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The restriction of the Fourier transform to some curves and surfaces

by

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Abstract. Given the curve $\gamma(t) = t^k$, $k \geqslant 2$, in a compact neighborhood of the origin, we prove for the restriction of the Fourier transform the inequality

$$\|\hat{f}\|_{L^q(\gamma, d\sigma)} \leqslant C_{p,q} \|f\|_{L^p(\mathbf{R}^2)}$$

where $1 \leqslant q \leqslant p'/(k+1)$, $p' > \max[4, k+1]$, $1/p+1/p' = 1$ and $d\sigma$ is the Lebesgue measure on γ. This result is sharp.

We also obtain for the "cone" surface Γ generated by γ:

$$\|\hat{f}\|_{L^q(\Gamma, d\sigma)} \leqslant C_p \|f\|_{L^p(\mathbf{R}^3)}$$

where $p' \geqslant 2k$ ($p' > 4$ if $k = 2$), $q = p'/(k+1)$ and $d\sigma$ is a natural homogeneous measure on Γ.

Introduction. In [11] A. Zygmund studied the restriction of the Fourier transform to the circle S^1 in $\mathbf{R}^2$. Later, P. Sjölin [5] generalized this result to more general curves in the plane, showing that in a compact neighborhood of the origin of the curve $\gamma(t) = t^k$, $k \geqslant 2$, we have the inequality

$$\left(\int |\hat{f}(\xi)|^q \, d\sigma(\xi)\right)^{1/q} \leqslant C_{p,q} \|f\|_{L^p(\mathbf{R}^2)} \tag{1}$$

where $1 \leqslant q < p'/(k+1)$, $p' \geqslant 2k$, $1/p+1/p' = 1$ and $d\sigma$ is the Lebesgue measure on γ.

The restriction of the Fourier transform depends mainly on the curvature at each point, and so we are interested in local results. The theorem remains true if γ is a curve with a contact of order $k-1$ with its tangent at the origin. Zygmund's result is recovered by taking $k = 2$.

A. Ruíz [4] improved the theorem by extending it to $1 \leqslant q < p'/(k+1)$ with $p' > 4$ and $q = p'/(k+1)$ if $p' > 2k$.

A. Ruíz' proof uses Geometrical Fourier Analysis, and this allows to obtain sharper results than the previous ones, which were obtained by analytic methods. Moreover, it shows more clearly the behavior of the Fourier transform that appears in the problem.

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In this paper we give a different geometrical decomposition of the curve and a more careful analysis of the inequalities involved, arriving at the following result:

THEOREM 1. *Let γ be the curve $\gamma(t) = t^k$, $k \geqslant 2$. Then in a compact neighborhood of the origin we have*

$$\|\hat{f}\|_{L^q(\gamma)} \leqslant C_{p,q}\|f\|_{L^p(\mathbf{R}^2)} \tag{2}$$

in the cases $1 \leqslant q \leqslant p'/(k+1)$ *and* $p' > \max[4, k+1]$.

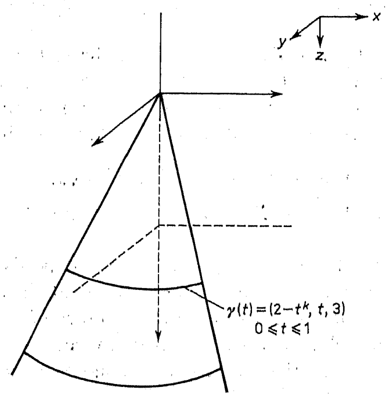
Then we show in Proposition 4 that $p' > k+1$ is a necessary condition. That $p' > 4$ is necessary in case $k = 2$ was already known (see [10], [11]). Also, the necessity of the condition $q \leqslant p'/(k+1)$ was known, by a simple homogeneity argument due to A. Knapp [8].

In the second part of this paper we study the restriction of the Fourier transform to the infinite surface of the "cone" whose sections are as the preceding curve $\gamma(t) = t^k$.

In order to state precisely the picture we build our surface Γ by taking in the space the curve

$$(2-t^k, t, 3), \qquad 0 \leqslant t \leqslant 1,$$

and then joining each point of this curve with the origin (0, 0, 0) by a straight line which is a generatrix of the surface (see the figure).



In fact we can take Γ to be a homogeneous surface whose section at height $z = 1$ has a contact of order $k-1$ with its tangent at the origin.

If we parametrize this surface with the cylindrical coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = r,$$
$$0 \leqslant \theta \leqslant 2\pi, \quad r > 0,$$

then the Lebesgue measure on Γ is comparable to $r\,dr\,d\theta$. We then obtain the following theorem:

THEOREM 2. *Given Γ as indicated above, we have the inequality*

$$\Big(\int_\Gamma |\hat{f}(\xi)|^q\, d\sigma(\xi)\Big)^{1/q} \leqslant C_p\|f\|_{L^p(\mathbf{R}^3)} \tag{3}$$

for every function $f \in S(\mathbf{R}^3)$ *and* $p' \geqslant 2k$ ($p' > 4$ *if* $k = 2$), $q = p'/(k+1)$, *and* $d\sigma$ *is the measure* $d\sigma = r^{(2-k)/(k+1)}\,dr\,d\theta$.

The measure $d\sigma$ that appears in the preceding statement may look strange at first sight, but it is natural due to the fact that Γ is an infinite surface and to the homogeneity of the Fourier transform. If we have only a compact piece of Γ we can take as $d\sigma$ the Lebesgue measure $r\,dr\,d\theta$.

In Propositions 5 and 6 we sketch the standard proof of these facts and also of the necessity of the condition $q \leqslant p'/(k+1)$, already known by Knapp's argument.

A particular case of Theorem 2 is the restriction of the Fourier transform to the right circular cone (take $k = 2$), which was obtained in [1]. Theorem 2 can also be thought of as a generalization of Theorem 1.

I am deeply grateful to Professors A. Córdoba and J. L. Rubio de Francia for their constant stimulus and advice in treating these problems.

1. The restriction to curves.

THEOREM 1. *Let γ be the curve $\gamma(t) = (t, t^k)$, $k \geqslant 2$, in a compact neighborhood of the origin in $\mathbf{R}^2$. Then for every function $f \in S(\mathbf{R}^2)$ we have the "a priori" inequality*

$$\|\hat{f}\|_{L^q(\gamma)} \leqslant C_{p,q}\|f\|_{L^p(\mathbf{R}^2)} \tag{4}$$

for $1 \leqslant q \leqslant p'/(k+1)$ *and* $p' > \max[4, k+1]$, *where p' is such that* $1/p + 1/p' = 1$.

Proof. We can suppose, without loss of generality, that the curve is $\gamma(t)$ with $0 \leqslant t \leqslant 1$.

We are now going to expand the curve a little by a thickness δ and divide the resulting "collar" in rectangles adapted to the geometry of the curve.

Because there is null curvature at the origin, the rectangles will become shorter as one moves away from the origin.

To compute the size of the rectangles, if R_n is the nth rectangle from the origin and its projection on the x-axis is the segment $[x_{n-1}, x_n]$, with $\Delta x_n = x_n - x_{n-1}$, letting $g(x) = x^k$, we have

$$g(x_{n+1}) = g(x_n) + g'(x_n)\Delta x_n + \frac{g''(x_n)}{2}(\Delta x_n)^2 + o(\Delta x_n)^2$$

and setting

$$\frac{g''(x_n)}{2}(\Delta x_n)^2 = \delta$$

we obtain the difference equation

$$\Delta x_n = c x_n^{(2-k)/k} \delta^{1/2}.$$

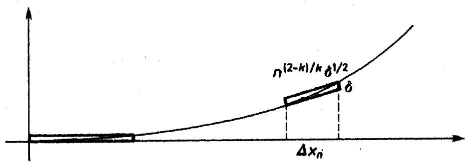
Solving it we obtain

$$x_n \approx n^{2/k} \delta^{1/k}, \qquad \Delta x_n \approx n^{(2-k)/k} \delta^{1/2}.$$

Now, since the length of the long side of R_n is approximately that of its projection, Δx_n, we can suppose that R_n is the rectangle of dimensions

$$n^{(2-k)/k} \delta^{1/k} \times \delta$$

and then there are $\delta^{-1/2}$ rectangles in the decomposition.



As we shall see in the course of the proof, it is important to observe at this point that to compute exactly the size and the shape of each rectangle only would change the constant C of the inequality (4) into a different one independently of δ.

If I_n is the segment of the curve γ inside R_n, χ_{I_n} is the characteristic function of I_n and φ_n is the characteristic function of R_n we have

$$\Big|\int_\gamma \hat{f}(\xi) \sum_{n=1}^{\delta^{-1/2}} a_n \chi_{I_n}(\xi)\, d\sigma(\xi)\Big| \leqslant \frac{C}{\delta}\Big|\int_{\mathbf{R}^2} \hat{f}(\xi) \sum_{n=1}^{\delta^{-1/2}} a_n \varphi_n(\xi)\, d\xi\Big|$$

$$= \frac{C}{\delta}\Big|\int_{\mathbf{R}^2} f(\xi) \sum_{n=1}^{\delta^{-1/2}} a_n \hat{\varphi}_n(\xi)\, d\xi\Big| \leqslant \frac{C}{\delta}\|f\|_p \Big\|\sum_{n=1}^{\delta^{-1/2}} a_n \hat{\varphi}_n(\xi)\Big\|_{p'},$$

where $1/p+1/p' = 1$. Then to prove (4) it is enough that, for every $\delta > 0$ sufficiently small, we have

$$\Big\|\sum_{n=1}^{\delta^{-1/2}} a_n \hat{\varphi}_n(\xi)\Big\|_{p'} \leqslant C\delta\Big(\sum_{n=1}^{\delta^{-1/2}} |a_n|^{q'} n^{(2-k)/k} \delta^{1/k}\Big)^{1/q'}. \tag{5}$$

For this, since $p' > 4$,

$$\Big\|\sum_{n=1}^{\delta^{-1/2}} a_n \hat{\varphi}_n(\xi)\Big\|_{p'} = \Big(\int_{\mathbf{R}^2} \Big|\sum_n a_n \hat{\varphi}_n(\xi) \cdot \sum_m a_m \hat{\varphi}_m(\xi)\Big|^{p'/2} d\xi\Big)^{2/p' \cdot 1/2} \tag{6}$$

$$\leqslant \Big(\int_{\mathbf{R}^2} \Big|\sum_{nm} a_n a_m \varphi_n(x) * \varphi_m(x)\Big|^s dx\Big)^{1/s \cdot 1/2}$$

by the Hausdorff–Young inequality, where $1/s+2/p' = 1$.

To handle this last expression, we observe that $\operatorname{supp}(\varphi_n * \varphi_m)$ has a finite overlapping when n, m vary (this is a particular case of Lemma 6). Then on applying Propositions 1 and 2 that follow, the last expression in (6) is less than or equal to

$$C\Big(\sum_{nm} |a_n|^s |a_m|^s \int |\varphi_n * \varphi_m(x)|^s dx\Big)^{1/2s}$$

$$\leqslant C\Bigg(\sum_{nm} \frac{|a_n|^s |a_m|^s}{n^{(k-2)/k} m^{(k-2)/k}} \cdot \frac{\delta^{(k+1)/k} \delta^{(k+1)/2ks}}{(|n^{(2k-2)/k} - m^{(2k-2)/k}|+1)^{s-1}}\Bigg)^{1/2s}$$

$$\leqslant C\Big[\delta^{(k+1)/k} \delta^{(k+1)/2ks} \Big(\sum_n |a_n|^{q'} n^{(2-k)/k}\Big)^{1/p_1} \Big(\sum_m |a_m|^{q'} m^{(2-k)/k}\Big)^{1/p_1}\Big]^{1/2s}$$

$$= C\Big[\delta\Big(\sum_n |a_n|^{q'} n^{(2-k)/k} \delta^{1/k}\Big)^{2/p_1}\Big]^{1/2s} \leqslant C\delta\Big(\sum_n |a_n|^{q'} n^{(2-k)/k} \delta^{1/k}\Big)^{1/q'}$$

and so (5) is proved.

PROPOSITION 1.

$$\int_{\mathbf{R}^2} |\varphi_n * \varphi_m(x)|^s dx \leqslant C \frac{\delta^{(k+1)/k} \delta^{(k+1)/2ks}}{n^{(k-2)/k} m^{(k-2)/k} (|n^{(2k-2)/k} - m^{(2k-2)/k}|+1)^{s-1}}. \tag{7}$$

Proof. We have

$$\int_{\mathbf{R}^2} |\varphi_n * \varphi_m(x)|^s dx \leqslant \|\varphi_n * \varphi_m\|_\infty^s \cdot |\operatorname{supp}(\varphi_n * \varphi_m)|.$$

To compute $|\operatorname{supp}(\varphi_n * \varphi_m)| = |R_n + R_m|$ observe that the thickness of R_n (resp. R_m) being negligible compared with the length of R_m (resp. R_n) if $n \neq m$, the size of $R_n + R_m$ will be smaller than two times the area of the parallelogram made up by the vectors u_n, u_m, where

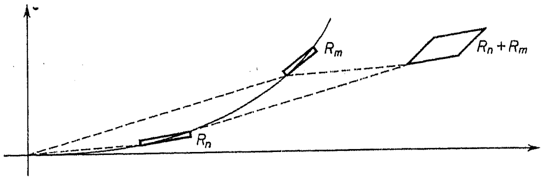
$$u_n = (\Delta x_n, \Delta y_n)$$

and Δy_n is the projection of R_n on the y-axis. We have

$$\Delta y_n = \Delta x_n \cdot g'(x_n) = n^{(2-k)/k} \delta^{1/k} k x_n^{k-1} \cong n\delta$$

and so

$$u_n = (n^{(2-k)/k} \delta^{1/k}, n\delta), \qquad u_m = (m^{(2-k)/k} \delta^{1/k}, m\delta);$$



therefore

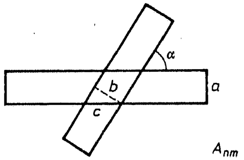
$$(8)\qquad |R_n+R_m| \leqslant C \begin{vmatrix} n^{(2-k)/k}\,\delta^{1/k} & n\delta \\ m^{(2-k)/k}\,\delta^{1/k} & m\delta \end{vmatrix} = C\delta^{(k+1)/k}\left(n^{(2-k)/k}m - nm^{(2-k)/k}\right).$$

The greatest intersection of R_n with R_m, A_{nm}, if $n \neq m$ is

$$A_{nm} \cong ac$$

where $a=\delta$, $c=b/\sin\alpha=\delta/\sin\alpha$ and

$$\sin\alpha = \frac{|u_1\times u_2|}{|u_1|\cdot|u_2|} \cong \frac{\delta^{(k+1)/k}\,|mn^{(2-k)/k}-nm^{(2-k)/k}|}{\delta^{2/k}\,m^{(2-k)/k}\,n^{(2-k)/k}} = \delta^{(k-1)/k}\,|n^{(2k-2)/k}-m^{(2k-2)/k}|.$$



Therefore

$$(9)\qquad A_{nm} \leqslant \frac{\delta^2}{\sin\alpha} \cong \frac{\delta^{(k+1)/k}}{|n^{(2k-2)/k}-m^{(2k-2)/k}|}$$

and since

$$\|\varphi_n * \varphi_m\|_\infty \leqslant A_{nm}$$

bearing in mind the case $n=m$, we finally obtain (7) from (8) and (9).

Proposition 2. *We have the inequality*

$$\sum_{nm} \frac{|a_n|^s\,|a_m|^s}{n^{(k-2)/k}\,m^{(k-2)/k}\left(|n^{(2k-2)/k}-m^{(2k-2)/k}|+1\right)^{s-1}} \leqslant C\Big(\sum_n |a_n|^{q'}\,n^{(2-k)/k}\Big)^{1/p_1}\Big(\sum_m |a_m|^{q'}\,m^{(2-k)/k}\Big)^{1/p_1}$$

whenever

$$\frac{1}{q}+\frac{1}{q'}=1,\qquad q=\frac{p'}{k+1},\qquad p_1 s = q',\qquad q>1.$$

Proof. The proposition can be thought of as a discrete analogue of the following Proposition 3, its continuous translation, where

$$\alpha = \frac{k-2}{k}\left(1-\frac{1}{p_1}\right),\qquad M=\frac{2k-2}{k},\qquad \beta = s-1,\qquad p=p_1.$$

Proposition 3. *Let $f, g \in L^p[1,\infty)$. Then*

$$(10)\qquad \left|\int_1^\infty\int_1^\infty \frac{f(x)\,g(y)}{x^\alpha y^\alpha(|x^M-y^M|+1)^\beta}\,dx\,dy\right| \leqslant C\,\|f\|_p\,\|g\|_p$$

whenever

$$1+\frac{1}{p'}=\frac{1}{p}+M\beta+2\alpha,\qquad \frac{1}{p}+\frac{1}{p'}=1,\qquad 0<\beta<1,\qquad p>1,\qquad M\geqslant 1.$$

Proof. Taking the bound $x^M-y^M \geqslant (x-y)^M$ in the denominator of the integrand and applying Hölder's inequality and fractional integration, we would obtain the same result but with the condition $0<M\beta<1$. This restricts to be $p'>2k$ in Theorem 1. But we have to bear in mind that the "singularity" is not $(x-y)^{M\beta}$ but, if for example M is an integer, it would be

$$(x^M-y^M)^\beta = (x-y)^\beta\,(x^{M-1}+x^{M-2}y+\ldots+y^{M-1})^\beta$$

and this allows us to take $0<\beta<1$, as we want.

Taking absolute values inside the integral (10) we can assume $f(x)\geqslant 0$ and $g(x)\geqslant 0$. We separate now the integral into three pieces:

$$\int_1^\infty\int_1^\infty \frac{f(x)\,g(y)}{x^\alpha y^\alpha(|x^M-y^M|+1)^\beta}\,dx\,dy = (\mathrm{a})+(\mathrm{b})+(\mathrm{c})$$

where

$$(\mathrm{a}) = \iint_{y>2x} \frac{f(x)\,g(y)}{x^\alpha y^\alpha(|x^M-y^M|+1)^\beta}\,dx\,dy,$$

$$(\mathrm{b}) = \iint_{1/2\leqslant x/y\leqslant 2} \frac{f(x)\,g(y)}{x^\alpha y^\alpha(|x^M-y^M|+1)^\beta}\,dx\,dy,$$

$$(\mathrm{c}) = \iint_{x>2y} \frac{f(x)\,g(y)}{x^\alpha y^\alpha(|x^M-y^M|+1)^\beta}\,dx\,dy.$$

We will bound (a); (c) can be treated in a similar way. For (a)

$$(\mathrm{a}) = \int_1^\infty f(x)\,h(x)\,dx \leqslant \|f\|_{q'}\,\|h\|_q$$

if $1/q+1/q'=1$ where we have put

$$h(x)=\int_{2x}^{\infty}\frac{g(y)}{x^{\alpha}y^{\alpha}(|x^M-y^M|+1)^{\beta}}\,dy\leqslant\frac{1}{x^{2\alpha}}\int_{2x}^{\infty}\frac{g(y)\,dy}{(|x-y|^M+1)^{\beta}}$$
$$=\frac{1}{x^{2\alpha}}h_1(x).$$

We will now show that $h_1(x)$ is of weak type (p,q_1):

LEMMA 1. *If* $1+1/q_1=1/p+M\beta$ *where* $0<\beta<1$, $M\geqslant 1$, $q_1<\infty$, *then*

$$|\{x|\ h_1(x)>\lambda\}|\leqslant C\left(\frac{\|g\|_p}{\lambda}\right)^{q_1}.$$

Proof. By the homogeneity of this inequality we can assume $\|g\|_p=1$. Then

$$h_1(x)=\int_{2x}^{\infty}\frac{g(y)}{(|x-y|^M+1)^{\beta}}\,dy\leqslant C\int_{|x-y|>1}\frac{g(y)}{|x-y|^{M\beta}}\,dy=k*g(x)$$

the kernel being

$$k(t)=\begin{cases}0 & \text{if } t\leqslant 1,\\ 1/t^{M\beta} & \text{if } t>1.\end{cases}$$

Given a real number μ that we shall fix later, we can truncate the kernel as $k=k_1+k_\infty$ where

$$k_1=k\cdot\chi_{[0,\mu]},\qquad k_\infty=k\cdot\chi_{[\mu,\infty)}.$$

But $k*g(x)=k_1*g(x)+k_\infty*g(x)$ and

$$k_\infty*g(x)\leqslant\|k_\infty\|_{p'}\cdot\|g\|_p=\|k_\infty\|_{p'},\qquad 1/p+1/p'=1,$$

with

$$\|k_\infty\|_{p'}^{p'}=\int_{\mu}^{\infty}\frac{dt}{t^{M\beta p'}}\cong\mu^{-M\beta p'+1}$$

because $M\beta p'<1$ is equivalent to $q_1<\infty$. Therefore putting

$$\|k_\infty\|_{p'}=\mu^{(-M\beta p'+1)\cdot 1/p'}=\lambda$$

we will have $k_\infty*g(x)\leqslant\lambda$ and

$$|\{x|\ h_1(x)>2\lambda\}|\leqslant|\{x|\ k_1*g(x)>\lambda\}|+|\{x|\ k_\infty*g(x)>\lambda\}|$$
$$=|\{x|\ k_1*g(x)>\lambda\}|\leqslant\frac{\|k_1*g\|_p^p}{\lambda^p}\leqslant\frac{\|k_1\|_1^p\,\|g\|_p^p}{\lambda^p}$$
$$=\frac{\|k_1\|_1^p}{\lambda^p}.$$

But

$$\|k_1\|_1=\int_1^{\mu}\frac{dt}{t^{M\beta}};$$

if $M\beta>1$, $\|k_1\|_1=\text{constant}$, but $M\beta>1$ implies $q_1<p$ and then $1/\lambda^p\leqslant 1/\lambda^{q_1}$, while if $M\beta\leqslant 1$, $\|k_1\|_1\cong\mu^{-M\beta+1}$ and

$$\frac{\|k_1\|_1^p}{\lambda^p}=\frac{\mu^{-M\beta+1}}{\lambda^p}=\frac{1}{\lambda^{q_1}}$$

because

$$\lambda=\mu^{(-M\beta p'+1)\cdot 1/p'}=\mu^{-1/q_1}\qquad\text{since } 1+1/q_1=1/p+M\beta.$$

We shall now see that $h(x)$ is of weak type (p,q):

LEMMA 2. *If* $1+1/q=1/p+M\beta+2\alpha$ *where* $0<\beta<1$, $q<\infty$, $M\geqslant 1$, *then*

$$|\{x|\ h(x)>\lambda\}|\leqslant C\left(\frac{\|g\|_p}{\lambda}\right)^{q}.$$

Proof. As in Lemma 1 we assume $\|g\|_p=1$. Since

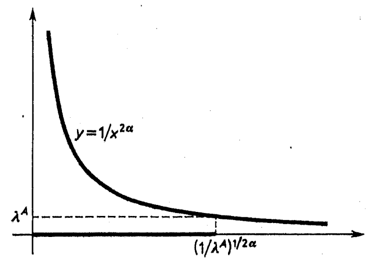
$$h(x)\leqslant\frac{1}{x^{2\alpha}}h_1(x),$$

for $A+B=1$ we have

$$|\{x|\ h(x)>\lambda\}|\leqslant|\{x|\ 1/x^{2\alpha}>\lambda^A\}|+|\{x|\ h_1(x)>\lambda^B\}|.$$

But

$$|\{x|\ 1/x^{2\alpha}>\lambda^A\}|\leqslant(1/\lambda^A)^{1/2\alpha}$$



and by Lemma 1

$$|\{x|\ h_1(x)>\lambda^B\}|\leqslant C(1/\lambda^B)^{q_1}$$

with $1+1/q_1=1/p+M\beta$. Taking $A=2\alpha q$, $B=q/q_1$, the condition of the statement of the Lemma, $1+1/q = 1/p + M\beta+2\alpha$, implies $A+B=1$ and then

$$|\{x|\ h(x)>\lambda\}| \leqslant \left(\frac{1}{\lambda^A}\right)^{1/2\alpha} + C\left(\frac{1}{\lambda^B}\right)^{q_1} \leqslant C\left(\frac{1}{\lambda}\right)^q.$$

LEMMA 3. *If, in addition to the assumptions of Lemma 2, $q>1$, we have the strong inequality*

$$\|h\|_q \leqslant C\|g\|_p.$$

Proof is a straightforward application of the Marcinkiewicz interpolation theorem.

According to Lemmas 1, 2 and 3 we obtain

$$\text{(11)}\qquad \text{(a)} = \iint\limits_{y>2x} \frac{f(x)g(y)}{x^\alpha y^\alpha(|x^M-y^M|+1)^\beta}\,dx\,dy \leqslant \|f\|_{q'}\|h\|_q \leqslant C\|f\|_{q'}\|g\|_p = C\|f\|_p\|g\|_p$$

because if $p'=q$, Lemma 3 works and also $q'=p$.

We will now bound the integral (b):

$$\text{(b)} = \iint\limits_{1/2\leqslant x/y\leqslant 2} \frac{f(x)g(y)}{x^\alpha y^\alpha(|x^M-y^M|+1)^\beta}\,dx\,dy = \int_1^\infty f(x)\,l(x)\,dx$$

where $l(x)$ is such that if $x\cong 2^n$ with $n\geqslant 1$ then

$$l(x)\cong \int_{2^{n-1}}^{2^{n+1}} \frac{g(y)\,dy}{x^\alpha y^\alpha(|x^M-y^M|+1)^\beta}.$$

Since

$$\min(x^{M-1}, y^{M-1})|x-y| \leqslant |x^M-y^M| \leqslant \max(x^{M-1}, y^{M-1})|x-y|,$$

for $x\cong 2^n$ we have

$$l(x)\cong \frac{1}{2^{n\alpha}2^{n(M-1)\beta}}\int_{2^{n-1}}^{2^{n+1}} \frac{g(y)}{(|x-y|+1)^\beta}\,dy$$

and therefore

$$l(x)\leqslant \frac{1}{x^{2\alpha+(M-1)\beta}}\int_1^\infty \frac{g(y)}{(|x-y|+1)^\beta}\,dy.$$

LEMMA 4. *If $1+1/q = 1/p+M\beta+2\alpha$, $0<\beta<1$, $M\geqslant 1$, $q<\infty$, then*

$$|\{x|\ l(x)>\lambda\}| \leqslant C\left(\frac{\|g\|_p}{\lambda}\right)^q.$$

Proof. We can assume $\|g\|_p = 1$ by the homogeneity of the inequality and we write

$$l(x)\leqslant \frac{1}{x^{2\alpha+(M-1)\beta}}h_2(x)$$

where

$$h_2(x) = \int_1^\infty \frac{g(y)}{(|x-y|+1)^\beta}\,dy.$$

By a fractional integration theorem ([7], p. 119) we see that $h_2(x)$ is of weak type (p, q_1):

$$|\{x|\ h_2(x)>\lambda\}| \leqslant C(1/\lambda)^{q_1}$$

provided that $q_1<\infty$, $1+1/q_1 = 1/p+\beta$.

Taking now

$$A = (2\alpha+(M-1)\beta)q,\qquad B = q/q_1,$$

by the hypothesis of the Lemma we obtain $A+B=1$. Then

$$|\{x|\ l(x)>\lambda\}| \leqslant |\{x|\ x^{1/(2\alpha+(M-1)\beta)}>\lambda^A\}| + |\{x|\ h_2(x)>\lambda^B\}| \leqslant C\left(\frac{1}{\lambda^A}\right)^{1/(2\alpha+(M-1)\beta)} + C\left(\frac{1}{\lambda^B}\right)^{q_1} = C\left(\frac{1}{\lambda}\right)^q.$$

By the Marcinkiewicz interpolation theorem, if we suppose also $q>1$, we obtain

$$\|l\|_q \leqslant C\|g\|_p,$$

so

$$\text{(12)}\qquad \text{(b)} = \int_1^\infty f(x)\,l(x)\,dx \leqslant \|f\|_{q'}\|l\|_q \leqslant C\|f\|_{q'}\|g\|_p = C\|f\|_p\|g\|_p$$

because $q'=p$.

Finally, joining (11) and (12) and remembering that (c) is similar to (a) we have

$$\int_1^\infty\int_1^\infty \frac{f(x)g(y)}{x^\alpha y^\alpha(|x^M-y^M|+1)^\beta}\,dx\,dy = \text{(a)}+\text{(b)}+\text{(c)} \leqslant C\|f\|_p\|g\|_p$$

and this concludes the proof of Proposition 3.

We are now going to show that Theorem 1 is sharp. We recall that the necessity of the conditions $q\leqslant p'/(k+1)$ and $p'>4$ if $k=2$ is already known. Thus in order to show the necessity of the condition $p'>\max[4, k+1]$, all that we need is the following result:

PROPOSITION 4. *Theorem 1 is false if $p' = k+1$, $q=1$.*

Proof. If we had the estimate

$$\|\hat{f}\|_{L^1(\gamma)} \leqslant C\|f\|_{(k+1)/k}$$

for every function $f \in L^{(k+1)/k}(\boldsymbol{R}^2)$, we would also have its dual bound

$$\|(g\,d\sigma)^{\widehat{}}\|_{L^{k+1}(\boldsymbol{R}^2)} \leqslant C\,\|g\|_{L^\infty(\gamma)}$$

for every function $g \in L^\infty(\gamma)$, where $d\sigma$ is the singular Lebesgue measure on γ. Taking $g \equiv 1$ we would have

$$\|(d\sigma)^{\widehat{}}\|_{L^{k+1}(\boldsymbol{R}^2)} \leqslant C$$

and this is false as we shall see now.

Indeed, if we parametrize the piece of the compact curve γ with

$$\gamma\colon [0,1] \to \boldsymbol{R}^2, \qquad \gamma(u) = (u, u^k),$$

then by the definition of the Fourier transform

$$\begin{aligned}(d\sigma)^{\widehat{}}\,(s,t) &= \int_0^1 e^{-2\pi i \gamma(u)\cdot(s,t)}\,\|\gamma'(u)\|\,du\\ &= \int_0^1 e^{-2\pi i(us+u^k t)}\sqrt{1+k^2u^{2k-2}}\,du\\ &= \int_0^1 \cos\big(2\pi(us+u^k t)\big)\sqrt{1+k^2u^{2k-2}}\,du\\ &\quad - i\int_0^1 \sin\big(2\pi(us+u^k t)\big)\sqrt{1+k^2u^{2k-2}}\,du.\end{aligned}$$

Since $|(d\sigma)^{\widehat{}}\,(s,t)|$ is greater than or equal to the absolute value of its imaginary part and $\sqrt{1+k^2u^{2k-2}}$ is bounded above and below by constants, to prove

$$\|(d\sigma)^{\widehat{}}\|_{L^{k+1}(\boldsymbol{R}^2)} = \infty$$

it is enough to prove

$$\|I(s,t)\|_{L^{k+1}(\boldsymbol{R}^2)} = \infty$$

where

$$I(s,t) = \int_0^1 \sin\big(2\pi(us+u^k t)\big)\,du.$$

If we assume $s \geqslant 0$, $t \geqslant 0$, then making in this last integral the change of variable

$$v = us+u^k t = u(s+u^{k-1}t),$$

$$dv = (s+ktu^{k-1})\,du, \qquad du = \frac{dv}{s+ktu^{k-1}}$$

gives

(13) $$I(s,t) = \int_0^{s+t} \frac{\sin(2\pi v)}{s+ktu^{k-1}}\,dv.$$

We separate this integral into two pieces:

(i) If $s \leqslant ktu^{k-1}$, then $ktu^k \leqslant v \leqslant 2ktu^k$, and we write $v \cong ktu^k$, because to put $v = ktu^k$ instead of its exact value does not affect the calculations that follow. In a similar way we can write

$$du \cong \frac{dv}{ktu^{k-1}} \cong \frac{dv}{t^{1/k}v^{(k-1)/k}}.$$

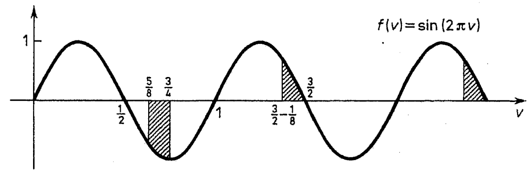
(ii) If $s \geqslant ktu^{k-1}$, then $v \cong us$, $du \cong dv/s$.

Now if $s = ktu^{k-1}$ then $u = (s/kt)^{1/(k-1)}$. So

$$v = us+u^k t = (k+1)tu^k = \frac{k+1}{k^{k/(k-1)}}\left(\frac{s^k}{t}\right)^{1/(k-1)} = C\left(\frac{s^k}{t}\right)^{1/(k-1)}$$

and the integral (13) is

$$I(s,t) = \frac{1}{t^{1/k}}\int_0^{C(s^k/t)^{1/(k-1)}} \frac{\sin(2\pi v)}{v^{(k-1)/k}}\,dv + \frac{1}{s}\int_{C(s^k/t)^{1/(k-1)}}^{s+t} \sin(2\pi v)\,dv.$$



But

$$\int_0^{C_1} \frac{\sin(2\pi v)}{v^{(k-1)/k}}\,dv \geqslant 0 \qquad \text{if } 0 \leqslant C_1 \leqslant \tfrac12+\tfrac14 = \tfrac34.$$

Thus if (s,t) is such that

(14) $$\tfrac58 = \tfrac12+\tfrac18 \leqslant C\left(\frac{s^k}{t}\right)^{1/(k-1)} \leqslant \tfrac12+\tfrac14 = \tfrac34$$

then

$$I(s,t) \geqslant \frac{1}{s}\int_{3/4}^{s+t} \sin(2\pi v)\,dv,$$

and

$$\int_{3/4}^{s+t} \sin(2\pi v)\,dv \geqslant \tfrac{1}{16}$$

for all pairs (s,t) such that

(15) $$\frac{2n+1}{2} - \frac18 \leqslant s+t \leqslant \frac{2n+1}{2}, \qquad n \geqslant 1 \text{ integer}.$$

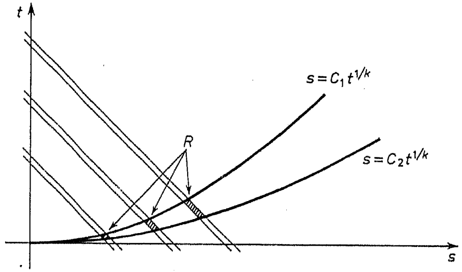
The condition (14) reads

$$C_1\, t^{1/k} \leqslant s \leqslant C_2\, t^{1/k} \quad \text{with } C_1 < C_2$$

and thus if R is the region of the plane formed by the pairs (s, t), $s \geqslant 0$, $t \geqslant 0$, fulfilling the conditions (14) and (15) (see the figure) we have

$$\|(d\sigma)\hat{}\,\|_{p'}^{p'} \geqslant \int_R |I(s,t)|^{p'}\, ds\, dt$$
$$\geqslant C_3 \int_{I_R} \int_{C_1 t^{1/k}}^{C_2 t^{1/k}} \left(\frac{1}{t^{1/k}}\right)^{p'} ds\, dt \geqslant C_4 \int_{I_R} \frac{t^{1/k}}{t^{p'/k}}\, dt$$

where $I_R = \{t \mid (s,t) \in R\}$.



The integral diverges if $(1-p')/k+1 \geqslant 0$, thus if $p' \leqslant k+1$ we have $\|(d\sigma)\hat{}\,\|_{p'} = \infty$.

2. The restriction to surfaces. Let Γ be the infinite cone surface whose sections are like the curve $\gamma(t) = t^k$ as stated in the introduction.

THEOREM 2. *We have the inequality*

$$\Big(\int_\Gamma |\hat{f}(\xi)|^q\, r^{(2-k)/(k+1)}\, dr\, d\theta\Big)^{1/q} \leqslant C_p \|f\|_{L^p(\mathbf{R}^3)} \tag{16}$$

for every function $f \in S(\mathbf{R}^3)$ *with* $p' \geqslant 2k$ $(p' > 4$ *if* $k = 2)$, $q = p'/(k+1)$.

Proof. In order to prove (16) we are going to make a suitable decomposition by cutting Γ into different pieces.

We can assume that Γ has its vertex at the origin. We divide Γ into dyadic blocks Γ_n, $n \in \mathbf{Z}$, where Γ_n is the part of the surface whose height z is such that $2^n \leqslant z \leqslant 2^{n+1}$.

We take the curve γ formed by the intersection of Γ with the plane perpendicular to the z-axis at height $z = 1$ and divide γ into $\delta^{-1/2}$ segments

$$I_\nu, \quad \nu = 1, 2, \ldots, \delta^{-1/2}$$

as in the proof of Theorem 1. We recall that I_ν is the piece of γ at angular direction ν and with length $\nu^{(2-k)/k}\,\delta^{1/k}$.

If $a_\nu = \nu^{2/k}\,\delta^{1/k}$ is the origin of I_ν, we join each a_ν with the cone vertex along generatrices and so the surface Γ is cut up into $\delta^{-1/2}$ triangular strips T_ν.

We cut now each Γ_n into $\delta^{-1/2}$ sections perpendicular to the z-axis and equidistant from each other, of length $2^n\delta^{1/2}$, and we expand Γ by homogeneity in such a way as to have thickness δr at height r.

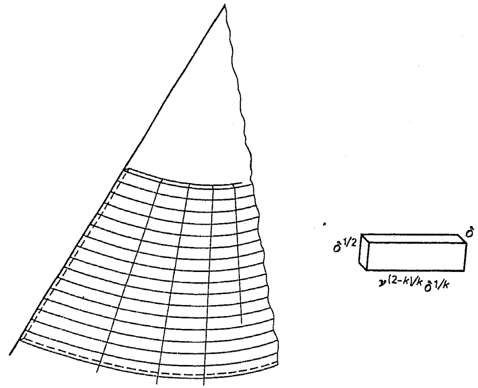
We then obtain the "fat" cone Γ^*, divided as we wish:

$$\Gamma^* = \bigcup_{nj\nu} Q_{nj\nu}$$

where $Q_{nj\nu}$ is the block of dimensions

$$2^n \nu^{(2-k)/k}\,\delta^{1/k} \times 2^n\delta^{1/2} \times 2^n\delta$$

which is at height $z \cong 2^n + 2^n\delta^{1/2} j$ and at angular direction $\theta \cong \nu^{2/k}\,\delta^{1/k}$.



To prove (16) it is enough to prove it for functions f that are constant in each $Q_{nj\nu}$ and show

$$\lim_{\delta\to 0}\left(\frac{1}{\delta}\int_{\Gamma^*}\left|\sum_{n=-\infty}^{+\infty}\sum_{j=1}^{\delta^{-1/2}}\sum_{\nu=1}^{\delta^{-1/2}}\frac{a_{nj\nu}}{2^{n\cdot 3k/(k+1)\cdot 1/q}}\,\hat{\varphi}_{nj\nu}(\xi)\right|^q d\xi\right)^{1/q} \tag{17}$$
$$\leqslant C\Big(\sum_{n=-\infty}^{+\infty}\sum_{j=1}^{\delta^{-1/2}}\sum_{\nu=1}^{\delta^{-1/2}} |a_{nj\nu}|^p\,\delta^2\, 2^{3n}\Big)^{1/p}$$

where $a_{nj\nu}$ is the value taken by f in $Q_{nj\nu}$ and $\varphi_{nj\nu}$ is the characteristic function of $Q_{nj\nu}$.

By duality, for $\delta > 0$ fixed and small enough, (17) is equivalent to

$$\Big\|\sum_{njv} \frac{a_{njv}}{2^{n\cdot 3k/(k+1)\cdot 1/q}}\, \hat{\varphi}_{njv}\Big\|_{p'} \leqslant C\delta^{1/q}\Big\|\sum_{njv} a_{njv}\,\varphi_{njv}\Big\|_{q'} \tag{18}$$

where p', q' are the conjugate exponents of p, q respectively.

In this way, since $p' > 4$ we can apply the Hausdorff–Young inequality and writing $\frac{3k}{k+1}\cdot\frac{1}{q} = \frac{3k}{p'}$ we obtain

$$\Big(\int\Big|\sum_{njv}\frac{a_{njv}}{2^{n\cdot 3k/p'}}\,\hat{\varphi}_{njv}(\xi)\Big|^{p'}d\xi\Big)^{1/p'}$$
$$\leqslant\Big(\int\Big|\sum_{nm}\sum_{jl}\sum_{v\mu}\frac{a_{njv}\,a_{ml\mu}}{2^{n\cdot 3k/p'}\,2^{m\cdot 3k/p'}}\,\varphi_{njv}*\varphi_{ml\mu}(x)\Big|^{s}dx\Big)^{1/2s}$$
$$\leqslant C\Big(\sum_{r\geqslant 0}\Big(\int\Big|\sum_{n}\sum_{jl}\sum_{v\mu}\frac{a_{njv}\,a_{n-r,l,\mu}}{2^{n\cdot 3k/p'}\,2^{(n-r)\cdot 3k/p'}}\,\varphi_{njv}*\varphi_{n-r,l,\mu}(x)\Big|^{s}dx\Big)^{1/s}\Big)^{1/2}$$
$$\leqslant C\Big(\sum_{(a)}+\sum_{(b)}\Big)^{1/2}$$

where $\sum_{(a)}$ is the sum in r with $r \geqslant |\log_2\delta|$ and $\sum_{(b)}$ is the sum in r with $0 \leqslant r \leqslant |\log_2\delta|$.

(i) *First case*. We are going to estimate $\sum_{(a)}$ which corresponds to all pairs (m, n) with $m \leqslant n$, $r = n-m \geqslant |\log_2\delta|$, that is, with $2^m \leqslant \delta 2^n$.

Therefore, for fixed $r = n-m$, we have except for finite overlappings independent of δ

$$(\varphi_{njv}*\varphi_{ml\mu})\cdot(\varphi_{nj'v'}*\varphi_{ml'\mu'}) = 0 \quad \text{if } j \neq j' \text{ or } v \neq v'$$

whereas

$$\operatorname{supp}(\varphi_{njv}*\varphi_{ml\mu}) \cong \operatorname{supp}(\varphi_{njv}*\varphi_{ml'\mu'})$$

for all l', μ', i.e. for $\delta^{-1/2}\cdot\delta^{-1/2} = \delta^{-1}$ pairs of indices. Consequently

$$\Big(\int\Big|\sum_{n}\sum_{jl}\sum_{v\mu}\frac{a_{njv}\,a_{ml\mu}}{2^{n\cdot 3k/p'}\,2^{m\cdot 3k/p'}}\,\varphi_{njv}*\varphi_{ml\mu}(x)\Big|^{s}dx\Big)^{1/s} \tag{19}$$
$$\leqslant C\Big(\sum_{njv}\frac{|a_{njv}|^s}{2^{n\cdot 3ks/p'}\,2^{m\cdot 3ks/p'}}\int\Big|\varphi_{njv}*\sum_{l\mu}a_{ml\mu}\,\varphi_{ml\mu}(x)\Big|^{s}dx\Big)^{1/s}$$
$$\leqslant C\Big(\sum_{n}\sum_{jv}\sum_{l\mu}\frac{\delta^{-1(s-1)}\,|a_{njv}|^s\,|a_{ml\mu}|^s}{2^{n\cdot 3ks/p'}\,2^{m\cdot 3ks/p'}}\int|\varphi_{njv}*\varphi_{ml\mu}(x)|^{s}dx\Big)^{1/s}.$$

But now

$$\int|\varphi_{njv}*\varphi_{ml\mu}(x)|^{s}dx \leqslant C\delta^{(1+1/2+1/k)(s+1)}\,\mu^{(2-k)s/k}\,v^{(2-k)/k}\,2^{3ms}\,2^{3n}$$

because

$$\|\varphi_{njv}*\varphi_{ml\mu}\|_\infty \cong |\operatorname{supp}(\varphi_{ml\mu})| = \delta^{1+1/2+1/k}\,2^{3m}\,\mu^{(2-k)/k}$$

and

$$|\operatorname{supp}(\varphi_{njv}*\varphi_{ml\mu})| \cong |\operatorname{supp}(\varphi_{njv})| = \delta^{1+1/2+1/k}\,2^{3n}\,v^{(2-k)/k}.$$

Thus the expression (19) is less than or equal to

$$C\Big(\sum_{n}\sum_{jl}\sum_{v\mu}\frac{\delta^{1-s}\,\delta^{(1+1/2+1/k)(s+1)}\,2^{3ms}\,2^{3n}}{2^{n\cdot 3ks/p'}\,2^{m\cdot 3ks/p'}}\cdot\frac{|a_{njv}|^s\,|a_{ml\mu}|^s}{\mu^{(k-2)s/k}\,v^{(k-2)/k}}\Big)^{1/2} = \mathrm{I}. \tag{20}$$

LEMMA 5. *We have the inequality*

$$\sum_{v\mu}\frac{|a_{njv}|^s\,|a_{ml\mu}|^s}{\mu^{(k-2)s/k}\,v^{(k-2)/k}} \leqslant C\delta^{-(s-1)/2}\Big(\sum_{v}|a_{njv}|^{q'}\,v^{(k-2)/k}\Big)^{1/p_1}\Big(\sum_{\mu}|a_{ml\mu}|^{q'}\,\mu^{(k-2)/k}\Big)^{1/p_1} \tag{21}$$

whenever $p_1 s = q'$, $q = p'/(k+1)$.

Proof. We write

$$\sum_{v\mu}\frac{|a_{njv}|^s\,|a_{ml\mu}|^s}{\mu^{(k-2)s/k}\,v^{(k-2)/k}} = \sum_{v\mu}A_v A_\mu\frac{1}{v^\alpha\mu^\beta} = \sum_{v}A_v B_v$$

where

$$A_v = \frac{|a_{njv}|^s}{v^{(k-2)/k\cdot 1/p_1}},$$
$$\alpha = \frac{k-2}{k}-\frac{k-2}{k}\cdot\frac{1}{p_1} = \frac{(k-2)(k-1)(s-1)}{2k},$$
$$\beta = \frac{k-2}{k}s-\frac{k-2}{k}\cdot\frac{1}{p_1} = \frac{(k-2)(k+1)(s-1)}{2k},$$
$$B_v = \sum_{\mu}\frac{A_\mu}{v^\alpha\mu^\beta}.$$

Because of $1 < p_1 < \infty$, by Hölder's inequality we have

$$\sum_{v}A_v B_v \leqslant \Big(\sum_{v}A_v^{p_1}\Big)^{1/p_1}\Big(\sum_{v}B_v^{p_1'}\Big)^{1/p_1'} \quad \text{with } \frac{1}{p_1}+\frac{1}{p_1'} = 1. \tag{22}$$

Now

$$\Big(\sum_{v}B_v^{p_1'}\Big)^{1/p_1'} = \Big(\sum_{v}\Big(\sum_{\mu}\frac{A_\mu}{v^\alpha\mu^\beta}\Big)^{p_1'}\Big)^{1/p_1'} \leqslant \sum_{\mu}\Big(\sum_{v}\Big(\frac{A_\mu}{v^\alpha\mu^\beta}\Big)^{p_1'}\Big)^{1/p_1'}$$
$$= \Big(\sum_{\mu}\frac{A_\mu}{\mu^\beta}\Big)\Big(\sum_{v}\frac{1}{v^{\alpha p_1'}}\Big)^{1/p_1'}$$

with

$$\Big(\sum_{\nu=1}^{\delta^{-1/2}} \frac{1}{\nu^{\alpha p_1'}}\Big)^{1/p_1'} = \Big(\sum_{\nu=1}^{\delta^{-1/2}} \nu^{(2-k)/k}\Big)^{1/p_1'} \cong \delta^{-1/2\cdot((2-k)/k+1)\cdot 1/p_1'} = \delta^{-(k-1)(s-1)/2k},$$

and

$$\sum_{\mu} \frac{A_\mu}{\mu^\beta} \leqslant \big(\sum_{\mu} A_\mu^{p_1}\big)^{1/p_1} \big(\sum_{\mu=1}^{\delta^{-1/2}} \mu^{-\beta p_1'}\big)^{1/p_1'} \leqslant \delta^{-(s-1)/2k} \big(\sum_{\mu} A_\mu^{p_1}\big)^{1/p_1}$$

since

$$\big(\sum_{\mu=1}^{\delta^{-1/2}} \mu^{-\beta p_1'}\big)^{1/p_1'} \cong \delta^{-1/2\cdot(-\beta p_1'+1)\cdot 1/p_1'} = \delta^{-(s-1)/2k}.$$

Therefore

$$\big(\sum_{\nu} B_\nu^{p_1'}\big)^{1/p_1'} \leqslant \delta^{-(k-1)(s-1)/2k} \delta^{-(s-1)/2k} \big(\sum_{\mu} A_\mu^{p_1}\big)^{1/p_1} = \delta^{-(s-1)/2} \big(\sum_{\mu} A_\mu^{p_1}\big)^{1/p_1}.$$

Combining this with (22) yields (21) and Lemma 5 is proved.

Bearing now in mind that

$$\sum_{j=1}^{\delta^{-1/2}} \big(\sum_{\nu} |a_{nj\nu}|^{q'} \nu^{(2-k)/k}\big)^{1/p_1} \leqslant \delta^{-1/2\cdot(1-1/p_1)} \big(\sum_{j\nu} |a_{nj\nu}|^{q'} \nu^{(2-k)/k}\big)^{1/p_1},$$

applying Hölder's inequality (since $p_1 > 1$), substituting (21) into (20) and grouping terms we finally obtain

$$\begin{aligned} \mathrm{I} &\leqslant C 2^{-r\cdot(3s-3)/2s} \delta^{k(s-1)/s} \Big[\sum_{n} \big(\sum_{j\nu} |a_{nj\nu}|^{q'} \nu^{(2-k)/k} \delta^{1+1/2+1/k} 2^{3n}\big)^{1/p_1} \\ &\quad \cdot \big(\sum_{l\mu} |a_{n-r,l,\mu}|^{q'} \mu^{(2-k)/k} \delta^{1+1/2+1/k} 2^{3(n-r)}\big)^{1/p_1}\Big]^{1/s} \\ &\leqslant C 2^{-r\cdot(3s-3)/2s} \delta^{k\cdot(s-1)/s} \big(\sum_{nj\nu} |a_{nj\nu}|^{q'} \nu^{(2-k)/k} \delta^{1+1/2+1/k} 2^{3n}\big)^{2/p_1 s} \end{aligned}$$

where we have used

$$\sum_{n} A_n^{2/p_1} \leqslant \big(\sum_{n} A_n\big)^{2/p_1}$$

with

$$A_n = \big(\sum_{j\nu} |a_{nj\nu}|^{q'} \nu^{(2-k)/k} \delta^{1+1/2+1/k} 2^{3n}\big)$$

which is correct if $2/p_1 \geqslant 1$, and this is equivalent to $p' \geqslant 2k$.

Then finally

$$\begin{aligned} \sum_{(a)} &= \sum_{r \geqslant |\log_2 \delta|} \Big(\int \Big|\sum_{n} \sum_{jl} \sum_{\nu\mu} \frac{a_{nj\nu}\, a_{n-r,l,\mu}}{2^{n\cdot 3k/p'}\, 2^{(n-r)\cdot 3k/p'}} \varphi_{nj\nu} * \varphi_{n-r,l,\mu}(x)\Big|^s dx\Big)^{1/s} \\ &\leqslant C \sum_{r \geqslant |\log_2 \delta|} 2^{-r(3s-3)/2s} \delta^{k(s-1)/s} \big(\sum_{nj\nu} |a_{nj\nu}|^{q'} \nu^{(2-k)/k} \delta^{1+1/2+1/k} 2^{3n}\big)^{2/q'} \\ &\leqslant C \delta^{(3+2k)(s-1)/2s} \big(\sum_{nj\nu} |a_{nj\nu}|^{q'} \nu^{(2-k)/k} \delta^{1+1/2+1/k} 2^{3n}\big)^{2/q'}. \end{aligned}$$

(ii) *Second case*. We are going to estimate

$$\sum_{(b)} = \sum_{0 \leqslant r \leqslant |\log_2 \delta|} \Big(\int \Big|\sum_{n} \sum_{jl} \sum_{\nu\mu} \frac{a_{nj\nu}\, a_{n-r,l,\mu}}{2^{n\cdot 3k/p'}\, 2^{(n-r)\cdot 3k/p'}} \varphi_{nj\nu} * \varphi_{n-r,l,\mu}(x)\Big|^s dx\Big)^{1/s}.$$

$r = n-m \geqslant |\log_2 \delta|$ means $2^m \geqslant \delta 2^n$, that is, the diameter of Γ_m^* is bigger than the thickness of any block in Γ_n^*.

For fixed $r = n-m$ and the angular directions ν, μ, we want to know to how many supports of the type

$$\operatorname{supp}(\varphi_{nj\nu} * \varphi_{ml\mu}), \qquad j, \nu = 1, 2, \ldots, \delta^{-1/2},$$

a point inside the integrand in $\sum_{(b)}$ belongs. This will be the vertical overlapping.

In order to do this, we observe that if $2^m \leqslant \delta^{1/2} 2^n$, that is, the diameter of Γ_m^* is smaller than the height of any block in Γ_n^*, then

$$\operatorname{supp}(\varphi_{nj\nu} * \varphi_{mk\mu}) \cap \operatorname{supp}(\varphi_{nj'\nu'} * \varphi_{mk'\mu'}) = \emptyset$$

except for

$$|j-j'| + |\nu-\nu'| \leqslant C$$

where C is a constant independent of δ.

On the other hand, if we keep $\varphi_{nj\nu}$ fixed, then

$$\operatorname{supp}(\varphi_{nj\nu} * \varphi_{mk\mu}) \cap \operatorname{supp}(\varphi_{nj\nu} * \varphi_{mk'\mu'}) \neq \emptyset$$

for certain indices k' and μ'.

If we fix the angular directions μ, μ' in this last intersection, it will not be empty for all k, k'. Thus, with fixed angular directions, a point will belong to at most $\delta^{-1/2}$ supports at the same time.

If $2^m \geqslant \delta^{1/2} 2^n$, we consider in Γ_m^* the $\dfrac{2^m}{\delta^{1/2} 2^n}$ rings Γ_{ml}^*, where Γ_{ml}^* is the ring of height $\delta^{1/2} 2^n$ obtained as the union $\bigcup_{k,\mu} Q_{mk\mu}$ with μ varying over all angular directions and k such that

$$\delta^{1/2} 2^n l \leqslant \delta^{1/2} 2^m k \leqslant \delta^{1/2} 2^n (l+1).$$

Hence

$$[\Gamma^*_{ml}+\operatorname{supp}(\varphi_{nj\nu})]\cap[\Gamma^*_{ml'}+\operatorname{supp}(\varphi_{nj'\nu'})]\neq\emptyset$$

for $l+j=l'+j'$, while this intersection has a finite overlapping independent of δ if $l+j\neq l'+j'$.

Hence, fixing n, m, a point in $\Gamma^*_{ml}+\operatorname{supp}(\varphi_{nj\nu})$ belongs to at most $\dfrac{2^m}{\delta^{1/2}2^n}$ different sums of this kind.

Since

$$\operatorname{supp}(\varphi_{nj\nu}*\sum_\mu\varphi_{mk\mu})\cap\operatorname{supp}(\varphi_{nj\nu}*\sum_\mu\varphi_{mk'\mu})\neq\emptyset$$

for all k, k' with $\operatorname{supp}(\sum_\mu\varphi_{mk\mu})\subset\Gamma^*_{ml}$ and $\operatorname{supp}(\sum_\mu\varphi_{mk'\mu})\subset\Gamma^*_{ml}$ and there are 2^{n-m} such pairs of indices, it follows that a point in $\operatorname{supp}(\varphi_{nj\nu}*\sum_\mu\varphi_{mk\mu})$ belongs to at most

$$\left(\frac{2^m}{\delta^{1/2}2^n}\right)\cdot\left(\frac{2^n}{2^m}\right)=\delta^{-1/2}$$

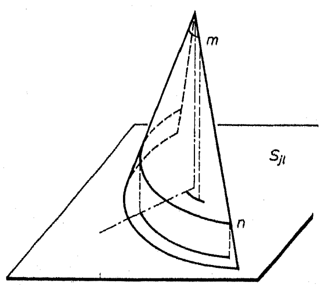
different supports of this class.

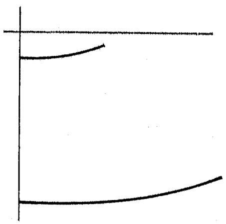
Therefore

$$\left(\int\left|\sum_n\sum_{jl}\sum_{\nu\mu}\frac{a_{nj\nu}a_{ml\mu}}{2^{n\cdot 3k/p'}2^{m\cdot 3k/p'}}\varphi_{nj\nu}*\varphi_{ml\mu}(x)\right|^s dx\right)^{1/s}\tag{23}$$
$$\leqslant C\left(\sum_n\sum_{jl}\frac{\delta^{-1/2\cdot(s-1)}}{2^{n\cdot 3k/p'\cdot s}2^{m\cdot 3k/p'\cdot s}}\int\left|\sum_\nu a_{nj\nu}\varphi_{nj\nu}*\sum_\mu a_{ml\mu}\varphi_{ml\mu}(x)\right|^s dx\right)^{1/s}.$$

Now, for fixed $r=n-m$ and j, l, we are going to study the horizontal overlapping and for this it is enough to observe the size and distribution of the different supports

$$\operatorname{supp}(\varphi_{nj\nu}*\varphi_{ml\mu})$$





in the plane S_{jl}, which is perpendicular to the z-axis at height

$$z=2^n+2^m+j\delta^{1/2}2^n+l\delta^{1/2}2^m.$$

We shall have

$$\int\left|\sum_\nu a_{nj\nu}\varphi_{nj\nu}*\sum_\mu a_{ml\mu}\varphi_{ml\mu}(x)\right|^s dx\leqslant\sum_{\nu\mu}M(\nu,\mu)^{s-1}|a_{nj\nu}|^s|a_{ml\mu}|^s\int|\varphi_{nj\nu}*\varphi_{ml\mu}(x)|^s dx$$

where $M=M(\nu,\mu)$ is the overlapping function. That is, for fixed $r=n-m$, j and l, $M(\nu,\mu)$ is the maximum number of other $\operatorname{supp}(\varphi_{nj\nu'}*\varphi_{ml\mu'})$'s a point in $\operatorname{supp}(\varphi_{nj\nu}*\varphi_{ml\mu})$ can lie in.

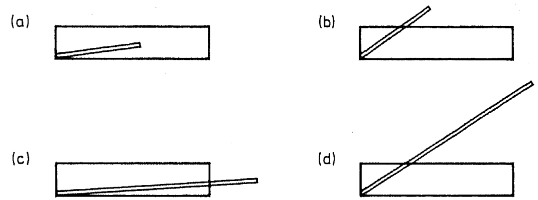
The geometrical situation is complicated because the large side of $Q_{nj\nu}$ may be smaller than the large side of $Q_{ml\mu}$. We are going to distinguish and treat separately each of the following four possible cases, depending on ν, μ:

(a) $\mu^{(2-k)/k}2^m\leqslant\nu^{(2-k)/k}2^n$, $\mu^{(2-k)/k}(\nu^{(2k-2)/k}-\mu^{(2k-2)/k})\leqslant 2^{n-m}$,

(b) $\mu^{(2-k)/k}2^m\leqslant\nu^{(2-k)/k}2^n$, $\mu^{(2-k)/k}(\nu^{(2k-2)/k}-\mu^{(2k-2)/k})\geqslant 2^{n-m}$,

(c) $\mu^{(2-k)/k}2^m\geqslant\nu^{(2-k)/k}2^n$, $\mu^{(2-k)/k}(\nu^{(2k-2)/k}-\mu^{(2k-2)/k})\leqslant 2^{n-m}$,

(d) $\mu^{(2-k)/k}2^m\geqslant\nu^{(2-k)/k}2^n$, $\mu^{(2-k)/k}(\nu^{(2k-2)/k}-\mu^{(2k-2)/k})\geqslant 2^{n-m}$.



LEMMA 6. (i) *In cases* (a), (c) *we have the bounds*

$$M\leqslant\frac{2^{n-m}\mu^{(k-2)/k}}{|\nu^{(2k-2)/k}-\mu^{(2k-2)/k}|+1}\quad\textit{if }\nu\leqslant\mu,\ \textit{or }\mu\leqslant\nu\textit{ with }M\leqslant\mu,$$

$$M\leqslant\frac{2^{(n-m)(1/2+(k-1)/k)}}{|\nu^{(2k-2)/k}-\mu^{(2k-2)/k}|+1}\quad\textit{if }\mu\leqslant\nu\textit{ with }M\geqslant\mu.$$

(ii) *In cases* (b), (d) *we have the bound*

$$M\leqslant 4.$$

Proof. The technical proof is done at the end of the proof of Theorem 2.

LEMMA 7. *For each of the preceding cases we have*:

(a) $\int|\varphi_{nj\nu}*\varphi_{ml\mu}(x)|^s dx\leqslant C\delta^{(1+1/2+1/k)(s+1)}2^{3n}2^{3ms}\nu^{(2-k)/k}\mu^{(2-k)s/k}$.

(b) $\int|\varphi_{nj\nu}*\varphi_{ml\mu}(x)|^s dx$

$$\leqslant C\delta^{(1+1/2+1/k)(s+1)}\frac{2^{n(2+s)}2^{m(2s+1)}}{\nu^{(k-2)/k}\mu^{(k-2)/k}(|\nu^{(2k-2)/k}-\mu^{(2k-2)/k}|+1)^{s-1}}.$$

(c) $\int|\varphi_{nj\nu}*\varphi_{ml\mu}(x)|^s\,dx \leqslant C\delta^{(1+1/2+1/k)(s+1)}\,2^{n(2+s)}\,2^{m(2s+1)}\,\nu^{(2-k)s/k}\,\mu^{(2-k)/k}$.

(d) $\int|\varphi_{nj\nu}*\varphi_{ml\mu}(x)|^s\,dx$

$$\leqslant C\delta^{(1+1/2+1/k)(s+1)}\frac{2^{n(2+s)}\,2^{m(2s+1)}}{\nu^{(k-2)/k}\,\mu^{(k-2)/k}\,(|\nu^{(2k-2)/k}-\mu^{(2k-2)/k}|+1)^{s-1}}.$$

Proof. The proof is similar to that of Proposition 1 with obvious modifications.

According to Lemmas 6 and 7 and considering that in case (c) we have $\mu^{(2-k)/k}2^m \geqslant \nu^{(2-k)/k}2^n$ and in case (d), $\mu^{(2-k)/k}2^m \leqslant \nu^{(2-k)/k}2^n$, after some computations we arrive at

$$\int\Big|\sum_{\nu} a_{nj\nu}\,\varphi_{nj\nu}*\sum_{\mu} a_{ml\mu}\,\varphi_{ml\mu}(x)\Big|^s dx$$

$$\leqslant C\frac{2^{n((3k-2)s/2k+(3k+2)/2k)}\,2^{m((3k-2)s/2k+(3k-2)/2k)}\,|a_{nj\nu}|^s\,|a_{ml\mu}|^s}{\mu^{(k-2)s/k}\,\nu^{(k-2)/k}\,(|\nu^{(2k-2)/k}-\mu^{(2k-2)/k}|+1)^{s-1}}.$$

Inserting now this expression in (23), applying the fractional integration of Proposition 2 and grouping terms, we have, noting that $p_1 s = q'$,

$$\Big(\int\Big|\sum_n\sum_{jl}\sum_{\nu\mu}\frac{a_{nj\nu}\,a_{n-r,l,\mu}}{2^{n\cdot 3k/p'}\,2^{(n-r)\cdot 3k/p'}}\,\varphi_{nj\nu}*\varphi_{n-r,l,\mu}(x)\Big|^s dx\Big)^{1/s}$$

$$\leqslant C2^{-r(s-1)/ks}\,\delta^{(k+1)(s-1)/s}\Big(\sum_n\Big(\sum_{j\nu}|a_{nj\nu}|^{q'}\,2^{3n}\,\nu^{(2-k)/k}\,\delta^{1+1/2+1/k}\Big)^{1/p_1}$$

$$\cdot\Big(\sum_{l\mu}|a_{n-r,l,\mu}|^{q'}\,2^{3(n-r)}\,\mu^{(2-k)/k}\,\delta^{1+1/2+1/k}\Big)^{1/p_1}\Big)^{1/s}.$$

Taking now $p' \geqslant 2k$, like in the first case, we obtain

$$\sum_n\Big(\sum_{j\nu}|a_{nj\nu}|^{q'}\,2^{3n}\,\nu^{(2-k)/k}\,\delta^{1+1/2+1/k}\Big)^{1/p_1}\Big(\sum_{l\mu}|a_{n-r,l,\mu}|^{q'}\,2^{3(n-r)}\,\mu^{(2-k)/k}\,\delta^{1+1/2+1/k}\Big)^{1/p_1}$$

$$\leqslant\Big(\sum_{nj\nu}|a_{nj\nu}|^{q'}\,2^{3n}\,\nu^{(2-k)/k}\,\delta^{1+1/2+1/k}\Big)^{2/p_1}.$$

From this it follows that

$$\sum_{\text{(b)}}\leqslant C\Big(\sum_{r\geqslant 0}2^{-r(s-1)/ks}\Big)\delta^{(k+1)(s-1)/s}\Big(\sum_{nj\nu}|a_{nj\nu}|^{q'}\,2^{3n}\,\nu^{(2-k)/k}\,\delta^{1+1/2+1/k}\Big)^{2/q'}.$$

But

$$\sum_{r\geqslant 0}2^{-r(s-1)/ks}\cong\frac{1}{\log 2}\cdot\frac{s-1}{ks}\leqslant C$$

and so, bearing in mind that $1/q=(k+1)(s-1)/2s$ because $q=p'/(k+1)$ and

$2/p'+1/s=1$, we finally obtain

$$\Big\|\sum_{nj\nu}\frac{a_{nj\nu}}{2^{n\cdot 3k/(k+1)\cdot 1/q}}\,\hat{\varphi}_{nj\nu}\Big\|_{p'}\leqslant C\Big(\sum_{\text{(a)}}+\sum_{\text{(b)}}\Big)^{1/2}$$

$$\leqslant C\delta^{1/q}\Big(\sum_{nj\nu}|a_{nj\nu}|^{q'}\,2^{3n}\,\nu^{(2-k)/k}\,\delta^{(1+1/2+1/k)}\Big)^{1/q'}$$

which is (18) and the proof of Theorem 2 is done.

Proof of Lemma 6. We study the whole geometrical situation in the S_{jl} plane. In cases (b), (d) it is clear that there is a finite overlapping and so M is bounded by a constant independent of the decomposition. Case (c) is similar to (a).

We thus treat case (a). For fixed ν, μ we project

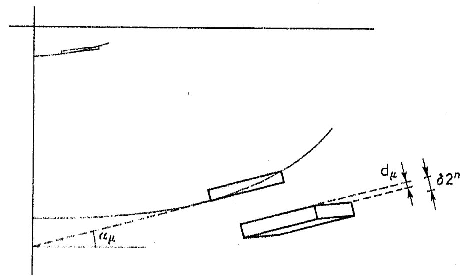
$$\operatorname{supp}(\varphi_{nj\nu}*\varphi_{ml\mu})$$

on a straight line r perpendicular to the direction of the larger side of $Q_{nj\nu}$. If we take $\operatorname{supp}(\varphi_{m,l,\mu+1})$, the projection of $\operatorname{supp}(\varphi_{nj\nu}*\varphi_{m,l,\mu+1})$ on r is displaced a distance d_μ with respect to the projection of $\operatorname{supp}(\varphi_{nj\nu}*\varphi_{ml\mu})$, where d_μ is the length of the larger side of $\operatorname{supp}(\varphi_{m,l,\mu})$ on the line r, that is,

$$d_\mu=\delta^{1/k}\,2^m\,\mu^{(2-k)/k}\sin|\alpha_\mu-\alpha_\nu|\cong\delta 2^m\,\mu^{(2-k)/k}\,|\mu^{(2k-2)/k}-\nu^{(2k-2)/k}|$$

where α_μ is the angle formed by $\operatorname{supp}(\varphi_{ml\mu})$ with the axis $x=0$, and so

$$\sin\alpha_\mu\cong C(\mu^{2/k}\,\delta^{1/k})^{k-1}.$$



The respective projections are further away or closer to the origin according as $\mu\leqslant\nu$ or $\nu\leqslant\mu$.

If for example $\mu\geqslant\nu$ and we take $\mu+1,\ldots,\mu+M_1$ (with $M_1\leqslant\delta^{-1/2}$), then

$$\operatorname{supp}(\varphi_{nj\nu}*\varphi_{ml\mu})\cap\operatorname{supp}(\varphi_{nj\nu}*\varphi_{m,l,\mu+M_1})=\emptyset$$

if

$$d_\mu+d_{\mu+1}+\ldots+d_{\mu+M_1}\geqslant \delta 2^n \tag{24}$$

because $\delta 2^n$ is the thickness of $\operatorname{supp}(\varphi_{nj\nu})$.

In order to find the overlapping function M we must take the first M_2 such that

$$d_\mu+d_{\mu-1}+\ldots+d_{\mu-M_2}\geqslant \delta 2^n \tag{25}$$

and then $M(\nu,\mu)=\max\{M_1, M_2\}$.

We must be careful, because it could be for example that in (24) no M_1 satisfies the relation because we arrive at the end $\delta^{-1/2}$ and still

$$d_\mu+d_{\mu+1}+\ldots+d_{\delta^{-1/2}}\leqslant \delta 2^n$$

or that in (25)

$$d_\mu+d_{\mu-1}+\ldots+d_\nu\leqslant \delta 2^n.$$

We must then consider these possibilities.

Similar considerations can be done in the case $\mu\leqslant\nu$.

(1) Let $\mu\leqslant\nu$; two cases can occur:

$$d_\mu+d_{\mu+1}+\ldots+d_\nu\geqslant \delta 2^n \tag{1.1}$$

or

$$d_\mu+d_{\mu+1}+\ldots+d_\nu\leqslant \delta 2^n. \tag{1.2}$$

(1.1) In this case, if we take $f(\mu)=d_\mu$, then f is decreasing because

$$f'(\mu)=\delta 2^m\left(\frac{2-k}{k}\nu^{(2k-2)/k}\mu^{(2-2k)/k}-1\right)\leqslant 0;$$

this tells us that

$$d_\mu+d_{\mu+1}+\ldots+d_{\mu+M}\leqslant d_\mu+d_{\mu-1}+\ldots+d_{\mu-M}$$

and so the most unfavorable case shall be to take the first M such that

$$d_\mu+d_{\mu+1}+\ldots+d_{\mu+M}\geqslant \delta 2^n.$$

As

$$d_\mu+d_{\mu+1}+\ldots+d_{\mu+M}=\delta 2^m\sum_{j=0}^{M}(\mu+j)^{(2-k)/k}\big(\nu^{(2k-2)/k}-(\mu+j)^{(2k-2)/k}\big)$$

we are looking for M such that

$$\sum_{j=0}^{M}(\mu+j)^{(2-k)/k}\big(\nu^{(2k-2)/k}-(\mu+j)^{(2k-2)/k}\big)\cong 2^{n-m}.$$

(1.1.1) If $M\leqslant 2\mu$ with $\mu\leqslant\nu/4$, then

$$\begin{aligned}2^{n-m}&\cong\sum_{j=0}^{M}(\mu+j)^{(2-k)/k}\big(\nu^{(2k-2)/k}-(\mu+j)^{(2k-2)/2}\big)\\&\geqslant M\mu^{(2-k)/k}\big(\nu^{(2k-2)/k}-\mu^{(2k-2)/k}\big)\end{aligned}$$

because

$$\mu\leqslant\mu+j\leqslant 3\mu\quad\text{and}\quad \nu^{(2k-2)/k}-(\mu+j)^{(2k-2)/k}\cong\nu^{(2k-2)/k}$$

for every j; so

$$M\leqslant\frac{2^{n-m}\mu^{(k-2)/k}}{|\nu^{(2k-2)/2}-\mu^{(2k-2)/k}|+1}.$$

(1.1.2) If $M\geqslant 2\mu$ with $\mu\leqslant\nu/4$ then

$$\begin{aligned}&\sum_{j=0}^{M}(\mu+j)^{(2-k)/k}\big(\nu^{(2k-2)/k}-(\mu+j)^{(2k-2)/k}\big)\\&\quad=\nu^{(2k-2)/k}\sum_{j=0}^{M}(\mu+j)^{(2-k)/k}-\sum_{j=0}^{M}(\mu+j)\\&\quad\cong\frac{k}{2}\nu^{(2k-2)/k}\big((\mu+M)^{2/k}-\mu^{2/k}\big)-\tfrac12\big((\mu+M)^2-\mu^2\big)\cong\nu^{(2k-2)/k}M^{2/k}-M^2.\end{aligned}$$

But $M\leqslant 2^{n-m}/\nu$ since

$$\begin{aligned}2^{n-m}&\cong\sum_{j=0}^{M}(\mu+j)^{(2-k)/k}\big(\nu^{(2k-2)/k}-(\mu+j)^{(2k-2)/k}\big)\\&\geqslant\sum_{j=0}^{M}(\mu+j)^{(2-k)/k}(\nu-\mu-j)(\mu+j)^{(k-2)/k}\\&=\sum_{j=0}^{M}(\nu-\mu-j)=(M+1)\big((\nu-\mu)-M/2\big).\end{aligned}$$

Hence $M\leqslant 2^{n-m}/\nu$ and then

$$\begin{aligned}2^{n-m}&\cong\nu^{(2k-2)/k}M^{2/k}-M^2=M^2\big(\nu^{(2k-2)/k}M^{(2-2k)/k}-1\big)\\&\geqslant M^2\left(\frac{2^{n-m}}{\nu}\right)^{(2-2k)/k}\nu^{(2k-2)/k},\end{aligned}$$

that is,

$$M\leqslant\frac{2^{(n-m)(1+(2k-2)/k)\cdot 1/2}}{\nu^{(2k-2)/k}}\cong\frac{2^{(n-m)(1+(2k-2)/k)\cdot 1/2}}{|\nu^{(2k-2)/k}-\mu^{(2k-2)/k}|+1}.$$

(1.1.3) $\mu \geqslant \nu/4$; in this case

$$2^{n-m} \cong \sum_{j=0}^{M} (\mu+j)^{(2-k)/k}\left(\nu^{(2k-2)/k}-(\mu+j)^{(2k-2)/k}\right) \geqslant \nu^{(2-k)/k}\sum_{j=0}^{M}(\nu-\mu-j)\,\mu^{(k-2)/k}$$

$$\cong M((\nu-\mu)-M/2),$$

so

$$M \leqslant \frac{2^{n-m}}{|\nu-\mu|+1}.$$

(1.2) In this case we shall take $M=\max\{M_1, M_2, M_3\}$, where $M_1 = \nu-\mu$, M_2 is the first M_2 such that

$$d_\mu+d_{\mu-1}+\ldots+d_{\mu-M_2}\geqslant\delta 2^n$$

and M_3 the first M_3 such that

$$d_\nu+d_{\nu+1}+\ldots+d_{\nu+M_3}\geqslant \delta 2^n.$$

(1.2.1) We are going to bound $M_1=\nu-\mu$.
As $d_\nu+d_{\nu-1}+\ldots+d_\mu \leqslant \delta 2^n$,

$$2^{n-m}\geqslant \sum_{j=0}^{\nu-\mu}(\nu-j)^{(2-k)/k}\left(\nu^{(2k-2)/k}-(\nu-j)^{(2k-2)/k}\right)$$

$$\cong \frac{k}{2}(\nu^{2/k}-\mu^{2/k})\,\nu^{(2k-2)/k}-\tfrac{1}{2}(\nu^2-\mu^2).$$

(1.2.1.1) If $\mu\leqslant\nu/4$ then

$$2^{n-m}\geqslant\frac{k-1}{2}\nu^2\cong\nu^2;$$

therefore

$$M_1\leqslant\nu\leqslant 2^{(n-m)/2}\leqslant \frac{2^{(n-m)(1+(2k-2)/k)/2}}{\nu^{(2k-2)/k}} \cong \frac{2^{(n-m)(1+(2k-2)/k)/2}}{|\nu^{(2k-2)/k}-\mu^{(2k-2)/k}|+1}.$$

(1.2.1.2) If $\mu\geqslant\nu/4$ then

$$\sum_{j=0}^{\nu-\mu}(\nu-j)^{(2-k)/k}\left(\nu^{(2k-2)/k}-(\nu-j)^{(2k-2)/2}\right)\cong\mu^{(2-k)/k}\,\mu^{(k-2)/2}\sum_{j=0}^{\nu-\mu}j\cong(\nu-\mu)^2,$$

so

$$M_1\leqslant\nu-\mu\leqslant\frac{2^{n-m}}{\nu-\mu}\cong\frac{2^{n-m}\,\mu^{(k-2)/k}}{|\nu^{(2k-2)/k}-\mu^{(2k-2)/k}|+1}.$$

(1.2.2) We are going to bound M_2. This is easy because

$$2^{n-m}\cong\sum_{j=0}^{M_2}(\mu-j)^{(2-k)/k}\left(\nu^{(2k-2)/k}-(\mu-j)^{(2k-2)/2}\right)$$

$$\geqslant\sum_{j=0}^{M_2}\mu^{(2-k)/k}\left(\nu^{(2k-2)/k}-\mu^{(2k-2)/k}\right) = (M_2+1)\,\mu^{(2-k)/k}\left(\nu^{(2k-2)/k}-\mu^{(2k-2)/k}\right)$$

from which we obtain the desired relation.

(1.2.3) We are looking for M_3 such that

$$2^{n-m}\cong\sum_{j=0}^{M_3}(\nu+j)^{(2-k)/k}\left((\nu+j)^{(2k-2)/k}-\nu^{(2k-2)/k}\right).$$

As

$$\sum_{j=0}^{M_3}(\nu+j)^{(2-k)/k}\left((\nu+j)^{(2k-2)/k}-\nu^{(2k-2)/k}\right)\geqslant \sum_{j=0}^{M_3}(\nu+j)^{(2-k)/k}\,j\,\nu^{(k-2)/k}$$

$$= \sum_{j=0}^{M_3}\left(\frac{\nu}{\nu+j}\right)^{(k-2)/k} j\geqslant M_3^2,$$

it follows that $M_3\leqslant(2^{n-m})^{1/2}$; therefore if $\mu\geqslant\nu/4$ then

$$(2^{n-m})^{1/2}\leqslant\frac{2^{n-m}\,\mu^{(k-2)/k}}{|\nu^{(2k-2)/k}-\mu^{(2k-2)/k}|+1}$$

because the quotient is as $2^{n-m}/(\nu-\mu)$ and $(\nu-\mu)^2\leqslant 2^{n-m}$ like in 1.2.1.2, while if $\mu\leqslant\nu/4$ then

$$M_3\leqslant(2^{n-m})^{1/2}\leqslant\frac{2^{(n-m)(1/2+(2k-2)/k)/2}}{|\nu^{(2k-2)/k}-\mu^{(2k-2)/k}|+1}$$

as in 1.2.1.1.

(2) Let now $\mu\geqslant\nu$; as before, there appear two following cases:

(2.1) $$d_\nu+d_{\nu+1}+\ldots+d_\mu\geqslant\delta 2^n$$

or

(2.2) $$d_\nu+d_{\nu+1}+\ldots+d_\mu\leqslant\delta 2^n.$$

(2.1) $f(\mu)=d_\mu$ is an increasing function because

$$f'(\mu)=\delta 2^m\left(1+\frac{k-2}{k}\nu^{(2k-2)/k}\,\mu^{(2-k)/k-1}\right)\geqslant 0$$

and then the most unfavorable case is to take the first M such that

$$d_\mu+d_{\mu-1}+\ldots+d_{\mu-M}\geqslant\delta 2^n.$$

As

$$d_\mu+d_{\mu-1}+\ldots+d_{\mu-M}=\sum_{j=0}^{M}(\mu-j)^{(2-k)/k}\big((\mu-j)^{(2k-2)/k}-\nu^{(2k-2)/k}\big)\delta 2^m,$$

we must find M such that

$$\sum_{j=0}^{M}(\mu-j)^{(2-k)/k}\big((\mu-j)^{(2k-2)/k}-\nu^{(2k-2)/k}\big)\cong 2^{n-m}.$$

We shall use Abel's summation formula, which reads

$$\sum_{j=0}^{M}a_j b_j=s_M b_0+\sum_{j=0}^{M}(s_M-s_j)(b_{j+1}-b_j),$$

where $s_j=a_0+a_1+\ldots+a_j$.

In this way

$$\sum_{j=0}^{M}(\mu-j)^{(2-k)/k}\big((\mu-j)^{(2k-2)/k}-\nu^{(2k-2)/k}\big)$$
$$=\sum_{j=0}^{M}(\mu-j)^{(2-k)/k}\big(\mu^{(2k-2)/k}-\nu^{(2k-2)/k}\big)$$
$$-\frac{k}{2}\sum_{j=0}^{M}\big((\mu-j)^{2/k}-(\mu-M)^{2/k}\big)(\mu-j)^{(k-2)/k}.$$

But

$$\big((\mu-j)^{2/k}-(\mu-M)^{2/k}\big)(\mu-j)^{(k-2)/k}\leqslant\big((\mu-j)^{(2k-2)/k}-\nu^{(2k-2)/k}\big)(\mu-j)^{(2-k)/k}$$

because for $\nu^{2/k}\leqslant(\mu-M)^{2/k}$

$$(\mu-j)^{2/k}-(\mu-M)^{2/k}\leqslant(\mu-j)^{2/k}-\nu^{2/k}$$

and

$$\big((\mu-j)^{2/k}-\nu^{2/k}\big)(\mu-j)^{(k-2)/k}\leqslant\big((\mu-j)^{(2k-2)/k}-\nu^{(2k-2)/k}\big)(\mu-j)^{(2-k)/k},$$

since this last inequality is equivalent to

$$(\mu-j)-\nu^{2/k}(\mu-j)^{(k-2)/k}\leqslant(\mu-j)-\nu^{(2k-2)/k}(\mu-j)^{(2-k)/k}$$

and

$$\nu^{(2k-2)/k}(\mu-j)^{(2-k)/k}\leqslant\nu^{2/k}(\mu-j)^{(k-2)/k}$$

because

$$\nu^{(2k-2)/k-2/k}=\nu^{2(k-2)/k}\leqslant(\mu-j)^{2(k-2)/k}$$

as $\nu\leqslant\mu-j$ for every j.

Therefore we have

$$2\cdot 2^{n-m}\geqslant 2\sum_{j=0}^{M}(\mu-j)^{(2-k)/k}\big((\mu-j)^{(2k-2)/2}-\nu^{(2k-2)/k}\big)$$
$$\geqslant\sum_{j=0}^{M}(\mu-j)^{(2-k)/k}\big(\mu^{(2k-2)/k}-\nu^{(2k-2)/k}\big)$$
$$\geqslant M\mu^{(2-k)/k}\big(\mu^{(2k-2)/k}-\nu^{(2k-2)/k}\big)$$

and so

$$M\leqslant\frac{2^{n-m}\mu^{(k-2)/k}}{|\mu^{(2k-2)/k}-\nu^{(2k-2)/k}|+1}.$$

(2.2) In this case

$$\sum_{j=0}^{\mu-\nu}(\nu+j)^{(2-k)/k}\big(\mu^{(2k-2)/k}-(\nu+j)^{(2k-2)/k}\big)\leqslant 2^{n-m}.$$

If $\mu\leqslant 2\nu$ then

$$\sum_{j=0}^{\mu-\nu}(\nu+j)^{(2-k)/k}\big(\mu^{(2k-2)/k}-(\nu+j)^{(2k-2)/k}\big)\cong\mu^{(2-k)/k}\sum_{j=0}^{\mu-\nu}(\mu-\nu-j)\,\mu^{(k-2)/k}$$
$$\cong(\mu-\nu)^2-\tfrac{1}{2}(\mu-\nu)^2=\tfrac{1}{2}(\mu-\nu)^2,$$

while if $\mu\geqslant 2\nu$ then

$$\sum_{j=0}^{\mu-\nu}(\nu+j)^{(2-k)/k}\big(\mu^{(2k-2)/k}-(\nu+j)^{(2k-2)/k}\big)$$
$$\cong\mu^{(2k-2)/k}\sum_{j=0}^{\mu-\nu}(\nu+j)^{(2-k)/k}-\sum_{j=0}^{\mu-\nu}(\nu+j)$$
$$\cong\mu^{(2k-2)/k}\frac{k}{2}\big(\mu^{2/k}-\nu^{2/k}\big)-\tfrac{1}{2}(\mu^2-\nu^2)\cong\frac{k-1}{2}\mu^2.$$

Now the reasoning follows as in 1.2.

PROPOSITION 5. *If inequality* (16) *in the statement of Theorem* 2 *is satisfied on Γ and the measure $d\sigma$ is as the Lebesgue measure on a compact piece of Γ, then necessarily*

$$q\leqslant\frac{p'}{k+1}.$$

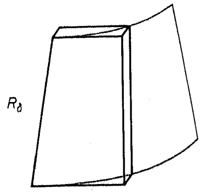
Proof. We can adapt Knapp's argument to this situation. Taking for example the truncated cone Γ_0, let R_δ be the rectangle of dimensions $\delta^{1/k}\times\delta\times 1$ adjusted to the surface as in the figure and let Ψ be a regularization of the characteristic function of this rectangle. We have

$$\|\hat{\Psi}\|_{L^q(\Gamma)}\geqslant c\delta^{1/k}$$

whereas

$$\|\Psi\|_p \approx \delta^{(1+1/k)(p-1)/p}.$$

This can be computed directly and also follows from the uncertainty principle.



Then if (16) is satisfied by every $f \in L^p(\boldsymbol{R}^3)$ we would have

$$c\delta^{1/kq} \leqslant \delta^{(1+1/k)(p-1)/p}$$

for every $\delta > 0$; letting δ go to zero we obtain the desired condition.

Proposition 6. *If for p, q fixed, inequality* (16) *in the statement of Theorem* 2 *is satisfied, and* $d\sigma$ *is as the Lebesgue measure on any compact piece of* Γ *then necessarily* $d\sigma = r^\alpha\, dr\, d\theta$ *where*

$$\alpha = 3q/p' - 1.$$

Proof. Note that in case of Theorem 2, $\alpha = (2-k)/(k+1)$.

The proof follows by the fact that the homogeneity of the Fourier transform forces inequality (16) to be satisfied by any function with compact support $f(x)$ and by all its dilations $g_\delta(x) = f(\delta x)$ for all $\delta > 0$.

Corollary 1. *If* Σ *is a compact piece of* Γ, *then*

$$\Big(\int_\Sigma |\hat{f}(\xi)|^q\, d\mu(\xi)\Big)^{1/q} \leqslant C_{p,q} \|f\|_{L^p(\boldsymbol{R}^3)} \tag{26}$$

for every function $f \in L^p(\boldsymbol{R}^3)$, $d\mu(\xi)$ *being the Lebesgue measure on* Σ, $1 \leqslant q \leqslant p'/(k+1)$, $p' > k+1$.

Proof. Note that there is a constant depending on Σ such that

$$d\mu(\xi) \leqslant C\, d\sigma(\xi) \qquad \forall\, \xi \in \Sigma;$$

then apply Theorem 2 and Hölder's inequality.

Corollary 2. *Given* Γ, *the surface of the cone with sections* (t, t^k), *we have*

$$\Big(\int_\Gamma |\hat{f}(\xi)|^q\, r^\alpha\, dr\, d\theta\Big)^{1/q} \leqslant C_{p,q} \|f\|_{L^p(\boldsymbol{R}^3)} \tag{27}$$

where $\alpha = 3q/p' - 1$, $1 \leqslant q \leqslant p'/(k+1)$, $p' \geqslant 2k$ ($p' > 4$ *if* $k = 2$).

Proof. The reasoning of Theorem 2 is to be repeated step by step changing the exponents q and α in a suitable way.

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