k)+\bar{l}_3(k)}{\lambda_{t_0+k+1}} & \text{if } v_{t_0}(k) > \frac{1}{2}. \end{cases}$$

Now we denote $\bar{r} = r_{t,k+1}$, $r = r_{t,k}$, $\bar{s} = s_{t,k+1}$, $s = s_{t,k}$. It follows from the definitions of $\bar{l}_2(k)$, $\bar{l}_3(k)$ and $\bar{l}_4(k)$ that

$$\frac{\bar{l}_2(k)+\bar{l}_4(k)}{\lambda_{t+k+1}} = d(\beta^{t+k+1}, b^{t+k+1}_{\bar{r}+r} b^{t+k+1}_{\bar{s}+r}[q'_{t+k+1}, q'_{t+k+1}+\lambda_{t+k+1}-1])$$

and

$$\frac{\bar l_2(k)+\bar l_3(k)}{\lambda_{t+k+1}} = d(\beta^{t+k+1},\ b_{\bar r+s}^{t+k+1}\, b_{\bar s+s}^{t+k+1}[q'_{t+k+1}+1,\ q'_{t+k+1}+\lambda_{t+k+1}]).$$

Finally we can define r_t, s_t for $t \geqslant t_0$. Put

$$r_{t_0+k+1} = r_{t_0,k+1}+r_{t_0,k}, \qquad s_{t_0+k+1} = s_{t_0,k+1}+r_{t_0,k}$$

whenever $v_{t_0}(k) \leqslant \frac{1}{2}$ and

$$r_{t_0+k+1} = r_{t_0,k+1}+s_{t_0,k}, \qquad s_{t_0+k+1} = s_{t_0,k+1}+s_{t_0,k}$$

if $v_{t_0}(k) > \frac{1}{2}$ and $q'_{t_0+k+1} < \lambda_{t_0+k+1}-1$. If $q'_{t_0+k+1} = \lambda_{t_0+k+1}-1$ then we take $r_{t_0+k+1} = s_{t_0,k+1}+s_{t_0,k}$ and s_{t_0+k+1} in any way. Recall that $q_{t_0+k+1} = 0$ in that case.

In the above formulas the addition of the symbols 0 and 1 is taken mod 2. It is easy to see that for sufficiently large t_0, $v_{t_0}(k) \leqslant \frac{1}{2}$ iff $l_{t_0+k} \leqslant n_{t_0+k}-l_{t_0+k}-1$. For $t = 0, 1, \ldots, t_0$ we may define r_t, s_t in an arbitrary way. Thus (44) implies

$$u_{t_0}(t-t_0) \geqslant u_{t_0}(t-t_0-1)+\tfrac{1}{4}\, d(\beta^t,\ b^t_{r_t} b^t_{s_t}[q_t,\ q_t+\lambda_t-1])$$

for $t > t_0+1$. By the above inequalities and by (42) the lemma follows.

LEMMA 6. *The series* $\sum_{t=0}^{\infty} \min(l_t/n_t,\ 1-l_t/n_t)$ *is convergent.*

Proof. We set

$$\bar m_t(k) = \frac{1}{n_{t+k}}\operatorname{card}\{i:\ 0 \leqslant i \leqslant n_{t+k}-l_{t+k}-1,\ A_{t+k}[i] \neq A_t^{(k)}[i]\}$$

$$\text{if } 2l_{t+k} < n_{t+k},$$

$$\bar m_t(k) = \frac{1}{n_{t+k}}\operatorname{card}\{i:\ n_{t+k}-l_{t+k} \leqslant i \leqslant n_{t+k}-1,\ A_{t+k}[i] \neq A_t^{(k)}[i]\}$$

$$\text{if } 2l_{t+k} > n_{t+k}.$$

Further we put

$$\eta_t = \min(l_t/n_t,\ 1-l_t/n_t), \qquad t \geqslant 0.$$

We will show that for sufficiently large t and for $k \geqslant 0$

$$\bar m_t(k+1) \geqslant (1-\eta_{t+k+1})\,\bar m_t(k)+\eta_{t+k}\,\varrho', \tag{45}$$

where ϱ' is a positive constant.

We consider the following cases:

(α) $l_{t+k+1} < n_{t+k+1}-l_{t+k+1}-1$ and $l_{t+k} < n_{t+k}-l_{t+k}-1$,

(β) $l_{t+k+1} < n_{t+k+1}-l_{t+k+1}-1$ and $l_{t+k} > n_{t+k}-l_{t+k}-1$,

(γ) $l_{t+k+1} > n_{t+k+1}-l_{t+k+1}-1$ and $l_{t+k} < n_{t+k}-l_{t+k}-1$,

(δ) $l_{t+k+1} > n_{t+k+1}-l_{t+k+1}-1$ and $l_{t+k} > n_{t+k}-l_{t+k}-1$.

In case (α),

$$\begin{aligned}\bar m_t(k+1) = {} & \bar m_t(k)\frac{w_1(k)}{\lambda_{t+k+1}}+\left(1-\frac{l_{t+k}}{n_{t+k}}-\bar m_t(k)\right)\frac{\lambda_{t+k+1}-q'_{t+k+1}-w_1(k)}{\lambda_{t+k+1}}\\ & +\frac{w_2(k)}{\lambda_{t+k+1}}(u_t(k)-\bar m_t(k))\\ & +\left(\frac{\lambda_{t+k+1}-q'_{t+k+1}-1-w_2(k)}{\lambda_{t+k+1}}\right)\left(\frac{l_{t+k}}{n_{t+k}}-u_t(k)+\bar m_t(k)\right),\end{aligned} \tag{46}$$

where

$w_1(k) = \operatorname{card}\{i:\ 0 \leqslant i \leqslant \lambda_{t+k+1}-q'_{t+k+1}-1,\ b^{t+k+1}_{r'_{t+k+1}}[i+q'_{t+k+1}] = r'_{t+k}$ and $\beta^{t+k+1}[i] = 0$ or $b^{t+k+1}_{r'_{t+k+1}}[i+q'_{t+k+1}] = \tilde r'_{t+k}$ and $\beta^{t+k+1}[i] = 1\}$,

$w_2(k) = \operatorname{card}\{i:\ 0 \leqslant i \leqslant \lambda_{t+k+1}-q'_{t+k+1}-2,\ b^{t+k+1}_{r'_{t+k+1}}[i+q'_{t+k+1}+1] = s'_{t+k}$ and $\beta^{t+k+1}[i] = 0$ or $b^{t+k+1}_{r'_{t+k+1}}[i+q'_{t+k+1}+1] = \tilde s'_{t+k}$ and $\beta^{t+k+1}[i] = 1\}$.

Setting

$$\bar u_t(k) = \frac{n_{t+k}}{l_{t+k}}u_t(k) \quad \text{and} \quad \bar{\bar m}_t(k) = \frac{n_{t+k}}{l_{t+k}}\bar m_t(k)$$

we have

$$\begin{aligned}\bar m_t(k+1) = {} & \frac{\lambda_{t+k+1}-q'_{t+k+1}}{\lambda_{t+k+1}}\left[\frac{w_1(k)}{\lambda_{t+k+1}-q'_{t+k+1}}\bar m_t(k)\right.\\ & \left.+\left(1-\frac{w_1(k)}{\lambda_{t+k+1}-q'_{t+k+1}}\right)\cdot\left(1-\frac{l_{t+k}}{n_{t+k}}-\bar m_t(k)\right)\right]+\frac{l_{t+k}}{n_{t+k}}\left[(\bar u_t(k)-\bar{\bar m}_t(k))\frac{w_2(k)}{\lambda_{t+k+1}}\right.\\ & \left.+\left(\frac{\lambda_{t+k+1}-q'_{t+k+1}-1-w_2(k)}{\lambda_{t+k+1}}\right)\cdot(1-\bar u_t(k)+\bar{\bar m}_t(k))\right].\end{aligned} \tag{47}$$

Choose t_0 such that

$$\bar m_t(k) < \frac{\eta}{8}, \quad \min\left(\frac{l_t}{n_t},\ 1-\frac{l_t}{n_t}\right) < \frac{\eta}{8} \quad \text{and} \quad \frac{1}{\lambda_t} < \frac{\eta}{8} \tag{48}$$

for $t \geqslant t_0$ and $k = 1, 2, \ldots$ Then the equality

$$\frac{l_{t+k+1}}{n_{t+k+1}} = \frac{l_{t+k}}{n_{t+k}}\cdot\frac{1}{\lambda_{t+k+1}}+\frac{q'_{t+k+1}}{\lambda_{t+k+1}}$$

and (46) imply

$$(49)\qquad \frac{\lambda_{t+k+1}-q'_{t+k+1}-w_1(k)}{\lambda_{t+k+1}}<\frac{\eta}{4}\quad\text{and}\quad \frac{q'_{t+k+1}}{\lambda_{t+k+1}}<\frac{\eta}{8}.$$

Using (48) and (49) it is easy to show the inequality

$$\frac{w_2(k)}{\lambda_{t+k+1}}\geqslant\frac{1}{\lambda_{t+k+1}}\operatorname{fr}(r'_{t+k}s'_{t+k},b^{t+k+1})+\frac{1}{\lambda_{t+k+1}}\operatorname{fr}(\tilde{r}'_{t+k}\tilde{s}'_{t+k},b^{t+k+1})$$
$$-\frac{\lambda_{t+k+1}-q'_{t+k+1}-w_1(k)}{\lambda_{t+k+1}}-\frac{q'_{t+k+1}}{\lambda_{t+k+1}}-\frac{1}{\lambda_{t+k+1}}>\frac{\eta}{2}.$$

In a similar way we obtain

$$\frac{\lambda_{t+k+1}-q'_{t+k+1}-1-w_2(k)}{\lambda_{t+k+1}}\geqslant\frac{1}{\lambda_{t+k+1}}\operatorname{fr}(r'_{t+k}\tilde{s}'_{t+k},b^{t+k+1})$$
$$+\frac{1}{\lambda_{t+k+1}}\operatorname{fr}(\tilde{r}'_{t+k}s'_{t+k},b^{t+k+1})-\frac{\lambda_{t+k+1}-q'_{t+k+1}-w_1(k)}{\lambda_{t+k+1}}-\frac{q'_{t+k+1}}{\lambda_{t+k+1}}$$
$$-\frac{1}{\lambda_{t+k+1}}>\frac{\eta}{2}.$$

The above inequalities and (47) give

$$(50)\qquad \bar{m}_t(k+1)\geqslant\left(1-\frac{q'_{t+k+1}}{\lambda_{t+k+1}}\right)\bar{m}_t(k)+\frac{l_{t+k}}{n_{t+k}}\cdot\frac{\eta}{2}.$$

Finally (50) implies

$$\bar{m}_t(k+1)\geqslant\left(1-\frac{l_{t+k+1}}{n_{t+k+1}}\right)\bar{m}_t(k)+\frac{l_{t+k}}{n_{t+k}}\cdot\frac{\eta}{2}.$$

This gives (45) with $\varrho'=\eta/2$.

Further we examine case (β). Then

$$(51)\quad \bar{m}_t(k+1)=\bar{m}_t(k)\frac{\bar{w}_1(k)}{\lambda_{t+k+1}}+\left(\frac{l_{t+k}}{n_{t+k}}-\bar{m}_t(k)\right)\frac{\lambda_{t+k+1}-q'_{t+k+1}-1-\bar{w}_1(k)}{\lambda_{t+k+1}}$$
$$+\frac{\lambda_{t+k+1}-q'_{t+k+1}-\bar{w}_2(k)}{\lambda_{t+k+1}}\left(1-\frac{l_{t+k}}{n_{t+k}}-u_t(k)+\bar{m}_t(k)\right)$$
$$+\frac{\bar{w}_2(k)}{\lambda_{t+k+1}}(u_t(k)-\bar{m}_t(k)),$$

where

$\bar{w}_1(k)=\operatorname{card}\{i\colon 0\leqslant i\leqslant\lambda_{t+k+1}-q'_{t+k+1}-2,\ b^{t+k+1}_{r'_{t+k+1}}[i+q'_{t+k+1}+1]=s'_{t+k}$

and $\beta^{t+k+1}[i]=0$ or $b^{t+k+1}_{r'_{t+k+1}}[i+q'_{t+k+1}+1]=\tilde{s}'_{t+k}$ and $\beta^{t+k+1}[i]=1\}$,

$\bar{w}_2(k)=\operatorname{card}\{i\colon 0\leqslant i\leqslant\lambda_{t+k+1}-q'_{t+k+1}-1,\ b^{t+k+1}_{r'_{t+k+1}}[i+q'_{t+k+1}]=r'_{t+k}$

and $\beta^{t+k+1}[i]=0$ or $b^{t+k+1}_{r'_{t+k+1}}[i+q'_{t+k+1}]=\tilde{r}'_{t+k}$ and $\beta^{t+k+1}[i]=1\}$.

The formula (51) may be written as

$$(52)\qquad \bar{m}_t(k+1)=\frac{\lambda_{t+k+1}-q'_{t+k+1}-1}{\lambda_{t+k+1}}\left[\frac{\bar{w}_1(k)}{\lambda_{t+k+1}-q'_{t+k+1}-1}\bar{m}_t(k)\right.$$
$$\left.+\left(1-\frac{\bar{w}_1(k)}{\lambda_{t+k+1}-q'_{t+k+1}-1}\right)\left(\frac{l_{t+k}}{n_{t+k}}-\bar{m}_t(k)\right)\right]$$
$$+\left(1-\frac{l_{t+k}}{n_{t+k}}\right)\left[(\bar{u}_t(k)-\bar{\bar{m}}_t(k))\frac{\bar{w}_2(k)}{\lambda_{t+k+1}}+(1-\bar{u}_t(k)+\bar{\bar{m}}_t(k))\frac{\lambda_{t+k+1}-q'_{t+k+1}-\bar{w}_2(k)}{\lambda_{t+k+1}}\right].$$

Reasoning as in case (α) we get the inequality

$$(53)\qquad \bar{m}_t(k+1)\geqslant\left(1-\frac{q'_{t+k+1}+1}{\lambda_{t+k+1}}\right)\bar{m}_t(k)+\left(1-\frac{l_{t+k}}{n_{t+k}}\right)\frac{\eta}{2}$$

for $t\geqslant t_0$ and $k\geqslant 1$.

In order to obtain (45) we use the equality

$$(54)\qquad \frac{l_{t+k+1}}{n_{t+k+1}}=\frac{q'_{t+k+1}+1}{\lambda_{t+k+1}}-\left(1-\frac{l_{t+k}}{n_{t+k}}\right)\frac{1}{\lambda_{t+k+1}}.$$

Putting (54) in (53) we have

$$\bar{m}_t(k+1)\geqslant\left(1-\frac{l_{t+k+1}}{n_{t+k+1}}\right)\bar{m}_t(k)+\left(1-\frac{l_{t+k}}{n_{t+k}}\right)\left(\frac{\eta}{2}-\frac{\bar{m}_t(k)}{\lambda_{t+k+1}}\right)$$
$$\geqslant\left(1-\frac{l_{t+k+1}}{n_{t+k+1}}\right)\bar{m}_t(k)+\left(1-\frac{l_{t+k}}{n_{t+k}}\right)\cdot\tfrac{3}{8}\eta.$$

Therefore (45) is obtained with $\varrho'=\frac{3}{8}\eta$.

In the remaining cases the proof of the inequality (45) is similar.

As a consequence of (45) we obtain

$$(55)\qquad \bar{m}_{t_0}(k+1)\geqslant\varrho'\sum_{i=2}^{k+1}\prod_{u=k+1}^{i}(1-\eta_{t_0+u})\eta_{t_0+i-1}=\varrho'\sum_{i=2}^{k+1}\prod_{u=i}^{k+1}(1-\eta_{t_0+u})\eta_{t_0+i-1}$$

for $k\geqslant 1$.

Next we put $\bar{b}_0=\eta_{t_0+k+1}$, $\bar{b}_1=\eta_{t_0+k}$, …, $\bar{b}_k=\eta_{t_0+1}$ and apply Lemma 8 below. If t_0 is a number such that $\bar{m}_{t_0}(k+1)<\frac{1}{2}$ for every $k=0,1,\ldots$, then we obtain

$$(56)\qquad \sum_{i=1}^{k+1} \eta_{t_0+i} < 1$$

for arbitrary $k = 1, 2, \ldots$, which means the convergence of the series $\sum_{t=0}^{\infty} \eta_t$. The lemma is proved.

In the next lemma we present a detailed proof that the codes $\{A_t, \tilde{A}_t\}$ constructed in § 3 (sufficiency) determine a partition Q of $X(y)$. At present we assume that conditions (A) and (B) of Theorem 1 are satisfied.

LEMMA 7. *The codes* $\{A_t, \tilde{A}_t\}$ *satisfy the conditions* (11).

Proof. Recall that

$$A_t = c_t^{r_t'} c_t^{s_t'} [l_t, l_t + n_t - 1].$$

Put

$$\delta_t = b_{r_t'}^t b_{s_t'}^t [q_t', q_t' + \lambda_t].$$

In order to show the lemma we use the following formula:

$$\begin{aligned} d(A_t^{(k+1)}, A_{t+k+1}) = {} & \frac{l_1}{\lambda_{t+k+1}} d(A_t^{(k)}, A_{t+k}) + \frac{l_2}{\lambda_{t+k+1}} (1 - d(A_t^{(k)}, A_{t+k})) \\ & + \frac{l_3}{\lambda_{t+k+1}} d(A_t^{(k)}, c_{t+k}^{r_t'+k} c_{t+k}^{\tilde{s}_t'+k} [l_{t+k}, l_{t+k} + n_{t+k} - 1]) \\ & + \frac{l_4}{\lambda_{t+k+1}} d(A_t^{(k)}, c_{t+k}^{\tilde{r}_t'+k} c_{t+k}^{s_t'+k} [l_{t+k}, l_{t+k} + n_{t+k} - 1]), \end{aligned}$$

where

$$\begin{aligned} l_1 = \operatorname{card} \{i\colon\ 0 \leqslant i \leqslant \lambda_{t+k+1} - 1, \\ & \beta^{t+k+1}[i] = 0 \text{ and } \delta_{t+k+1}[i, i+1] = r_{t+k}' s_{t+k}' \\ & \text{or } \beta^{t+k+1}[i] = 1 \text{ and } \delta_{t+k+1}[i, i+1] = \tilde{r}_{t+k}' \tilde{s}_{t+k}'\}, \end{aligned}$$

$$\begin{aligned} l_2 = \operatorname{card} \{i\colon\ 0 \leqslant i \leqslant \lambda_{t+k+1} - 1, \\ & \beta^{t+k+1}[i] = 0 \text{ and } \delta_{t+k+1}[i, i+1] = \tilde{r}_{t+k}' \tilde{s}_{t+k}' \\ & \text{or } \beta^{t+k+1}[i] = 1 \text{ and } \delta_{t+k+1}[i, i+1] = r_{t+k}' s_{t+k}'\}, \end{aligned}$$

$$\begin{aligned} l_3 = \operatorname{card} \{i\colon\ 0 \leqslant i \leqslant \lambda_{t+k+1} - 1, \\ & \beta^{t+k+1}[i] = 0 \text{ and } \delta_{t+k+1}[i, i+1] = r_{t+k}' \tilde{s}_{t+k}' \\ & \text{or } \beta^{t+k+1}[i] = 1 \text{ and } \delta_{t+k+1}[i, i+1] = \tilde{r}_{t+k}' s_{t+k}'\}, \end{aligned}$$

$$\begin{aligned} l_4 = \operatorname{card} \{i\colon\ 0 \leqslant i \leqslant \lambda_{t+k+1} - 1, \\ & \beta^{t+k+1}[i] = 0 \text{ and } \delta_{t+k+1}[i, i+1] = \tilde{r}_{t+k}' s_{t+k}' \\ & \text{or } \beta^{t+k+1}[i] = 1 \text{ and } \delta_{t+k+1}[i, i+1] = r_{t+1}' \tilde{s}_{t+k}'\}. \end{aligned}$$

If $\frac{l_{t+k}}{n_{t+k}} < 1 - \frac{l_{t+k}}{n_{t+k}}$ then we have

$$\begin{aligned} (57)\qquad d(A_t^{(k+1)}, A_{t+k+1}) \leqslant {} & \frac{l_1}{\lambda_{t+k+1}} d(A_t^{(k)}, A_{t+k}) + \frac{l_2}{\lambda_{t+k+1}} [1 - d(A_t^{(k)}, A_{t+k})] \\ & + \frac{l_3}{\lambda_{t+k+1}} \left[d(A_t^{(k)}, A_{t+k}) + \frac{l_{t+k}}{n_{t+k}} \right] + \frac{l_4}{\lambda_{t+k+1}} \left[1 - d(A_t^{(k)}, A_{t+k}) + \frac{l_{t+k}}{n_{t+k}} \right] \\ & \leqslant d(A_t^{(k)}, A_{t+k}) + \frac{l_2 + l_4}{\lambda_{t+k+1}} + \frac{l_{t+k}}{n_{t+k}}. \end{aligned}$$

It is not difficult to see that

$$\frac{l_2 + l_4}{\lambda_{t+k+1}} = d(\beta^{t+k+1}, b_{r_{t+k+1}}^{t+k+1} b_{s_{t+k+1}}^{t+k+1} [q_{t+k+1}, q_{t+k+1} + \lambda_{t+k+1} - 1]).$$

The above equality and (57) imply

$$\begin{aligned} (58)\qquad d(A_t^{(k+1)}, A_{t+k+1}) \leqslant {} & d(A_t^{(k)}, A_{t+k}) + \eta_{t+k} \\ & + d(\beta^{t+k+1}, b_{r_{t+k+1}}^{t+k+1} b_{s_{t+k+1}}^{t+k+1} [q_{t+k+1}, q_{t+k+1} + \lambda_{t+k+1} - 1]). \end{aligned}$$

If $\frac{l_{t+k}}{n_{t+k}} > 1 - \frac{l_{t+k}}{n_{t+k}}$ then we obtain

$$d(A_t^{(k+1)}, A_{t+k+1}) \leqslant d(A_t^{(k)}, A_{t+k}) + \frac{l_2 + l_3}{\lambda_{t+k+1}} + \left(1 - \frac{l_{t+k}}{n_{t+k}}\right)$$

which also gives (58).

In the same manner we can show that

$$\begin{aligned} d(A_{t_0}^{(1)}, A_{t_0+1}) \leqslant {} & \min(l_{t_0}/n_{t_0}, 1 - l_{t_0}/n_{t_0}) \\ & + d(\beta^{t_0+1}, b_{r_{t_0+1}}^{t_0+1} b_{s_{t_0+1}}^{t_0+1} [q_{t_0+1}, q_{t_0+1} + \lambda_{t_0+1} - 1]). \end{aligned}$$

Therefore we have

$$d(A_{t_0}^{(k+1)}, A_{t_0+k+1}) \leqslant \sum_{t=t_0+1}^{\infty} d(\beta^t, b_{r_t}^t b_{s_t}^t [q_t, q_t + \lambda_t - 1] + \sum_{t=t_0}^{\infty} \min(l_t/n_t, 1 - l_t/n_t)$$

for $k = 0, 1, \ldots$

The above inequalities, (A) and (B) imply that the sequence of codes $\{A_t, \tilde{A}_t\}$ satisfies the condition (11) which ends the proof of the lemma.

LEMMA 8. *Let k be a positive integer and let $\bar{b}_0, \bar{b}_1, \ldots, \bar{b}_k$ be numbers such that $0 < \bar{b}_j < 1$, $j = 0, 1, \ldots, k$. If for each $m = 0, 1, \ldots, k$ the numbers*

$$D_0 = \bar{b}_0, \qquad D_m = \bar{b}_0 + \sum_{i=1}^{m} \bar{b}_i \prod_{j=0}^{i-1} (1 - \bar{b}_j)$$

satisfy the condition $D_m \leqslant M < 1$ *then*

$$\sum_{i=0}^{k} \bar{b}_i \leqslant \frac{M}{1-M}. \tag{59}$$

Proof. We have

$$D_{m+1} = D_m + (1-D_m)\bar{b}_{m+1}, \quad m = 0, 1, \ldots, k-1.$$

This implies that $D_m < 1$ and

$$D_{m+1} > D_m + (1-M)\bar{b}_{m+1}.$$

Hence

$$D_k > (1-M)\sum_{i=0}^{k} \bar{b}_i$$

which gives (59).

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INSTYTUT MATEMATYKI UNIWERSYTETU MIKOŁAJA KOPERNIKA
INSTITUTE OF MATHEMATICS, NICHOLAS COPERNICUS UNIVERSITY
Chopina 12/18, 87-100 Toruń, Poland

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On metric isomorphism of Morse dynamical systems

by

TADEUSZ ROJEK (Toruń)

Abstract. For each continuous Morse sequence x, the class of all continuous Morse sequences y such that the dynamical systems induced by x and y are metrically isomorphic is described.

Introduction. J. Kwiatkowski in [3] gave sufficient and necessary conditions for two Morse dynamical systems $\theta(x)$ and $\theta(y)$ induced by $x = b^0 \times b^1 \times \ldots$ and $y = \beta^0 \times \beta^1 \times \ldots$ to be metrically isomorphic, assuming that the lengths of the b^t and β^t are the same for $t = 0, 1, 2, \ldots$ and x and y are regular sequences. It is also proved in [3] that for a given Morse sequence x there exist a continuum of Morse sequences y such that the systems $\theta(x)$ and $\theta(y)$ are metrically isomorphic but the corresponding shift invariant measures on the space $X = \prod_{-\infty}^{+\infty} \{0, 1\}$ are pairwise orthogonal. For a given regular Morse sequence x Kwiatkowski defines a class $\mathscr{M}(x)$ of Morse sequences y such that the dynamical systems $\theta(x)$ and $\theta(y)$ are metrically isomorphic.

However, the procedure of obtaining the class $\mathscr{M}(x)$ which is described there can be applied to a continuous Morse sequence x (without the assumption of regularity). In this paper we show that $\mathscr{M}(x)$ is the class of all continuous Morse sequences y such that $\theta(y)$ is metrically isomorphic to $\theta(x)$.

To prove this, we use the same technique of coding as in [3], but we omit the assumption that the lengths of the blocks b^t and β^t are equal and thus codes have different form. In order to prove the main result, for given Morse sequences $x = b^0 \times b^1 \times \ldots$ and $y = \beta^0 \times \beta^1 \times \ldots$ such that $\theta(x)$ is metrically isomorphic to $\theta(y)$ we construct a Morse sequence $z = a_0 \times \bar{a}_0 \times a_1 \times \bar{a}_1 \times \ldots$ satisfying

$$|a_0| = |b^0|, \quad |\bar{a}_0 \times a_1| = |b^1|, \quad |\bar{a}_1 \times a_2| = |b^2|, \quad \ldots$$
$$|a_0 \times \bar{a}_0| = |\beta^0|, \quad |a_1 \times \bar{a}_1| = |\beta^1|, \quad |a_2 \times \bar{a}_2| = |\beta^2|, \quad \ldots$$