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REGULAR GENERATORS FOR MULTIDIMENSIONAL DYNAMICAL SYSTEMS

BY

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Introduction. Perfect partitions play an important role in the ergodic theory. In particular, they give information on mixing and spectral properties of dynamical systems. The existence of such partitions for one-dimensional systems has been proved by V. A. Rohlin and Y. G. Sinai [5], and for multidimensional systems by B. Kamiński [2].

It is easy to construct a perfect partition for a one-dimensional system with finite entropy (e.g. see [4]). The situation is more complicated for multidimensional systems with finite entropy. It is natural to seek such a partition in a way similar to that used in the one-dimensional case. This procedure leads to a problem of existence of a special kind of generators — so-called regular generators.

Let $(X, \mathscr{B}, \mu)$ be a Lebesgue space and let G be an abelian free group of rank 2 of automorphisms of $(X, \mathscr{B}, \mu)$. We assume in the sequel that G is aperiodic.

We denote by $h(G)$ the entropy of G and by $\pi(G)$ the Pinsker partition of G. For the definitions the reader is referred to [1].

Let (T, S) be an ordered pair of generators of G.

LEMMA 1 ([2]). *If ζ is a measurable partition of X with*

(a) $S^{-1}\zeta < \zeta,\ T^{-1}\zeta_S < \zeta,$

(b) $\bigvee_{n=0}^{\infty} T^n \zeta_S = \varepsilon,$

(c) $\bigwedge_{n=0}^{\infty} S^{-n}\zeta = T^{-1}\zeta_S,$

then

$$\bigwedge_{n=0}^{\infty} T^{-n}\zeta_S \geqslant \pi(G).$$

Now we recall the concept of a perfect partition of X [2].

A measurable partition ζ of X is said to be *(T, S)-perfect* if

(i) $S^{-1}\zeta < \zeta,\ T^{-1}\zeta_S < \zeta$,

(ii) $\bigvee_{n=0}^{\infty} T^n \zeta_S = \varepsilon$,

(iii) $\bigwedge_{n=0}^{\infty} S^{-n}\zeta = T^{-1}\zeta_S$,

(iv) $\bigwedge_{n=0}^{\infty} T^{-n}\zeta_S = \pi(G)$,

(v) $H(\zeta \mid S^{-1}\zeta) = h(G)$.

It has been shown that for every pair (T, S) of generators of G such a partition exists [2].

Now, let us suppose $h(G) < \infty$.

It is known [1] that there exists a measurable partition ξ with $H(\xi) < \infty$ and $\xi_G = \bigvee_{(k,l)\in Z^2} T^k S^l \xi = \varepsilon$.

Any such partition is called a *generator* for (X, μ, G).

DEFINITION. A generator ξ for (X, μ, G) is said to be *(T, S)-regular* if

$$\bigwedge_{n=0}^{\infty} (S^{-n}\xi_S^- \vee (\xi_S)_T^-) = (\xi_S)_T^- .$$

PROPOSITION. *A generator ξ is (T, S)-regular iff $\zeta = \xi_S^- \vee (\xi_S)_T^-$ is (T, S)-perfect.*

Proof. Since ξ is a generator for (X, μ, G) we have $h(\xi, G) = h(G)$([1]). Hence it is easy to see that ζ satisfies (i), (ii) and (v). From the regularity of ξ we obtain (iii). It follows from [1] that

$$\bigwedge_{n=0}^{\infty} T^{-n}\zeta_S = \bigwedge_{n=0}^{\infty} T^{-n}(\xi_S)_T^- \leqslant \pi(G).$$

Using Lemma 1 and (iii) we obtain $\bigwedge_{n=0}^{\infty} T^{-n}\zeta_S \geqslant \pi(G)$. Thus ζ is (T, S)-perfect.

The converse implication is trivial.

Regular generators for dynamical systems with zero entropy. Let $h(G) = 0$ and let (T, S) be an ordered pair of generators of G.

THEOREM 1. *A generator ξ is (T, S)-regular iff ξ_S is a strong generator for (X, μ, T), i.e. $(\xi_S)_T^- = \varepsilon$.*

Proof. If ξ_S is a strong generator for (X, μ, T), then obviously ξ is (T, S)-regular.

Now, let us suppose that ξ is (T, S)-regular. Since $h(G) = 0$ we have

$$S^{-1}\xi_S^- \vee (\xi_S)_T^- = \xi_S^- \vee (\xi_S)_T^-$$

and so

$$\bigwedge_{n=0}^{\infty}(S^{-n}\xi_S^- \vee (\xi_S)_T^-) = \xi_S^- \vee (\xi_S)_T^- .$$

From the regularity of ξ

$$\xi_S^- \vee (\xi_S)_T^- = (\xi_S)_T^- ,$$

i.e. $\xi_S^- \leqslant (\xi_S)_T^-$.

Hence $\xi_S \leqslant (\xi_S)_T^-$ and so $T^{-1}(\xi_S)_T^- = (\xi_S)_T^-$. Thus

$$\varepsilon = \xi_G = (\xi_S)_T = \bigvee_{n=0}^{\infty} T^n(\xi_S)_T^- = (\xi_S)_T^- ,$$

which completes the proof.

COROLLARY. *If $h(T) = 0$ then every generator for (X, μ, G) is (T, S)-perfect.*

The validity of the Corollary is an easy consequence of the fact that $h(T) = 0$ implies $\xi_S \leqslant (\xi_S)_T^-$.

We shall later see (Remark after Theorem 3) that there exists a dynamical system with zero entropy and a generator of this system that is not regular.

Regular partitions for Conze dynamical systems with positive entropy. It has been shown [2] that the independent generator for a Bernoulli dynamical system is regular with respect to a pair of shifts defining this system.

Our aim is to find regular generators for a class of dynamical systems considered by Conze [1] that includes Bernoulli systems.

For this purpose we first give some results concerning decreasing sequences of σ-algebras.

Let $(X, \mathscr{B}, \mu)$ be a probability space. For every $f \in L^1(X, \mu)$ and every σ-algebra $\mathscr{A} \subset \mathscr{B}$ the conditional expectation of f with respect to $\mathscr{A}$ is denoted by $E_{\mathscr{A}} f$. The conditional probability of a set $B \in \mathscr{B}$ with respect to $\mathscr{A}$ is defined as usual:

$$\mu(B \mid \mathscr{A}) = E_{\mathscr{A}}(1_B).$$

For $f \in L^1(X, \mu)$ we put

$$\|f\| = \int_X |f(x)|\, \mu(dx).$$

We denote by $\mathscr{Q}$ the trivial σ-algebra.

For σ-algebras $\mathscr{A}_1, \mathscr{A}_2 \subset \mathscr{B}$ the symbol $a(\mathscr{A}_1, \mathscr{A}_2)$ means the algebra generated by $\mathscr{A}_1$ and $\mathscr{A}_2$, and the symbol $\mathscr{A}_1 \vee \mathscr{A}_2$ – the σ-algebra generated by $\mathscr{A}_1$ and $\mathscr{A}_2$.

Let $\mathscr{C}$ be a fixed σ-subalgebra of $\mathscr{B}$ and let $\mathscr{B}_n$, $n \geqslant 1$, be σ-subalgebras of $\mathscr{B}$ such that $\mathscr{B}_n \supset \mathscr{B}_{n+1} \supset \mathscr{C}$, $n \geqslant 1$.

LEMMA 2. $\bigcap_{n=1}^{\infty} \mathscr{B}_n = \mathscr{C}$ *iff for every* $B \in \mathscr{B}_1$

$$\lim_{n\to\infty} \sup_{A\in\mathscr{B}_n} \|\mu(A\cap B|\mathscr{C}) - \mu(A|\mathscr{C})\cdot\mu(B|\mathscr{C})\| = 0.$$

Proof. Let $B \in \mathscr{B}_1$, $n \geqslant 1$, $A \in \mathscr{B}_n$ be arbitrary. We define

$$f = 1_A - \mu(A|\mathscr{C}), \qquad g = 1_B - \mu(B|\mathscr{C}).$$

It is easy to check that

$$E_{\mathscr{C}}(f\cdot g) = \mu(A\cap B|\mathscr{C}) - \mu(A|\mathscr{C})\cdot\mu(B|\mathscr{C}). \tag{1}$$

Since $\mathscr{C} \subset \mathscr{B}_n$ and f is $\mathscr{B}_n$-measurable we have

$$E_{\mathscr{C}}(f\cdot g) = E_{\mathscr{C}}\big(E_{\mathscr{B}_n}(f\cdot g)\big) = E_{\mathscr{C}}(f\cdot E_{\mathscr{B}_n} g). \tag{2}$$

From (1), (2) and the inequality $|f| \leqslant 1$ we obtain

$$\|\mu(A\cap B|\mathscr{C}) - \mu(A|\mathscr{C})\cdot\mu(B|\mathscr{C})\| = \|E_{\mathscr{C}}(f\cdot E_{\mathscr{B}_n} g)\| \leqslant \|E_{\mathscr{B}_n} g\|. \tag{3}$$

Hence

$$\sup_{A\in\mathscr{B}_n} \|\mu(A\cap B|\mathscr{C}) - \mu(A|\mathscr{C})\cdot\mu(B|\mathscr{C})\| \leqslant \|E_{\mathscr{B}_n} g\|, \qquad n \geqslant 1. \tag{4}$$

If $\bigcap_{n=1}^{\infty} \mathscr{B}_n = \mathscr{C}$ then the Doob convergence theorem implies

$$\lim_{n\to\infty} \|E_{\mathscr{B}_n} g\| = 0. \tag{5}$$

Thus (4) and (5) prove the necessity.

Now, let $C \in \bigcap_{n=1}^{\infty} \mathscr{B}_n$ and $\varepsilon > 0$ be arbitrary. Let

$$\lim_{n\to\infty} \sup_{A\in\mathscr{B}_n} \|\mu(A\cap B|\mathscr{C}) - \mu(A|\mathscr{C})\cdot\mu(B|\mathscr{C})\| = 0 \quad \text{for every } B \in \mathscr{B}_1.$$

Hence there exists $n \geqslant 1$ such that

$$\|\mu(A\cap C|\mathscr{C}) - \mu(A|\mathscr{C})\cdot\mu(B|\mathscr{C})\| < \varepsilon \quad \text{for every } A \in \mathscr{B}_n. \tag{6}$$

Taking $A = C$ in (6) we obtain

$$\|\mu(C|\mathscr{C}) - \mu^2(C|\mathscr{C})\| < \varepsilon, \quad \text{i.e.} \quad \mu^2(C|\mathscr{C}) = \mu(C|\mathscr{C}).$$

This means that $C \in \mathscr{C}$.

Thus we have proved the inclusion $\bigcap_{n=1}^{\infty} \mathscr{B}_n \subset \mathscr{C}$. Since the converse inclusion is trivial we have proved the sufficiency.

From the above lemma we obtain readily

COROLLARY. *If $\mathscr{B}_n$, $n \geqslant 1$, are σ-subalgebras of $\mathscr{B}$ such that $\mathscr{B}_n \supset \mathscr{B}_{n+1}$, $n \geqslant 1$, and $\bigcap_{n=1}^{\infty} \mathscr{B}_n = \mathscr{2}$ then for every $B \in \mathscr{B}_1$ we have*

$$\lim_{n\to\infty} \sup_{A \in \mathscr{B}_n} |\mu(A \cap B) - \mu(A)\cdot\mu(B)| = 0.$$

THEOREM 2. *If $\mathscr{A}_n$, $n \geqslant 1$, and $\mathscr{C}$ are σ-subalgebras of $\mathscr{B}$ such that*

(i) $\mathscr{A}_n \supset A_{n+1}$, $n \geqslant 1$,

(ii) *$\mathscr{A}_1$ and $\mathscr{C}$ are independent,*

then

$$\bigcap_{n=1}^{\infty} \mathscr{A}_n = \mathscr{2} \quad \textit{iff} \quad \bigcap_{n=1}^{\infty} (\mathscr{A}_n \vee \mathscr{C}) = \mathscr{C}.$$

Proof. The fact that (i), (ii) and the equality $\bigcap_{n=1}^{\infty} (\mathscr{A}_n \vee \mathscr{C}) = \mathscr{C}$ imply $\bigcap_{n=1}^{\infty} \mathscr{A}_n = \mathscr{2}$ is trivial.

Now we shall prove that (i), (ii) and $\bigcap_{n=1}^{\infty} \mathscr{A}_n = \mathscr{2}$ imply $\bigcap_{n=1}^{\infty} (\mathscr{A}_n \vee \mathscr{C}) = \mathscr{C}$.

Let $\mathscr{B}_n = \mathscr{A}_n \vee \mathscr{C}$, $n \geqslant 1$. Evidently $\mathscr{B}_n \supset \mathscr{B}_{n+1} \supset \mathscr{C}$, $n \geqslant 1$. Let $B \in a(\mathscr{A}_1, \mathscr{C})$ be arbitrary. First we shall show that

$$\lim_{n\to\infty} \sup_{A \in a(\mathscr{A}_n, \mathscr{C})} \|\mu(A \cap B \mid \mathscr{C}) - \mu(A \mid \mathscr{C}) \cdot \mu(B \mid \mathscr{C})\| = 0. \tag{7}$$

Let $n \geqslant 1$ and $A \in (\mathscr{A}_n, \mathscr{C})$ be arbitrary. There exist sets $A_i^{(n)} \in \mathscr{A}_n$ and pairwise disjoint sets $C_i \in \mathscr{C}$, $i = 1, \ldots, k_n$, such that $A = \bigcup_{i=1}^{k_n} A_i^{(n)} \cap C_i$. There exist also sets $\tilde{A}_j \in \mathscr{A}_1$ and pairwise disjoint sets $\tilde{C}_j \in \mathscr{C}$, $j = 1, \ldots, p$, such that $B = \bigcup_{j=1}^{p} \tilde{A}_j \cap \tilde{C}_j$. Since C_i, $\tilde{C}_j \in \mathscr{C}$, $i = 1, \ldots, k_n$, $j = 1, \ldots, p$, are disjoint, $\mathscr{A}_1$ and $\mathscr{C}$ are independent, we have

$$\begin{aligned}
&\|\mu(A \cap B \mid \mathscr{C}) - \mu(A \mid \mathscr{C}) \cdot \mu(B \mid \mathscr{C})\| \\
&= \int_X \Big|\sum_{i=1}^{k_n} \sum_{j=1}^{p} \big(\mu(A_i^{(n)} \cap \tilde{A}_j \mid \mathscr{C}) - \mu(A_i^{(n)} \mid \mathscr{C}) \cdot \mu(\tilde{A}_j \mid \mathscr{C})\big) \cdot 1_{C_i \cap \tilde{C}_j}\Big| \, d\mu \\
&= \int_X \Big|\sum_{i=1}^{k_n} \sum_{j=1}^{p} \big(\mu(A_i^{(n)} \cap \tilde{A}_j) - \mu(A_i^{(n)}) \cdot \mu(\tilde{A}_j)\big) \cdot 1_{C_i \cap \tilde{C}_j}\Big| \, d\mu \\
&\leqslant \sum_{i=1}^{k_n} \sum_{j=1}^{p} |\mu(A_i^{(n)} \cap \tilde{A}_j) - \mu(A_i^{(n)}) \cdot \mu(\tilde{A}_j)| \cdot \mu(C_i \cap \tilde{C}_j) \\
&\leqslant \max_{1 \leqslant j \leqslant p} \sup_{E \in \mathscr{A}_n} |\mu(E \cap \tilde{A}_j) - \mu(E) \cdot \mu(\tilde{A}_j)|.
\end{aligned}$$

Hence

$$(8)\qquad \sup_{A\in a(\mathscr{A}_n,\mathscr{C})} \|\mu(A\cap B\,|\,\mathscr{C})-\mu(A\,|\,\mathscr{C})\cdot\mu(B\,|\,\mathscr{C})\|$$
$$\leqslant \max_{1\leqslant j\leqslant p}\ \sup_{E\in\mathscr{A}_n} |\mu(E\cap \tilde{A}_j)-\mu(E)\cdot\mu(\tilde{A}_j)|.$$

Thus using the assumption $\bigcap_{n=1}^{\infty} \mathscr{A}_n = \mathscr{Q}$ and the last corollary we get (7).

Now we want to show

$$(9)\qquad \lim_{n\to\infty}\ \sup_{A\in\mathscr{A}_n\vee\mathscr{C}} \|\mu(A\cap B\,|\,\mathscr{C})-\mu(A\,|\,\mathscr{C})\cdot\mu(B\,|\,\mathscr{C})\| = 0.$$

Let $\varepsilon > 0$ be arbitrary. The equality (7) implies that there exists $N \geqslant 1$ such that for every $n \geqslant N$ and $E \in a(\mathscr{A}_n, \mathscr{C})$ the inequality

$$(10)\qquad \|\mu(E\cap B\,|\,\mathscr{C})-\mu(E\,|\,\mathscr{C})\cdot\mu(B\,|\,\mathscr{C})\| < \varepsilon/3$$

holds.

Now, let $n > N$ and $A \in \mathscr{A}_n \vee \mathscr{C}$ be arbitrary. There exists $A_\varepsilon \in a(\mathscr{A}_n, \mathscr{C})$ such that

$$(11)\qquad \mu(A \dot{-} A_\varepsilon) < \varepsilon/3.$$

From (10) and (11) we obtain

$$\|\mu(A\cap B\,|\,\mathscr{C})-\mu(A\,|\,\mathscr{C})\cdot\mu(B\,|\,\mathscr{C})\|$$
$$\leqslant \|\mu(A\cap B\,|\,\mathscr{C})-\mu(A_\varepsilon\cap B\,|\,\mathscr{C})\| + \|\mu(A_\varepsilon\cap B\,|\,\mathscr{C})-\mu(A_\varepsilon\,|\,\mathscr{C})\cdot\mu(B\,|\,\mathscr{C})\| +$$
$$+ \big\|\big(\mu(A_\varepsilon\,|\,\mathscr{C})-\mu(A\,|\,\mathscr{C})\big)\cdot\mu(B\,|\,\mathscr{C})\big\|$$
$$\leqslant 2\mu(A \dot{-} A_\varepsilon)+\varepsilon/3 < \varepsilon,$$

i.e. (9) is satisfied.

Using similar arguments as above we may prove that (9) is fulfilled for all $B \in \mathscr{A}_1 \vee \mathscr{C}$.

Hence, in view of Lemma 2, we obtain the equality $\bigcap_{n=1}^{\infty} (\mathscr{A}_n \vee \mathscr{C}) = \mathscr{C}$ which completes the proof.

Now let us recall the definition of a dynamical system considered by Conze in [1].

Let $(Y, \mathscr{F}, \lambda)$ be a Lebesgue space and let S_0 be an automorphism of Y. We denote by $(X, \mathscr{B}, \mu)$ the product measure space $\prod_{i=-\infty}^{+\infty} (Y_i, \mathscr{F}_i, \lambda_i)$ where $Y_i = Y$, $\mathscr{F}_i = \mathscr{F}$, $\lambda_i = \lambda$, $i \in Z$. Let T be the shift transformation on X: $(Tx)(n) = x(n+1)$ and let S be the transformation on X induced by S_0:

$$(Sx)(n) = S_0\, x(n), \qquad n \in Z.$$

It is easy to see that $T \circ S = S \circ T$. We denote by G the group of automorphisms of X generated by T and S.

Conze [1] has proved that $h(G) = h(S_0)$.

Let us suppose $h(G) < \infty$.

THEOREM 3. *If G is a K-group, then there exists a (T, S)-regular generator for (X, μ, G).*

Proof. Since G is a K-group then S_0 is a K-automorphism (and vice versa [3]). We take an arbitrary generator α of (Y, λ, S_0). Thus $\bigwedge_{n=0}^{\infty} S_0^{-n} \alpha_{S_0}^{-} = \nu$. Let us put $\xi = \pi_0^{-1}(\alpha)$ where $\pi_0 \colon X \to Y$ denotes the projection on the null axis. It is easy to see that

$$\bigwedge_{n=0}^{\infty} S^{-n} \xi_S^{-} = \nu .$$

Since μ is a product measure we see that the partitions ξ_S^{-} and $(\xi_S)_T^{-}$ are independent.

Let $\mathscr{A}_n(\mathscr{C})$ denote the σ-algebra generated by $S^{-n}\xi_S^{-}$ $((\xi_S)_T^{-})$. Hence $\bigcap_{n=1}^{\infty} \mathscr{A}_n = \mathscr{2}$, $\mathscr{A}_n$ and $\mathscr{C}$, $n \geqslant 1$, are independent. From this and Theorem 2 we infer that $\bigcap_{n=0}^{\infty} (\mathscr{A}_n \vee \mathscr{C}) = \mathscr{C}$, i.e.

$$\bigwedge_{n=0}^{\infty} (S^{-n} \xi_S^{-} \vee (\xi_S)_T^{-}) = (\xi_S)_T^{-} .$$

Since ξ is a generator for (X, μ, G) it follows from the above equality that ξ is a (T, S)-regular generator.

Remark. If $\nu \neq \pi(S_0)$ then ξ is not (T, S)-regular.

Proof. Let α be a measurable partition of Y defined in the proof of Theorem 3. Since $\bigwedge_{n=0}^{\infty} S_0^{-n} \alpha_{S_0}^{-} = \pi(S_0) \neq \nu$, we have $\bigwedge_{n=0}^{\infty} S^{-n} \xi_S^{-} \neq \nu$. The independence of ξ_S and $(\xi_S)_T^{-}$ and Theorem 2 imply $\bigwedge_{n=0}^{\infty} (S^{-n} \xi^{-} \vee (\xi_S)_T^{-}) \neq (\xi_S)_T^{-}$, i.e. ξ is not (T, S)-regular.

It would be interesting to know whether for every two-dimensional dynamical system regular generators exist. **(P 1289)**

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