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A NOTE ON "TWO CLASSES OF MEASURES" (BY J. K. PACHL)

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In the paper *Two classes of measures* J. K. Pachl raised the following question (P 1147):

Does there exist a set of semicompact measures which are not all regular with respect to a common semicompact lattice? (for definitions see [1]).

The aim of this note is to give a positive answer to this question.

Let I be an uncountable set, and $X \subseteq \{0, 1\}^I$ the set of points all but finitely many coordinates of which are 0. For $i \in I$ let $A(i)$ be the set of points in X whose ith coordinate is 1. Let $\mathscr{B}$ be the σ-algebra on X generated by the family $\{A(i)\colon i \in I\}$. The following properties of $\mathscr{B}$ are easily established: For any set $B \in \mathscr{B}$ there exists a countable set $J \subseteq I$ such that if two points $x, y \in X$ have ith coordinate equal for all $i \in J$, then either $\{x, y\} \subseteq B$ or $\{x, y\} \cap B = \emptyset$. The least set J with this property is called the *set of essential coordinates* for B. If a set $B \in \mathscr{B}$ is nonempty and $i \in I$ is not an essential coordinate for B, then $B \cap A(i) \in \mathscr{B}$ is also nonempty.

For any $x \in X$ let δ_x denote a measure on $\mathscr{B}$ defined by $\delta_x(B) = 1$ if $x \in B$, and $\delta_x(B) = 0$ otherwise. Any such measure is semicompact ([1], p. 335).

PROPOSITION. *The set $\{\delta_x\colon x \in X\}$ of semicompact measures on $\mathscr{B}$ is not regular with respect to a common semicompact lattice.*

Proof. Assume, *a contrario*, that such a semicompact lattice $\mathscr{K} \subseteq \mathscr{B}$ exists. For $x \in X$, δ_x is $\mathscr{K}$-regular, i.e. if $x \in B \in \mathscr{B}$, then there is $K \in \mathscr{K}$ such that $x \in K \subseteq B$. As a consequence we obtain that any nonempty set $B \in \mathscr{B}$ contains a nonempty set $K \in \mathscr{K}$. Take arbitrary $i_0 \in I$; for $A(i_0)$ there is $K_0 \in \mathscr{K}$, $\emptyset \neq K_0 \subseteq A(i_0)$. Take $i_1 \in I$ which is not an essential coordinate for K_0. The set $K_0 \cap A(i_1)$ is nonempty, then we can find $K_1 \in \mathscr{K}$, $\emptyset \neq K_1 \subseteq K_0 \cap A(i_1)$. Inductively we can choose $K_0, K_1, \ldots$ a decreasing sequence of nonempty sets in $\mathscr{K}$ and $i_0, i_1, \ldots$ a countable infinite subset of I such that $K_n \subseteq K_{n-1} \cap A(i_n)$ for $n = 1, 2, \ldots$ This gives a contradiction

$$\emptyset \neq \bigcap \{K_n\colon n = 1, 2, \ldots\} \subseteq \bigcap \{A(i_n)\colon n = 1, 2, \ldots\} = \emptyset$$

(the first inequality is because of semicompactness of $\mathscr{K}$, the last equality is because any $x \in X$ has only finitely many coordinates equal to 1), and ends the proof.

The construction actually shows that there is a "uniformly semicompact" collection of measures on a σ-algebra (namely, the set $\{\delta_x: x \in X\}$ on the σ-algebra of all subsets of X) such that the restriction of the collection to a sub-σ-algebra is not "uniformly semicompact". This contrasts and complements the results of Section 2 in [1].

REFERENCE

[1] J. K. Pachl, *Two classes of measures*, Colloquium Mathematicum 42 (1979), p. 331–340; *Correction*, Colloquium Mathematicum 45 (1982), p. 331–333.

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