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Actions of sets of integers on irrationals

by

DANIEL BEREND (Los Angeles, Cal.)

Introduction. It is well known that if α is any irrational number then the sequence $(n\alpha)_{n=1}^{\infty}$ is dense (and even uniformly distributed) modulo 1. It was further established by Hardy and Littlewood [5] that the same holds for the sequence $(n^r\alpha)_{n=1}^{\infty}$, r being any positive integer. The ultimate result in this direction, for multiplicative semigroups of integers, is as follows. Calling such a semigroup Σ *lacunary* if all the elements of $\{\sigma \in \Sigma: \sigma > 0\}$ form integer powers of a single integer a, and *non-lacunary* otherwise, Furstenberg [2, Th. IV.1] obtained

THEOREM A. *If Σ is non-lacunary and α is an irrational, then $\Sigma\alpha$ is dense modulo* 1.

It is easy to see that lacunary semigroups Σ admit irrationals α for which $\Sigma\alpha$ is not dense modulo 1.

The dynamical aspects of the foregoing result are discussed in [2]. The approach to be taken in this paper is of a more number-theoretical nature. As is noted in [3], the theorem provides a partial answer to the following question: Which subgroups Γ of $\boldsymbol{Q}$ have the property that, given any irrational α and $\varepsilon > 0$, we have

$$|\alpha - m/n| < \varepsilon/n \tag{1}$$

for some $m/n \in \Gamma$? Suppose Γ is a group of an *idempotent type*, i.e. (isomorphic to) a group composed of all those rationals whose denominators are divisible only by primes belonging to a certain set of primes. Employing Theorem A it can be inferred that Γ admits approximations as in (1) iff this set of primes consists of at least two elements.

Now let Γ be any subgroup of $\boldsymbol{Q}$. We may assume that $\Gamma \supseteq \boldsymbol{Z}$. Let $P = \{p_1, p_2, \ldots\}$ be the set of all primes. For some sequence (r_i) in $\{0, 1, 2, \ldots, \infty\}$, Γ is the group of all rationals m/n, for which in the decomposition $n = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$ we have $0 \leqslant e_i \leqslant r_i$ for each i. Denote by Δ the set of all positive integers forming permissible denominators for elements of Γ. Such a set Δ may be called, using the terminology of [4, Ch. 8], a *multiplicative IP-system* (the generators of which are in this case primes.) Γ

possesses the aforementioned property iff Δ has the property that 0 is an accumulation point modulo 1 of the set $\Delta\alpha$ for every irrational α.

The main problem to be studied in this paper is more directly related to Theorem A. Let $(a_i)_{i=1}^{\infty}$ be a sequence of integers. Form the multiplicative IP-system (or "approximate semigroup") Δ consisting of all integers of the form $a_{i_1} a_{i_2} \ldots a_{i_n}$ with $i_1 < i_2 < \ldots < i_n$. Under what conditions on (a_i) is $\Delta\alpha$ dense modulo 1 for every irrational α?

In Chapter I we approach the problem in its full generality employing elementary tools. Roughly speaking, we show that, under some restrictions on the rate of growth of (a_i), the only reason which may prevent Δ from having the property sought for is that (a_i) is "approximately" a sequence of powers of a single real number (see Theorem I.1). This implies, for the problem of approximations of irrationals by rationals lying in a group Γ, that if in the sequence (r_i) corresponding to Γ one of the terms is infinite and the sequence comprised of all others is unbounded above, then Γ admits approximations as in (1) (see Theorem I.4).

In Chapter II the case, where some integer a appears infinitely often in the sequence (a_i), is investigated more carefully. We view (a_i) as a sequence of a-adic numbers and, using analysis on the ring of a-adic numbers, arrive at analogues of results established in Chapter I while viewing (a_i) as a sequence of real numbers (compare Theorem I.3 and Theorem II.1). Roughly speaking, we show that, unless (a_i) is "approximately" a sequence of rational powers of a in the ring of b-adic numbers for some divisor b of a, Δ has the property in question.

Chapter III deals with some sequences (a_i) of special forms. Topological dynamics machinery is employed to show that some sequences, to which the results of the former two chapters are inapplicable, still possess the desired property.

Negative results, i.e., conditions on (a_i) guaranteeing that Δ does not have the property in question, are presented in Section I.2 and in Section II.2.

The problems described above were proposed to me by H. Furstenberg. I wish to express my deepest gratitude to him for many long discussions related to this topic. The paper has been written while I spent a summer at the Mathematical Sciences Research Institute, Berkeley. I thank the Institute for its hospitality.

Chapter I. Elementary techniques

1. Definitions; the main theorems. Let Δ be a set of integers.

Definition I.1. Δ is a *DD set* if $\Delta\alpha$ is dense modulo 1 for every irrational α. (DD – Dense Dilatations.)

It will be convenient, instead of considering the set $\Delta\alpha$ modulo 1, to view Δ and α as a set of endomorphisms and as an element, respectively, of the circle group $\boldsymbol{T} = \boldsymbol{R}/\boldsymbol{Z}$. Thus $\Delta\alpha$ is the "Δ-orbit" of α. Δ is a DD set if the Δ-orbit of every non-torsion element of $\boldsymbol{T}$ is dense in $\boldsymbol{T}$. We shall usually not distinguish between a point of $\boldsymbol{T}$ and real numbers lying above it. When distinction is necessary, we shall write $\{\alpha\}$ for the projection in $\boldsymbol{T}$ of an $\alpha \in \boldsymbol{R}$.

Given two sets of integers Δ_1 and Δ_2, we put:

$$\Delta_1 \Delta_2 = \{ab\colon\ a \in \Delta_1,\ b \in \Delta_2\}.$$

The product $\prod_{n=1}^{N} \Delta_n$ of finitely many sets is similarly defined. If $(\Delta_n)_{n=1}^{\infty}$ is a sequence of sets, with $1 \in \Delta_n$ for each n, we denote

$$\prod_{n=1}^{\infty} \Delta_n = \bigcup_{N=1}^{\infty} \prod_{n=1}^{N} \Delta_n.$$

The 2-element set $\{1, a\}$ will be denoted by $\langle a \rangle$. Thus $\{1, a, \ldots, a^n\}$ may be written as $\langle a \rangle^n$ and the semigroup $\{1, a, a^2, \ldots\}$ generated by a as $\langle a \rangle^{\infty}$.

The main question dealt with in this paper is under what conditions on a sequence $(a_n)_{n=1}^{\infty}$ of integers is $\Delta = \prod_{n=1}^{\infty} \langle a_n \rangle$ a DD set. We usually assume that all the a_n's are positive, but the modifications of our results to the general case are straightforward. We shall also be interested in the property to be defined now.

Definition I.2. Δ is a *WA set* if given any irrational α and $\varepsilon > 0$ there exist $m \in \boldsymbol{Z}$ and $n \in \Delta$ with $|\alpha - m/n| < \varepsilon/n$. (WA – Well Approximating.)

Denote by $BA(\Delta)$ the set of all non-torsion $\alpha \in \boldsymbol{T}$ for which $0 \notin \overline{\Delta\alpha}$. Δ is a WA set iff $BA(\Delta) = \emptyset$.

We obviously have

Lemma I.1. *A DD set is a WA set.*

The converse is false in general. For example, let Δ be a *difference set*, i.e., a set of the form $\{a_n - a_m\colon\ n > m\}$ for some sequence (a_n) [4, p. 176]. It is evident that Δ is a WA set, but not necessarily a DD set.

Definition I.3. A group $\Gamma \subseteq \boldsymbol{Q}$ is a *WA group* if for any irrational α and $\varepsilon > 0$ there exists an $m/n \in \Gamma$ with $|\alpha - m/n| < \varepsilon/n$.

Lemma I.2. *The WA property is an isomorphism-invariant.*

The proof is straightforward. In view of the lemma we may confine our attention to groups Γ with $1 \in \Gamma$.

Before we state the main theorems of this chapter, it will be convenient to introduce some more definitions and notations.

Definition I.4. (1) Let q be a prime. Δ is a PR_q *set* if all its elements are relatively prime to q.

(2) Δ is a *PR set* if it is a PR_q set for some q.

The notion of a PR_q (or a PR) *sequence* is analogously defined. Of course, $\Delta = \prod_{n=1}^{\infty} \langle a_n \rangle$ is a PR_q (resp. PR) *set* iff (a_n) is a PR_q (resp. PR) sequence.

DEFINITION I.5. (a_n) is an *SM sequence* if given any positive integers d and s there exists a constant $C = C(d, s)$ such that

$$a_{(n+1)d+s} \leqslant C a_s a_{d+s} \ldots a_{nd+s} \quad \text{for every } n \in N.$$

(SM – Sub-Multiplicative.)

Remark I.1. In the definition it is implicitly assumed that (a_n) is a non-decreasing sequence. When studying sets of the form $\Delta = \prod_{n=1}^{\infty} \langle a_n \rangle$ this may be always assumed to be the case; for, if a certain integer a appears infinitely often in (a_n), then we obtain the same set Δ by letting each of the powers a, a^2, $a^3, \ldots$ appear just once in (a_n). It is clear that in this case we always get an SM sequence.

Denote by $\vec{a}$ the sequence $(a_n)_{n=1}^{\infty}$ and by $\vec{a}/\vec{a}$ the double sequence $(a_m/a_n)_{m,n=1}^{\infty}$. Let $\operatorname{Lim} \vec{a}/\vec{a}$ be the set of limit points of $\vec{a}/\vec{a}$.

DEFINITION I.6. (1) Let (x_n) be a sequence of positive numbers converging to 0. (x_n) *converges rapidly to* 0, and we denote $x_n \xrightarrow[n\to\infty]{R} 0$, if there exists a constant C such that for every $\varepsilon > 0$ we have $\varepsilon \leqslant x_n < 2\varepsilon$ for at most C indices n. (2) Let (X, ϱ) be a metric space and (x_n) a sequence in X with $x_n \xrightarrow[n\to\infty]{} x$. (x_n) *converges rapidly to* x, and we denote $x_n \xrightarrow[n\to\infty]{R} x$, if $\varrho(x_n, x) \xrightarrow[n\to\infty]{R} 0$. Otherwise we denote $x_n \xrightarrow[n\to\infty]{NR} x$.

DEFINITION I.7. Let G be a group and $x, y \in G$. x and y are *rationally dependent* if there exist integers m and n, not both of which are 0, such that $x^m = y^n$, and *rationally independent* otherwise. Now let G be a metric group, (x_n) a sequence in G and $y \in G$. (x_n) is *rationally dependent of* y if there exist an integer d and a sequence (k_n), with either $d \neq 0$ or $k_n \neq 0$ for each n, such that $x_n^d y^{-k_n} \xrightarrow[n\to\infty]{R} 1$, and *rationally independent* otherwise.

For a ring R, denote by R^* the multiplicative group of units of R.

THEOREM I.1. *Let* $\Delta = \prod_{n=1}^{\infty} \langle a_n \rangle$. *Assume that*:

(1) (a_n) *is a PR sequence.*

(2) (a_n) *is an SM sequence.*

(3) *There exists a* $\lambda \in \operatorname{Lim} \vec{a}/\vec{a} \cap (1, \infty)$.

(4) (a_n) *is rationally independent of* λ *in* $\boldsymbol{R}^*$.

Then Δ *is a DD set.*

$\|x\|$ denotes the distance from x to the nearest integer.

Remark I.2. Condition (4) is equivalent to there not existing a positive integer d such that

$$\|\log_\theta a_n\| \xrightarrow[n\to\infty]{R} 0, \quad \text{where} \quad \theta = \lambda^{1/d}.$$

Let $P = \{p_1, p_2, \ldots\}$ be the set of all primes. Given a group $\Gamma \subseteq \boldsymbol{Q}$ with $1 \in \Gamma$, set

$$r_i = \sup\{n\colon 1/p_i^n \in \Gamma\} \in \{0, 1, \ldots, \infty\}.$$

The sequence $(r_1, r_2, \ldots)$ is called the *type* of Γ and denoted by $t(\Gamma)$ (see [1, Ch.XII] for more details). Note that Γ is a *WA* group iff $\Delta = \prod_{n=1}^{\infty} \langle p_n \rangle^{r_n}$ is a WA set.

Let Γ be a subgroup of $\boldsymbol{Q}$. Set $D = \{i \in N\colon r_i > 0\}$. Suppose $D = \{i_1, i_2, \ldots\}$. Γ is of a *frequently non-zero type* if $i_{n+1}/i_n \xrightarrow[n\to\infty]{} 1$. A special case of Theorem I.1 is

THEOREM I.2. *If* (a_n) *is a PR sequence and* $a_{n+1}/a_n \xrightarrow[n\to\infty]{} 1$, *then* $\Delta = \prod_{n=1}^{\infty} \langle a_n \rangle$ *is a DD set. In particular, a group of a frequently non-zero type is a WA group.*

Of particular importance for us will be the case where a certain integer a appears infinitely often in (a_n). We shall refer to this as the *homogeneous case.* We then rewrite Δ in the form $\Delta = \langle a \rangle^\infty \prod_{n=1}^{\infty} \langle a_n \rangle$. It will be convenient to put $a_0 = a$. Theorem I.1 gives in particular

THEOREM I.3. *Let* $\Delta = \langle a \rangle^\infty \prod_{n=1}^{\infty} \langle a_n \rangle$, *where* $(a_n)_{n=0}^{\infty}$ *is a PR sequence. If* $(a_n)_{n=1}^{\infty}$ *is rationally independent of* a, *then* Δ *is a DD set.*

Remark I.3. The theorem can be interpreted as stating that if Δ is not a DD set then the base a expansions of the a_n's look as follows. There exists a positive integer d such that a large initial block of the expansion of a_n coincides with the initial block of the expansion of one of the numbers 1, $a^{1/d}$, $a^{2/d}, \ldots, a^{1-1/d}$ (where both $10\ldots0$ and $a-1, a-1, \ldots, a-1$ may be considered as the initial blocks of the expansion of 1). In Example II.2 we shall see an analogue concerning the terminating block of the expansion of a_n.

COROLLARY I.1. *Theorem* A.

In fact, if Σ is a non-lacunary semigroup we can find $a, b \in \Sigma$ with $\log_a b$ irrational. The subsemigroup of Σ generated by a and b, which can be written as $\langle a \rangle^\infty \prod_{n=1}^{\infty} \langle b_n \rangle$, is then easily seen to be a DD set.

EXAMPLE I.1. $\Delta = \langle 2 \rangle^\infty \prod_{n \in S} \langle 3 \cdot 2^n + 1 \rangle$ is a DD set for any infinite set S. (Note that $(3 \cdot 2^n + 1)$ is a PR_{17} sequence.)

Let Γ be a subgroup of $\boldsymbol{Q}$ with $t(\Gamma) = (r_1, r_2, \ldots)$. Γ is of an *infinite type* if $r_{i_0} = \infty$ for some i_0, of an *unbounded type* – if the sequence (r_i) is unbounded above, of an *infinite unbounded type* – if $r_{i_0} = \infty$ for some i_0 and

$(r_i)_{i \neq i_0}$ is unbounded above, and of a *doubly infinite type* — if $r_i = \infty$ for at least two indices i. As noted earlier, Theorem A implies that a group of a doubly infinite type is a WA group. By Lemmas I.1 and I.2, a special case of Theorem I.3 is

THEOREM I.4. *A group of an infinite unbounded type is a WA group.*

In Section 2 we introduce a class of multiplicative IP-systems, which are not DD sets due to a too rapid rate of growth of the sequence of generators. The basic tool employed in the proof of Theorem I.1 is presented in Section 3. Section 4 is devoted to the proof of Theorem I.1. In Section 5 we deal with two examples, $\Delta_1 = \prod_{n=1}^{\infty} \langle n^n \rangle$ and $\Delta_2 = \prod_{n=1}^{\infty} \langle n! \rangle$, and show that these form DD sets even though Theorem I.1 cannot be applied (at least not directly.) The techniques employed there may well be useful in other cases not falling within the framework of Theorem I.1.

2. A counter-example: lacunarity. In this section we show that if condition (2) in Theorem I.1 is "severely violated", then the conclusion of that theorem is false.

The Hausdorff dimension of a set B will be denoted by $\dim B$.

THEOREM I.5. *Let* $\Delta = \prod_{n=1}^{\infty} \langle a_n \rangle^{r_n}$. *Suppose that*

$$a_{n+1} \geqslant \delta a_1^{r_1} a_2^{r_2} \dots a_n^{r_n}, \quad n = 1, 2, \dots$$

for a certain $\delta > 0$. *Then* $\dim BA(\Delta) = 1$, *and in particular* Δ *is not a DD set.*

LEMMA I.3. *Let* $\Delta \subseteq N$ *and* $d_1, d_2, \dots, d_l \in N$. *Then* $BA(\Delta)$ *is* (1) *non-empty,* (2) *uncountable,* (3) *of Hausdorff dimension* 1, *iff* $BA\big(\bigcup_{i=1}^{l} d_i \Delta\big)$ *is such.*

The lemma follows easily from the inclusions

$$BA(\Delta)/\prod_{i=1}^{l} d_i \subseteq BA\Big(\bigcup_{i=1}^{l} d_i \Delta\Big) \subseteq BA(\Delta).$$

A sequence (x_n) in $\boldsymbol{R}_+$ is *lacunary* if

$$x_{n+1} \geqslant \lambda x_n, \quad n = 1, 2, \dots$$

for some $\lambda > 1$. (For multiplicative semigroups this definition coincides with the one given in the Introduction.)

Proof of Theorem I.5. Select k such that $\delta a_1^{r_1} a_2^{r_2} \dots a_k^{r_k} > 1$, and set $\Delta_1 = \prod_{n=k+1}^{\infty} \langle a_n \rangle^{r_n}$. It is easy to see that Δ_1 is lacunary, whence by [7] or [8] we have $\dim BA(\Delta_1) = 1$. Since Δ is a union of finitely many dilatations of Δ_1, Lemma I.3 completes the proof.

Remark I.4. It follows from the last theorem that neither of the assumptions of Theorem I.4 is redundant. In fact, if $t(\Gamma) = (0, \dots, 0, \infty, 0, \dots)$ then Γ is a non-WA group of an infinite type, whereas if $t(\Gamma) = (0, \dots, 0, 1, 0, \dots, 0, 2, 0, \dots, 0, 3, 0, \dots)$ with the lengths of the 0's blocks increasing sufficiently fast then Γ is a non-WA group of an unbounded type.

3. The basic tool. The set of accumulation points of a set $E \subseteq \boldsymbol{T}$ will be denoted by E'. The set-theoretical difference of two sets E_1 and E_2 will be denoted by $E_1 \setminus E_2$ and their algebraic difference $\{x_1 - x_2 \colon x_1 \in E_1, x_2 \in E_2\}$ by $E_1 - E_2$.

DEFINITION I.8. Δ *is a* DD_0 *set if* $0 \in E' \Rightarrow \overline{\Delta E} = \boldsymbol{T}$.

PROPOSITION I.1. *Let* $(\Delta_n)_{n=1}^{\infty}$ *be a sequence of* DD_0 *sets with* $1 \in \Delta_n$ *for each* n. *Assume that* $\bigcup_{n=1}^{\infty} \Delta_n$ *is a PR set. Then* $\prod_{n=1}^{\infty} \Delta_n$ *is a DD set.*

Proof. Let α be a non-torsion element of $\boldsymbol{T}$. Define an ascending sequence (E_n) of subsets of $\boldsymbol{T}$ by:

$$E_0 = \{\alpha\}, \quad E_n = \overline{\Delta_n E_{n-1}}, \quad n \geqslant 1.$$

We have to show that $\overline{\bigcup_{n=1}^{\infty} E_n} = \boldsymbol{T}$.

Assume first that some E_k contains a torsion element, say $r/s \in E_k$. Then clearly $r/s \in E_k'$, and so $0 \in sE_k'$. Since Δ_{k+1} is a DD_0 set we obtain $sE_{k+1} = \Delta_{k+1}(sE_k) = \boldsymbol{T}$. It follows that

$$E_{k+1} \cup (E_{k+1} + 1/s) \cup \dots \cup (E_{k+1} + (s-1)/s) = \boldsymbol{T}.$$

Since the sets in the union on the left-hand side are all closed, one of them has a non-empty interior. Hence E_{k+1} itself has a non-empty interior. This implies that $lE_{k+1} = \boldsymbol{T}$ for all sufficiently large l, and so we certainly have $E_{k+2} = \overline{\Delta_{k+2} E_{k+1}} = \boldsymbol{T}$. We may assume therefore that $\bigcup_{n=1}^{\infty} E_n$ contains no rational points.

Select a prime q for which $\bigcup_{n=1}^{\infty} \Delta_n$ is a PR_q set. Denote

$$T[s] = \{0, 1/s, 2/s, \dots, (s-1)/s\} \quad \text{for} \quad s \in \boldsymbol{N}.$$

It suffices to show that if l is any positive integer and $0 < k \leqslant q^l$, then there exist $x \in \boldsymbol{T}$ and integers $(m_i)_{i=1}^{k-1}$ with $0 < m_1 < m_2 < \dots < m_{k-1} < q^l$ such that

$$x,\ x + m_1/q^l, \dots, x + m_{k-1}/q^l \in E_{2k}. \tag{1}$$

In fact, this is trivial for $k = 1$. Let us show that if $k < q^l$ and E_{2k} contains a k-point configuration as in (1), then E_{2k+2} contains a $(k+1)$-point configuration of that form. Since Δ_{2k+1} is a PR_q set, its elements are invertible modulo q^l. Consequently any element of Δ_{2k+1} maps the configuration appearing in (1) into another configuration of the same general form, i.e., into a translate of a k-element subset of $\boldsymbol{T}[q^l]$. There exist therefore integers

$$(m_i^{(1)})_{i=1}^{k-1}, \quad 0 < m_1^{(1)} < \ldots < m_{k-1}^{(1)} < q^l,$$

for which the set

$$F = \{x \in \boldsymbol{T}\colon\ x,\ x+m_1^{(1)}/q^l, \ldots, x+m_{k-1}^{(1)}/q^l \in E_{2k+1}\} \subseteq E_{2k+1}$$

is infinite. Since Δ_{2k+2} is a DD_0 set we have $\overline{\Delta_{2k+2}(F-F)} = \boldsymbol{T}$, and in particular $1/q^l \in \overline{\Delta_{2k+2}(F-F)}$. Put:

$$\Delta_{2k+2}^{(j)} = \{n \in \Delta_{2k+2}\colon\ n \equiv j \pmod{q^l}\}, \quad 0 < j < q^l,\ (j, q) = 1.$$

Pick j such that $1/q^l \in \overline{\Delta_{2k+2}^{(j)}(F-F)}$. Let $(m_i^{(2)})_{i=1}^{k-1}$ be the residues modulo q^l of $(jm_i^{(1)})_{i=1}^{k-1}$, ordered so that $0 < m_1^{(2)} < \ldots < m_{k-1}^{(2)} < q^l$. If $y \in \overline{\Delta_{2k+2}^{(j)} F}$, then

$$y,\ y+m_1^{(2)}/q^l, \ldots, y+m_{k-1}^{(2)}/q^l \in \Delta_{2k+2}^{(j)} E_{2k+1}. \tag{2}$$

Choose x with $x,\ x+1/q^l \in \overline{\Delta_{2k+2}^{(j)} F}$. Then (2) holds both for $y = x$ and for $y = x+1/q^l$. The set $\{x,\ x+1/q^l, \ldots, x+m_{k-1}^{(2)}/q^l,\ x+(m_{k-1}^{(2)}+1)/q^l\}$ contains at least $k+1$ elements. There exist therefore $(m_i^{(3)})_{i=1}^{k}$, $0 < m_1^{(3)} < \ldots < m_k^{(3)} < q^l$, such that

$$x,\ x+m_1^{(3)}/q^l, \ldots, m_k^{(3)}/q^l \in E_{2k+2}.$$

This completes the proof.

4. Proof of Theorem I.1. The main part of the proof consists of showing that the last three conditions of Theorem I.1 imply that Δ is a DD_0 set. After establishing this we shall explain why (a_n) can be splitted into infinitely many sequences, each satisfying again the same conditions, whereby Proposition I.1 will conclude the proof. Several definitions are needed first.

DEFINITION I.8. Δ is a set *of bounded ratios* if for every $d \in \Delta$ there exists a $d' \in \Delta$ with $d < d' \leqslant Cd$, where C is a certain constant. Such a C is a *ratio bound* for Δ.

DEFINITION I.9. Δ is an AG_1 *set* if given any $l \in N$ there exist an arbitrarily small $\varepsilon > 0$ and $d_1, d_2, \ldots, d_{l+1} \in \Delta$ such that

$$1+\varepsilon \leqslant d_{j+1}/d_j \leqslant 1+2\varepsilon, \quad 1 \leqslant j \leqslant l \tag{3}$$

(AG_1 – contains Almost Geometric progressions of ratio close to 1.)

Note that the term 2ε on the right-hand side of (3) can be replaced by $C\varepsilon$, where C is any constant greater than 1.

LEMMA I.4. *The product of a set of bounded ratios by an AG_1 set forms a DD_0 set.*

Proof. Let Δ_1 be a set of bounded ratios with a ratio bound C, Δ_2 an AG_1 set and $\Delta = \Delta_1 \Delta_2$. Let E be a subset of $\boldsymbol{T}$ with $0 \in E'$. Replacing E by $-E$ if necessary, we may assume that E contains arbitrarily small positive numbers. Let l be any positive integer. Choose $\varepsilon < 1/2l$ and $d_1, d_2, \ldots, d_{l+1} \in \Delta_2$ for which (3) holds. Take $b \in \Delta_1$ and $\alpha \in E$ with $0 < \alpha < C/\varepsilon l b d_1$. Increasing b we may assume that $1/\varepsilon l \leqslant \alpha b d_1 \leqslant C/\varepsilon l$. Then for every $1 \leqslant j \leqslant l$ we have on the one hand

$$\alpha b d_{j+1} - \alpha b d_j = \alpha b d_j (d_{j+1}/d_j - 1) \geqslant \alpha b d_1 \varepsilon \geqslant 1/l$$

and on the other hand

$$\alpha b d_{j+1} - \alpha b d_j \leqslant \alpha b d_1 (1+2\varepsilon)^{j-1} \cdot 2\varepsilon \leqslant 6C/l.$$

The set $\{\alpha b d_j\colon\ 1 \leqslant j \leqslant l+1\} \subseteq \Delta E$ forms therefore a $6C/l$-net in $\boldsymbol{T}$. Since l is arbitrary, this means that $\overline{\Delta E} = \boldsymbol{T}$, which proves the lemma.

DEFINITION I.10. Let $\lambda > 1$. Δ is an AG_λ *set* if given any $l \in N$ and $\varepsilon > 0$ there exist $d_1, d_2, \ldots, d_{l+1} \in \Delta$ such that

$$\lambda - \varepsilon \leqslant d_{j+1}/d_j \leqslant \lambda + \varepsilon, \quad 1 \leqslant j \leqslant l.$$

Notice that the last definition would make no sense for $\lambda = 1$; the definition of an AG_1 set is not analogous to that of an AG_λ set.

LEMMA I.5. *Let Δ_1 be a set of bounded ratios, Δ_2 an AG_λ set and $\Delta = \Delta_1 \Delta_2$. Given a set $E \subseteq \boldsymbol{T}$ with $0 \in E'$, there exists a non-zero α such that $\alpha\lambda^j \in \overline{\Delta E}$ for every $j \in \boldsymbol{Z}$.*

Proof. We may assume E to contain arbitrarily small positive numbers. Let C be a ratio bound for Δ_1 and l and m any positive integers. Pick $d_{-l}, \ldots, d_0, \ldots, d_l \in \Delta_2$ with $\lambda - 1/m \leqslant d_{j+1}/d_j \leqslant \lambda + 1/m$ for $-l \leqslant j < l$. Take $b \in \Delta_1$ and $\alpha_{lm} \in E$ such that $0 < \alpha_{lm} \leqslant C/bd_0$. Replacing b by a greater number in Δ_1, we may assume that $1 \leqslant \alpha_{lm} b d_0 \leqslant C$. Let α_l be a limit point of the sequence $(\alpha_{lm})_{m=1}^{\infty}$. Obviously, $1 \leqslant \alpha_l \leqslant C$ and $\alpha_l \lambda^j \in \overline{\Delta E}$ for $-l \leqslant j \leqslant l$. Taking now any limit point of $(\alpha_l)_{l=1}^{\infty}$, we obtain an α satisfying the required conditions. This proves the lemma.

Let $(\varrho_n)_{n=1}^{\infty}$ be a sequence in $\boldsymbol{T}$. Denote by $\mathrm{Lim}\,\vec{\varrho}$ the set of all limit points of the sequence. Consider the set $B = \sum_{n=1}^{\infty} \{0, \varrho_n\}$ consisting of all sums of the form $\sum_{j=1}^{k} \varrho_{i_j}$ with $i_1 < i_2 < \ldots < i_k$.

LEMMA I.6. *If $\bar{B}$ is properly contained in $\boldsymbol{T}$, then* $\mathrm{Lim}\,\vec{\varrho} \subseteq \boldsymbol{T}[k]$ *for some $k \in N$.*

Proof. Let H be the closed subgroup of $\boldsymbol{T}$ generated by $\mathrm{Lim}\,\vec{\varrho}$. It is clear that $H \subseteq \bar{B}$, whence H is a proper subgroup of $\boldsymbol{T}$, say $H = \boldsymbol{T}[k]$. Thus $\mathrm{Lim}\,\vec{\varrho} \subseteq \boldsymbol{T}[k]$, which proves the lemma.

DEFINITION I.11. A set $S \subseteq N$ is *syndetic* if there exists an $s \in N$ such that at least one out of any s consecutive positive integers belongs to S.

LEMMA I.7. *If (a_n) is an SM sequence and S is syndetic, then $(a_n)_{n\in S}$ is an SM sequence as well.*

LEMMA I.8. *If (a_n) is an SM sequence, then $\Delta = \prod_{n=1}^{\infty} \langle a_n \rangle$ is a set of bounded ratios.*

LEMMA I.9. *If $\lambda \in \operatorname{Lim} \vec{a}/\vec{a} \cap (1, \infty)$, then $\Delta = \prod_{n=1}^{\infty} \langle a_n \rangle$ is an AG_λ set.*

The proofs of all three lemmas are routine.

LEMMA I.10. *Suppose (a_n) satisfies conditions* (2) *and* (3) *of Theorem* I.1 *and also:*

$$\operatorname{Lim}(\{\log_\lambda a_n\})_{n=1}^{\infty} \notin T[k], \quad k \in N. \tag{4'}$$

Then $\Delta = \prod_{n=1}^{\infty} \langle a_n \rangle$ is a DD_0 set.

Proof. Choose sequences (l_i) and (m_i) such that $a_{l_i}/a_{m_i} \xrightarrow[i\to\infty]{} \lambda$. Set $S = \{l_1, m_1, l_2, m_2, \ldots\}$. Denote by $\vec{b}$ and by $\vec{b}'$ the sequences $(a_n)_{n\in S}$ and $(a_n)_{n\notin S}$, respectively. Passing to subsequences of (l_i) and (m_i) we may assume that S^C is syndetic, whence by Lemma I.7 $\vec{b}'$ is an SM sequence. It may also be assumed that every limit point modulo 1 of the sequence $(\log_\lambda a_n)_{n=1}^{\infty}$ is also a limit point modulo 1 of $(\log_\lambda b'_n)_{n=1}^{\infty}$. Similarly, we can split $\vec{b}'$ into two sequences $\vec{c}$ and $\vec{d}$ such that

$$\operatorname{Lim}(\{\log_\lambda c_n\})_{n=1}^{\infty} = \operatorname{Lim}(\{\log_\lambda b'_n\})_{n=1}^{\infty}$$

and $\vec{d}$ is an SM sequence. Put

$$\Delta_1 = \prod_{n=1}^{\infty} \langle b_n \rangle, \quad \Delta_2 = \prod_{n=1}^{\infty} \langle c_n \rangle \quad \text{and} \quad \Delta_3 = \prod_{n=1}^{\infty} \langle d_n \rangle.$$

Let E be a subset of T with $0 \in E'$. By Lemmas I.5, I.8 and I.9 the set $\overline{\Delta_1 \Delta_3 E}$ contains a set of the form $E_1 = \{\alpha\lambda^j \colon j \in Z\}$ for some, say, $\alpha > 0$. We want to show that $\overline{\Delta_2 E_1} = R_+$. In fact, take any $x > 0$. Given $\varepsilon > 0$ we can find, in view of Lemma I.6, elements $c_{n_1}, c_{n_2}, \ldots, c_{n_k}$ in $\vec{c}$ such that

$$\Big|\log_\lambda (x/\alpha) - \sum_{j=1}^{k} \log_\lambda c_{n_j} - l\Big| < \varepsilon$$

for an appropriate $l \in Z$. It follows that $x = \alpha\lambda^{l+\varepsilon_1} c_{n_1} \ldots c_{n_k}$ where $|\varepsilon_1| < \varepsilon$. Since ε is arbitrary, this means that $x \in \overline{\Delta_2 E_1}$, and so $\overline{\Delta_2 E_1} = R_+$, or, modulo 1, $\overline{\Delta_2 E_1} = T$. This proves the lemma.

LEMMA I.11. *If (a_n) satisfies conditions* (2)–(4) *of Theorem* I.1, *then $\Delta = \prod_{n=1}^{\infty} \langle a_n \rangle$ is a DD_0 set.*

Proof. According to Lemma I.10 it may be assumed that

$$\|\log_\theta a_n\| \xrightarrow[n\to\infty]{} 0, \quad \text{where} \quad \theta = \lambda^{1/e} \text{ for some } e \in N.$$

Similarly to the proof of Lemma I.10, the sequence $\vec{a}$ can be splitted into three sequences $\vec{b}$, $\vec{c}$ and $\vec{d}$ such that $\lambda \in \operatorname{Lim} \vec{b}/\vec{b}$, $\|\log_\theta c_n\| \xrightarrow[n\to\infty]{NR} 0$ and $\vec{d}$ is an SM sequence. Set

$$\Delta_1 = \prod_{n=1}^{\infty} \langle b_n \rangle, \quad \Delta_2 = \prod_{n=1}^{\infty} \langle c_n \rangle \quad \text{and} \quad \Delta_3 = \prod_{n=1}^{\infty} \langle d_n \rangle.$$

Let $I_j = [(j-1)/2e, j/2e]$ for $1 \leqslant j \leqslant 2e$. Split $\vec{c}$ into $2e$ sequences $\vec{c}^{(1)}, \vec{c}^{(2)}, \ldots, \vec{c}^{(2e)}$, where $\vec{c}^{(j)}$ consists of those terms c_n with $\{\log_\lambda c_n\} \in I_j$. At least one of these sequences, say $\vec{c}^{(k)}$, still has the property that $\|\log_\theta c_n^{(k)}\| \xrightarrow[n\to\infty]{NR} 0$. Replacing $\vec{c}$ by $\vec{c}^{(k)}$ we may assume that $\{\log_\lambda c_n\} \in I_k$ for each n. The sequence $(\{\log_\lambda c_n\})$ is convergent, all its elements lying to the same side of its limit. Reordering $\vec{c}$ we may assume therefore that $(\{\log_\lambda c_n\})_{n=1}^{\infty}$ is a monotone sequence. Define a sequence $\vec{c}'$ by:

$$c'_n = c_{(e-1)n+1}\, c_{(e-1)n+2} \ldots c_{en}, \quad n = 1, 2, \ldots$$

We readily see that $\{\log_\lambda c'_n\} \xrightarrow[n\to\infty]{NR} 0$, where the sequence on the left-hand side is monotonous. Since $\prod_{n=1}^{\infty} \langle c'_n \rangle \subseteq \prod_{n=1}^{\infty} \langle c_n \rangle$ we may assume, substituting $\vec{c}'$ for $\vec{c}$, that, say, $(\{\log_\lambda c_n\})_{n=1}^{\infty}$ is monotone decreasing and converges non-rapidly to 0.

We claim that $\Delta_1 \Delta_2$ is an AG_1 set. In fact, let l be a positive integer and $\varepsilon > 0$. Dropping finitely many terms from $\vec{c}$ we may assume that

$$\lambda^{n_i + 4\delta/3} \leqslant c_i \leqslant \lambda^{n_i + 5\delta/3}, \quad i = 1, 2, \ldots, l \tag{4}$$

for suitably chosen positive integers $(n_i)_{i=1}^{l}$, where $0 < \delta < \varepsilon$. Set $N = \sum_{i=1}^{l} n_i$. Pick $b_1^{(1)}, b_1^{(2)}, b_2^{(1)}, b_2^{(2)}, \ldots, b_N^{(1)}, b_N^{(2)}$ in $\vec{b}$ such that

$$\lambda^{1-\delta/3N} \leqslant b_j^{(2)}/b_j^{(1)} \leqslant \lambda^{1+\delta/3N}, \quad j = 1, 2, \ldots, N. \tag{5}$$

Let $m_i = \sum_{k=1}^{i} n_k$ for $0 \leqslant i \leqslant l$. Consider the following $l+1$ elements of $\Delta_1 \Delta_2$:

$$e_i = b_1^{(1)} b_2^{(1)} \ldots b_{m_i}^{(1)} b_{m_i+1}^{(2)} \ldots b_N^{(2)} c_1 c_2 \ldots c_i, \quad i = 0, 1, \ldots, l.$$

We have

$$e_{i+1}/e_i = c_{i+1} / \prod_{j=m_i+1}^{m_{i+1}} (b_j^{(2)}/b_j^{(1)}), \quad i = 0, 1, \ldots, l-1$$

so that by (4) and (5)

$$\lambda^{\delta} \leqslant e_{i+1}/e_i \leqslant \lambda^{2\delta}, \quad i = 0, 1, \ldots, l-1.$$

$\Delta_1 \Delta_2$ is therefore an AG_1 set, which implies by Lemmas I.4 and I.8 that $\Delta_1 \Delta_2 \Delta_3$ is a DD_0 set. This proves the lemma.

LEMMA I.12. *Suppose that (a_n) satisfies either one of the following conditions:*

(1) $\lambda \in \operatorname{Lim} \vec{a}/\vec{a}$.

(2) *(a_n) is rationally independent of λ in $\boldsymbol{R}^*$.*

(3) *(a_n) is an SM sequence.*

Then (a_n) can be splitted into infinitely many subsequences, each satisfying the same condition.

The proof is straightforward.

It is now easy to conclude the proof of Theorem I.1. As in the proofs of Lemmas I.10 and I.11, we can split $\vec{a}$ into three sequences $\vec{b}$, $\vec{c}$ and $\vec{d}$ such that:

(1) $\lambda \in \operatorname{Lim} \vec{b}/\vec{b}$.

(2) (c_n) is rationally independent of λ in $\boldsymbol{R}^*$.

(3) (d_n) is an SM sequence.

By Lemma I.12 we can split $\vec{b}$, $\vec{c}$ and $\vec{d}$ into infinitely many sequences $(\vec{b}^{(j)})$, $(\vec{c}^{(j)})$ and $(\vec{d}^{(j)})$, respectively, each having the same property as the original sequence. Define:

$$\Delta_j = \prod_{n=1}^{\infty} \langle b_n^{(j)} \rangle \prod_{n=1}^{\infty} \langle c_n^{(j)} \rangle \prod_{n=1}^{\infty} \langle d_n^{(j)} \rangle, \quad j = 1, 2, 3, \ldots.$$

By Lemmas I.10 and I.11 each Δ_j is a DD_0 set. Condition (1) of the theorem enables us to employ Proposition I.1 to get that $\Delta = \prod_{j=1}^{\infty} \Delta_j$ is a DD set. This completes the proof.

5. Examples. We shall give in this section two examples of sequences (a_n), for which the conditions of Theorem I.1 are not satisfied, and show that $\Delta = \prod_{n=1}^{\infty} \langle a_n \rangle$ is still a DD set.

EXAMPLE I.2. $\prod_{n=1}^{\infty} \langle n^n \rangle$ is a DD set. Since (n^n) is not a PR sequence, we shall consider $\prod_{n=1}^{\infty} \langle (2n+1)^{2n+1} \rangle$, which is a PR_2 set, and show that even the latter is a DD set. Let $a_n = (2n+1)^{2n+1}$. It is easy to verify that (a_n) is an SM sequence. Define (b_n) by $b_n = a_{n+1}/a_n$. We have $b_n \nearrow \infty$, so that condition (3) of Theorem I.1 is violated. We proceed therefore as follows. Since $b_{n+1}/b_n \to 1$, if (l_k) is any sequence we can find a sequence (m_k) such that $a_{m_k+1}/a_{l_k} a_{m_k} \xrightarrow[k\to\infty]{} 2$. Now replace for each k the two terms a_{l_k} and a_{m_k} in $\vec{a}$ by the single term $a_{l_k} a_{m_k}$. Denote the resulting sequence by $\vec{c}$. Obviously, $2 \in \operatorname{Lim} \vec{c}/\vec{c}$ and, if (l_k) is chosen so as to be increasing fast enough, $\vec{c}$ forms an SM sequence. Since $\prod_{n=1}^{\infty} \langle c_n \rangle \subseteq \prod_{n=1}^{\infty} \langle a_n \rangle$, it suffices to show that $\prod_{n=1}^{\infty} \langle c_n \rangle$ is an SM sequence. A similar modification of $\vec{c}$ leads us to an SM sequence $\vec{d}$ with $\prod_{n=1}^{\infty} \langle d_n \rangle \subseteq \prod_{n=1}^{\infty} \langle c_n \rangle$ and $2, 3 \in \operatorname{Lim} \vec{d}/\vec{d}$. Since $\log_2 3$ is irrational, Theorem I.1 implies that $\prod_{n=1}^{\infty} \langle d_n \rangle$ is a DD set, whence so is $\prod_{n=1}^{\infty} \langle n^n \rangle$ as well.

EXAMPLE I.3. $\prod_{n=1}^{\infty} \langle n! \rangle$ is a DD set. In this case for any prime p only finitely many a_n's are relatively prime to p, so that Theorem I.1 is inapplicable. Set

$$\Delta = \prod_{n=1}^{\infty} \langle n! \rangle, \quad \Delta_1 = \prod_{n=1}^{\infty} \langle (100n)! \rangle \quad \text{and} \quad \Delta_2 = \prod_{n=1}^{\infty} \langle (100n+1)! \rangle.$$

Similarly to the preceding example we show, using this time Lemma I.10 instead of Theorem I.1, that Δ_1 is a DD_0 set. Let α be an irrational. Consider the sequence (x_k) given by

$$x_k = \prod_{n=1}^{k} \big((100n+2)!\,(200n+3)!\,(100n+6)!\,(300n+17)!\big)\alpha, \quad k \in N.$$

Assume first that (x_k) has some rational limit point. Since Δ_1 is a DD_0 set, we get that $\overline{\Delta_1 \{x_k \colon k \in N\}}$ has a non-empty interior, which implies $\overline{\Delta\alpha} \supseteq \overline{\Delta_2 \Delta_1 \{x_k \colon k \in N\}} = T$. We may consequently assume that (x_k) has no rational limit point, so that for a certain sequence (k_l) we have $x_{k_l} \xrightarrow[l\to\infty]{} x$, where x is an irrational. Given any non-negative integers r and s we have

$$\prod_{n=1}^{r} \big((100n+1)!\,(200n+4)!\big) \prod_{n=1}^{s} \big((100n+5)!\,(300n+18)!\big) \times \prod_{n=r+1}^{k_l} \big((100n+2)!\,(200n+3)!\big) \prod_{n=s+1}^{k_l} \big((100n+6)!\,(300n+17)!\big)\alpha = 2^r 3^s x_{k_l} \xrightarrow[l\to\infty]{} 2^r 3^s x.$$

By Theorem A we obtain $\overline{\Delta\alpha} = T$. Thus Δ is a DD set.

Chapter II. a-adic techniques

1. Definitions; the main theorem. In this chapter we shall deal only with the homogeneous case, that is with sets Δ of the form

(1) $$\Delta = \langle a\rangle^{\infty} \prod_{n=1}^{\infty} \langle a_n\rangle.$$

This case is often more convenient. Denote by T_a the endomorphism of T given by:

$$T_a(x) = ax, \quad x \in T.$$

If $E = \overline{\Delta\alpha}$ then E is T_a-*invariant* (or, more briefly, a-invariant), that is $x \in E \Rightarrow T_a(x) \in E$. This implies, for example,

LEMMA II.1. *If Δ is a DD set, then so is $\langle a\rangle^{\infty} \prod_{n=s}^{\infty} \langle a_n\rangle$ for any $s \in N$.*

Proof. Let α be an irrational. Set $E = \overline{\langle a\rangle^{\infty} \prod_{n=s}^{\infty} \langle a_n\rangle \alpha}$. Since Δ is a DD set $\prod_{n=1}^{s-1} \langle a_n\rangle E = T$, which implies that E has a non-empty interior. It follows that $a^r E = T$ for sufficiently large r, and so $E = T$. This proves the lemma.

The lemma shows in particular that we may assume a not to be a non-trivial power of some other integer. For, if $a = b^r$ and $\langle b\rangle^{\infty} \prod_{n=1}^{\infty} \langle a_n\rangle$ is a DD set, then so is $\langle a\rangle^{\infty} \prod_{n=1}^{\infty} \langle a_n\rangle$ as well.

Our results in the preceding chapter were obtained with the a_n's being viewed as real numbers. In this chapter they will be considered as elements of the ring Q_a of a-adic numbers. Decompose a into a product of primes

$$a = p_1^{e_1} p_2^{e_2} \ldots p_h^{e_h}.$$

Denote $H = \{1, 2, \ldots, h\}$. A typical non-empty subset of H will be denoted by J, where for simplicity we shall usually deal with $G = \{1, 2, \ldots, g\}$. If $b = \prod_{j\in J} p_j^{e_j}$ then we denote the ring Q_b of b-adic numbers and its subring Z_b of b-adic integers by Q_J and by Z_J respectively. Let us recall a few facts concerning the rings Q_J and Z_J (see, for example, [6]). We may assume $J = G$. The topological rings Q_b and Z_b can be viewed as $\prod_{j=1}^{g} Q_{p_j}$ and $\prod_{j=1}^{g} Z_{p_j}$, respectively. Considered as a subset of Z_b, Z_{p_j} is the set of all numbers divisible by p_i^n for every $n \in N$ and $i \in \{1, \ldots, j-1, j+1, \ldots, g\}$. The b-adic norm on Q_b is defined as follows. Given $x \in Q_b$ take the least integer k for which $b^k x \in Z_b$ and put $|x|_b = b^k$ (and $|0|_b = 0$). Writing $x = (x_1, x_2, \ldots, x_g) \in \prod_{j=1}^{g} Q_{p_j}$, and letting k_j be the least integer for which $p_j^{k_j e_j} x_j \in Z_{p_j}$, we get $k = \max_{1\leqslant j\leqslant g} k_j$. The b-adic metric is given by

$$\varrho_b(x, y) = |y-x|_b \quad \text{for} \quad x, y \in Q_b.$$

If $c = \prod_{j=1}^{g} p_j^{f_j}$ for some positive integers f_j, then Q_b and Q_c are isomorphic as topological rings, but they differ with respect to questions involving rapid convergence. It may also happen that a sequence of integers converges non-rapidly to some integer in each Q_{p_j} and converges to it rapidly in Q_b.

EXAMPLE II.1. Define (a_n) by

$$a_n = \begin{cases} 2^n 5^{2^{2k}}, & 2^{2k-1} \leqslant n < 2^{2k}, \\ 2^{2^{2k+1}} 5^n, & 2^{2k} \leqslant n < 2^{2k+1}. \end{cases}$$

It is easy to check that $a_n \xrightarrow[n\to\infty]{NR} 0$ in Q_2, in Q_5, in Q_{20} and in Q_{50}, but $a_n \xrightarrow[n\to\infty]{R} 0$ in Q_{10}.

The following lemma will be useful later.

LEMMA II.2. *Suppose $x_n \xrightarrow[n\to\infty]{} x$ in Q_b, and let $k \in N$. Then*

$$x_n \xrightarrow[n\to\infty]{R} x \quad \textit{iff} \quad x_n^k \xrightarrow[n\to\infty]{R} x^k.$$

The proof is routine.

Our main result in this chapter is

THEOREM II.1. *Let $\Delta = \langle a\rangle^{\infty} \prod_{n=1}^{\infty} \langle a_n\rangle$. Assume that:*

(1) $(a_n)_{n=0}^{\infty}$ is a PR sequence (where $a_0 = a$).

(2) (a_n) is rationally independent of a in Q_J^* for every non-empty $J \subseteq H$.

Then Δ is a DD set.

COROLLARY II.1. *Theorem* A.

COROLLARY II.2. *Theorem* I.4.

It will be helpful to provide equivalent formulations of the second condition in the theorem. Fix a non-empty $J \subseteq H$, say $J = G$. For any $1 \leqslant j \leqslant g$, the set $D_j(a)$ of those elements of $Q_{p_j}^*$ which are rationally dependent of a (or, for that matter, of any fixed element of $Q_{p_j}^*$) forms a subgroup of $Q_{p_j}^*$. Denote by l_j the least positive integer l for which $D_j(a) \cap p_j^l Z_{p_j}^* \neq \emptyset$. Since $a \in D_j(a) \cap p_j^{e_j} Z_{p_j}^*$ we have $l_j | e_j$. For $\xi \in Q_p^*$ let $n_p(\xi)$ be the maximal k for which $\xi \in p^k Z_p$. Evidently, if $x \in D_j(a)$ and $n_{p_j}(x) = n_{p_j}(y)$, then $y \in D_j(a)$ iff x/y is a root of unity. Pick any $\xi \in D_j(a)$ with $n_{p_j}(\xi) = l_j$, and let $\omega_1 = 1, \omega_2 = -1, \omega_3, \ldots, \omega_{p_j-1}$ (ω_2 if $p_j = 2$) be the roots of unity in Q_{p_j}. Denote by Ω_j the set of roots of unity in Q_{p_j}. We have

$$D_j(a) = \{\xi^s \omega\colon\ s \in Z,\ \omega \in \Omega_j\}.$$

In particular, if $x \in D_j(a)$ then $x^{(p_j-1)e_j}$ (x^{2e_j} if $p_j = 2$) is an integer power of a.

Consider the subgroup $D_G(a)$ of Q_G^* consisting of those elements of Q_G^* which are rationally dependent of a. Let $x = (x_1, x_2, \ldots, x_g) \in Q_G^*$. It is easily verified that $x \in D_G(a)$ iff $x_j \in D_j(a)$ for $1 \leqslant j \leqslant g$ and $n_{p_1}(x_1)/e_1 = n_{p_2}(x_2)/e_2 = \ldots = n_{p_g}(x_g)/e_g$. Set $e = \text{g.c.d.}(e_1, e_2, \ldots, e_g)$ and $r = \text{l.c.m.}(2, p_1-1, \ldots, p_g-1)$. If $x \in D_G(a)$ then x^{re} is a power of a.

Lemma II.3. *The following conditions are equivalent*:

(1) (a_n) *is rationally dependent of* a *in* Q_G^*.

(2) $a_n^{re}/a^{k_n} \xrightarrow[n\to\infty]{R} 1$ *in* Q_G *for some sequence* (k_n).

(3) *There exists a sequence* (η_n) *in* $D_G(a)$ *with* $a_n/\eta_n \xrightarrow[n\to\infty]{R} 1$ *in* Q_G.

Proof. (2)$\Rightarrow$(1) follows from the definition of rational dependence, while (3)$\Rightarrow$(2) follows from the discussion preceding the lemma and from Lemma II.2. Let us show that (1)$\Rightarrow$(3). Take d and (k_n) such that $a_n^d/a^{k_n} \xrightarrow[n\to\infty]{R} 1$ in Q_G. Write $k_n = dl_n + s_n$, with $0 \leqslant s_n < d$. We have

$$(a_n/a^{l_n})^d/a^{s_n} \xrightarrow[n\to\infty]{R} 1 \quad \text{in} \quad Q_G,$$

so that the sequence (a_n/a^{l_n}) has only finitely many limit points in Q_G, each of which is rationally dependent of a. There exists therefore a sequence (η_n) in $D_G(a)$ such that $a_n/\eta_n \xrightarrow[n\to\infty]{} 1$. Obviously, $\eta_n^d = a^{k_n}$ for all sufficiently large n, and so $(a_n/\eta_n)^d \xrightarrow[n\to\infty]{R} 1$ in Q_G. By Lemma II.2 we now get $a_n/\eta_n \xrightarrow[n\to\infty]{R} 1$. This proves the lemma.

The lemma shows by the way that it is easier, in principle at least, to determine whether or not (a_n) is rationally dependent of a in Q_G^* than to determine whether or not this is the case in R^*. In the former case we easily find an integer d such that if (a_n) is rationally dependent of a then $a_n^d/a^{k_n} \xrightarrow[n\to\infty]{R} 1$ for some (k_n), whereas in the latter case no such d exists. This is due to the fact that, denoting by a^Z the multiplicative group generated by a, $D_G(a)/a^Z$ is a finite group of an effectively computable order, while the analogous quotient group for R_+^* is Q/Z.

To find out whether or not (a_n) is rationally dependent of a in Q_G^*, one proceeds as follows. Taking d as before, there exists for each n at most one non-negative integer k_n such that $a_n^d/a^{k_n} \in Q_G^*$. If for infinitely many indices n no such k_n exists, then (a_n) is certainly rationally independent of a. Otherwise,

let $\xi_n = a_n^d/a^{k_n} - 1$. We have to check if $\xi_n \xrightarrow[n\to\infty]{R} 0$ in Q_G. Put $s_n = \min_{1 \leqslant j \leqslant g} n_{p_j}(\xi_n)/e_j$ for each n. It is easy to see that $\xi_n \xrightarrow[n\to\infty]{} 0$ iff $s_n \xrightarrow[n\to\infty]{} \infty$, and that $\xi_n \xrightarrow[n\to\infty]{R} 0$ iff there exists a constant C such that the sequence (s_n) attains no value more than C times.

Example II.2. Suppose that $a = p$ is a prime, $(a_n, p) = 1$ for every $n \in N$ and $(a_n)_{n=0}^{\infty}$ is a PR sequence. For each n, let ω_n be that root of unity for which $a_n/\omega_n \in 1 + pZ_p$. ($a_n/\omega_n \in 1 + 4Z_2$ if $p = 2$.) Set $s_n = \max\{k\colon\ p^k | (a_n - \omega_n)\}$. It follows from Theorem II.1 that, if $\Delta = \langle p\rangle^\infty \prod_{n=1}^{\infty} \langle a_n\rangle$ is not a DD set, then $s_n \to \infty$ and, moreover, (s_n) assumes each value at most C times for a certain constant C. This condition is particularly easy to visualize if $p = 2$ or $p = 3$, because in these cases the only roots of unity in Q_p are 1 and -1. In any case we can make a similar statement regarding the sequence (s_n') given by $s_n' = \max\{k\colon\ p^k | a_n^{p-1} - 1\}$. ($s_n' = \max\{k\colon\ 2^k | a_n^2 - 1\}$ if $p = 2$.) Combining this with Remark I.3 we see that, if Δ is not a DD set, then there exists a $d \in N$ such that the base p expansion of a_n^d is either of the form $10\ldots0{*}\ldots{*}0\ldots01$ or of the form $p-1, \ldots, p-1, * \ldots * 0 \ldots 01$. Here the lengths of the initial and terminal blocks go to infinity with n (each length being realized at most C times for a certain constant C).

Similarly to the real case, rational dependence can also be characterized in terms of p-adic logarithms. Namely, let Log_p be the p-adic logarithmic function, which constitutes an isomorphism between the multiplicative group $1 + pZ_p$ ($1 + 4Z_2$ if $p = 2$) and the additive group pZ_p ($4Z_2$ if $p = 2$). Extend Log_p to a function on Z_p^* by

$$\mathrm{Log}_p x = \begin{cases} \dfrac{1}{p-1}\mathrm{Log}_p x^{p-1}, & p \neq 2, \\ \tfrac{1}{2}\mathrm{Log}_2 x^2, & p = 2. \end{cases}$$

A routine verification shows that $x_n \xrightarrow[n\to\infty]{R} x$ iff $\mathrm{Log}_p x_n \xrightarrow[n\to\infty]{R} \mathrm{Log}_p x$. Hence in our case (a_n) is rationally dependent of p iff $\mathrm{Log}_p a_n \xrightarrow[n\to\infty]{R} 0$.

Example II.3. $\Delta = \langle 2\rangle^\infty \prod_{n\in S} \langle 2^n + 3\rangle$ is a DD set for any infinite set S. Note that this does not follow from Theorem I.3 even if $S = N$.

Example II.4. Consider the set $\Delta = \langle 10\rangle^\infty \prod_{n\in S} \langle 2^n + 1\rangle$. Employing Theorem I.3, one can show that if $S = \langle k\rangle^\infty$ for some $k \geqslant 2$ then Δ is a DD set. However, since $2^n + 1 \xrightarrow[n\to\infty]{R} 1$ in Q_2, Theorem II.1 does not imply the same result even for $S = N$.

In Section 2 we give a general example of non-DD sets in the homogeneous case. Section 3 establishes an analogue of Proposition I.1, and in Section 4 we prove Theorem II.1.

2. A counter-example; homogeneous case. In view of Theorem I.3 and Theorem II.1 it is natural, when looking for a non-DD set Δ of the form given in (1), to try for (a_n) a sequence of the form $(a^n+1)_{n\in S}$. It will follow from Theorem III.1 that if $S = N$, for example, then Δ is a DD set. We shall now show that, for sufficiently thin sets S, non-DD sets are in fact obtained.

THEOREM II.2. *Let* $\Delta = \langle a\rangle^{\infty} \prod_{k=1}^{\infty} \langle a^{n_k}+1\rangle$. *Denote*

$$N_k = n_{k+1} - \sum_{i=1}^{k} n_i \quad \text{for} \quad k \in N.$$

Assume that (N_k) *is bounded below and that* $\overline{\lim_{k\to\infty}} N_k = \infty$. *Then* $BA(\Delta)$ *is uncountable, and in particular* Δ *is not a DD set.*

Note that if $n_{k+1} > 2n_k$ for each k, then both conditions of the theorem are satisfied.

Remark II.1. We do not know whether or not there exist non-WA groups of an infinite non-idempotent type. However, it follows from the theorem that if there are infinitely many Fermat primes then such groups do exist. Similarly, one can show that if there are infinitely many Mersenne primes then such groups exist.

LEMMA II.4. *There exists a prime* q *such that* $(a, q) = 1$ *and* $(a^n+1, q) = 1$ *for every* $n \in N$.

In fact, it is easy to verify that if q is any prime divisor of a^2+a+1 then it satisfies the required conditions.

Proof of Theorem II.2. Choose q as in Lemma II.4. Consider the following two subsets of $\boldsymbol{T}$:

$$A = \boldsymbol{T}[q]\setminus\{0\}, \qquad B = \{1/a^n \colon n \in N\}.$$

Set $\delta = \varrho(A, B)$, where ϱ is the metric on $\boldsymbol{T}$. Evidently $\delta > 0$, and if $x \in A+B$ then $\|x\| \geq \delta$. Take an integer r with $\sum_{n=1}^{\infty} a^{-rn} < \delta/2$.

According to Lemma I.3 we may assume, omitting finitely many terms from the sequence (n_k), *that* $n_{k+1} \geq \sum_{i=1}^{k} n_i + r$ for each k. Define a sequence $(m_s)_{s=0}^{\infty}$ inductively as follows. Let $m_0 = 0$. Suppose $m_0, m_1, \ldots, m_s$ have already been defined. Choose k for which $n_{k+1} > \sum_{i=1}^{k} n_i + m_s + r$. Set $m_{s+1} = n_{k+1} - 1$. Consider the point $x = \sum_{s=1}^{\infty} a^{-m_s}$. Given any $x' \in \Delta x$ we can write $x' = \sum_{j=1}^{\infty} a^{-l_j}$, with (l_j) non-decreasing. We claim that $l_{j+1} \geq l_j + r$ for each j.

In fact, any l_j is a positive integer of the form $m_s - \sum_{t=1}^{u} n_{i_t} - e$. It has to be proved that any two such non-identical expressions differ by at least r. If the expressions correspond to the same s, then this follows from the properties of (n_k), whereas otherwise the construction of (m_s) implies it.

Consequently, if $x' \in \Delta x$ then $\varrho(x', B) \leq \sum_{n=1}^{\infty} a^{-rn} < \delta/2$, and so the same holds for every $x' \in \overline{\Delta x}$. Setting $y = x+1/q$ we readily observe then that $\|y'\| \geq \delta/2$ for any $y' \in \overline{\Delta y}$, which implies that $y \in BA(\Delta)$. It is easy to see that, substituting for (m_s) any subsequence thereof, we obtain another point of $BA(\Delta)$. Hence $BA(\Delta)$ is uncountable, which completes the proof.

3. The basic tool. Analyzing the proof of Proposition I.1, we observe that its main part consists of a sequence of steps, each having two parts. After each step we obtain a set, known to contain a certain finite configuration. The first part of the next step is to get from our set another one, containing infinitely many copies of the same, or of a similar, configuration. The second part consists of obtaining a set which contains a "better" configuration than the one with which we started. The key to the analogue of the first part is given by

PROPOSITION II.1. *Let E be an infinite closed a-invariant subset of $\boldsymbol{T}$. Then for each* $l \in N$ *there exist* $x^{(1)}, x^{(2)} \in E$ *with* $x^{(2)} - x^{(1)} \in \boldsymbol{T}[a^l]\setminus \boldsymbol{T}[a^{l-1}]$.

Proof. Given any $y \in E$, consider its base a expansion $y = 0.y_1 y_2 \ldots$. An n-tuple $(c_1, c_2, \ldots, c_n)$, with $0 \leq c_i < a$ for each i, is an *n-block appearing in E* if there exists a $y \in E$ with

$$(2) \qquad y_k = c_1, \quad y_{k+1} = c_2, \quad \ldots, \quad y_{k+n-1} = c_n$$

for some k. $(c_1, c_2, \ldots, c_n)$ is an *n-block appearing in arbitrarily distant places in E* if for every $k \in N$ there exists a $y \in E$ satisfying (2). Let $B_a(n)$ denote the number of n-blocks appearing in arbitrarily distant places in E. It is easy to see that, since E is infinite, $B_a(n)$ is strictly increasing as a function of n. For any $m \in N$ we can find therefore two points $x^{1,m}$, $x^{2,m} \in E$ such that $x_l^{1,m} \neq x_l^{2,m}$ and $x_{l+j}^{1,m} = x_{l+j}^{2,m}$ for $1 \leq j \leq m$. Take a sequence $(m_s)_{s=1}^{\infty}$ such that

$$x^{1,m_s} \xrightarrow[s\to\infty]{} x^{(1)} \quad \text{and} \quad x^{2,m_s} \xrightarrow[s\to\infty]{} x^{(2)}.$$

It is readily seen that $x^{(1)}$ and $x^{(2)}$ satisfy the required properties. This proves the proposition.

Denote $\boldsymbol{Z}(a^{\infty}) = \bigcup_{k=1}^{\infty} \boldsymbol{T}[a^k]$.

DEFINITION II.1 Δ is a *DD_a set* if $\overline{\Delta E} = \boldsymbol{T}$ for every infinite T_a-invariant $E \subseteq \boldsymbol{Z}(a^{\infty})$.

DEFINITION II.2. A closed T_a-invariant set $E \subseteq T$ is *T_a-minimal* (or, more briefly, *a-minimal*) if it contains no proper closed T_a-invariant subset.

PROPOSITION II.2. *Let* $(\Delta_n)_{n=1}^{\infty}$ *be a sequence of* DD_a *sets with* $1 \in \Delta_n$ *for each n. Assume that* $\bigcup_{n=1}^{\infty} \Delta_n \cup \{a\}$ *is a PR set. Then* $\langle a\rangle^{\infty} \prod_{n=1}^{\infty} \Delta_n$ *is a DD set.*

Proof. Let α be an irrational. Set:

$$E_0 = \overline{\langle a\rangle^{\infty}\alpha},$$
$$E_n = \overline{\Delta_n E_{n-1}}, \quad n \geqslant 1.$$

We shall show that $\overline{\bigcup_{n=1}^{\infty} E_n} = T$.

Assume first that $0 \in E_0$. This means that the base a expansion of α contains arbitrarily long blocks consisting of 0's or arbitrarily long blocks consisting of $(a-1)$'s. From this we infer that E_0 contains an infinite subset of $Z(a^{\infty})$, whence $E_1 = T$. If E_0 is known to contain a torsion element, then we similarly get that E_1 has a non-empty interior, so that $E_2 = T$. We may assume therefore that E_0 contains no torsion elements and, passing to a subset of E_0, that E_0 is an infinite a-minimal set.

Now assume that $0 \in E_n$ for some n. Fix $m \in N$, and take $b \in \prod_{j=1}^{n} \Delta_j$ and $x^{(0)} \in E_0$ with $\|bx^{(0)}\| < 1/m$. Since E_0 is a-minimal, the restriction of T_a to E_0 is onto. Find a sequence $(x^{(i)})_{i=1}^{\infty}$ in E such that $ax^{(i)} = x^{(i-1)}$ for each $i \in N$. If $bx^{(i)} \xrightarrow[i\to\infty]{} 0$, then, taking a limit point of $(x^{(i)})$, we see that $E_0 \cap T[b] \neq \emptyset$, which is a contradiction. Hence for every l the set E_n contains a point $y^{l,m}$ with $\varrho(y^{l,m}, T[a^l] \setminus T[a^{l-1}]) < 1/m$. Taking a limit point of $(y^{l,m})_{m=1}^{\infty}$ we see that $E_n \cap (T[a^l] \setminus T[a^{l-1}]) \neq \emptyset$, and therefore E_n contains an infinite subset of $Z(a^{\infty})$. This implies that $E_{n+1} = T$. If E_n is assumed to contain an arbitrary torsion element, then, similarly to the first part of the proof, we get $E_{n+2} = T$. We may thus assume that $\bigcup_{n=1}^{\infty} E_n$ contains no torsion elements.

Take a prime q for which our set is a PR_q set. We proceed to show that for each l the set E_{q^l} contains a translate of $T[q^l]$. Suppose that for some $0 \leqslant k < q^l$ we have

$$x,\ x+m_1/q^l, \ldots, x+m_{k-1}/q^l \in E_k.$$

Take s with $a^s \equiv 1 \pmod{q^l}$. E_k is a-invariant, and hence a^s-invariant as well. It follows that

$$F = \{x \in T\colon\ x,\ x+m_1/q^l, \ldots, x+m_{k-1}/q^l \in E_k\}$$

is an infinite closed a^s-invariant set. By Proposition II.1, and since Δ_{k+1} is a DD_a set, we have $\overline{\langle a\rangle^{s-1}\Delta_{k+1}(F-F)} = T$. The set $\overline{\Delta_{k+1}(F-F)}$ is a^s-invariant, and so we easily obtain $\overline{\Delta_{k+1}(F-F)} = T$. Similarly to the last part of the proof of Proposition I.1 we now show that E_{k+1} contains a translate of a $(k+1)$-element subset of $T[q^l]$. This completes the proof.

4. Proof of Theorem II.1. Given a $\xi \in Q_a$, say $\xi = \sum_{i=-k}^{\infty} \xi_i a^i$, the *fractional part of* ξ is defined by $\{\xi\} = \sum_{i=-k}^{-1} \xi_i a^i$.

LEMMA II.5. *Let E be an infinite a-invariant subset of* $Z(a^{\infty})$. *Then there exists a* $\xi \in Z_a \setminus aZ_a$ *such that* $\{\xi/a^n\} \in E$ *for every* $n \in N$.

Proof. Select a sequence $(x^{(n)})$ in E with $x^{(n)} \in T[a^n] \setminus T[a^{n-1}]$ for each n. Consider $(x^{(n)})$ as a sequence of real numbers, and define a sequence $(y^{(n)})$ by $y^{(n)} = a^n x^{(n)}$. Evidently, $(y^{(n)})$ is a sequence of integers, none of which is divisible by a. Let ξ be a limit point of $(y^{(n)})$ in Z_a. It is clear that ξ satisfies the required properties, and thereby the lemma is proved.

For $\xi \in Z_a$ we shall denote

$$\xi/a^{\infty} = \{\{\xi/a^n\}\colon\ n \in N\} \subseteq Z(a^{\infty}).$$

Obviously $a^k \xi/a^{\infty} = \xi/a^{\infty}$, whence if $E = \xi/a^{\infty}$ for some $\xi \neq 0$ then there exists a positive integer s such that $sE = \xi'/a^{\infty}$, where $\xi' \in Z_J^*$ for some non-empty $J \subseteq H$. Hence, given an infinite a-invariant $E \subseteq Z(a^{\infty})$, we shall usually be able to assume that $E = \xi/a^{\infty}$, where $\xi \in Z_J^*$.

LEMMA II.6. *If condition* (2) *of Theorem* II.1 *is satisfied, then* Δ *is a* DD_a *set.*

Proof. We have to show that, if E is an infinite a-invariant subset of $Z(a^{\infty})$, then $\overline{\Delta E} = T$. It may be assumed that $E = \xi/a^{\infty}$, where $\xi \in Z_J^*$ for a certain $J \subseteq H$, say $J = G$. Suppose that $\xi = (\xi_1, \ldots, \xi_g)$, and put $\underline{a} = \prod_{j=1}^{g} p_j^{e_j}$.

We may assume that

(3) $$n_{p_1}(a_n)/e_1 = n_{p_2}(a_n)/e_2 = \ldots = n_{p_g}(a_n)/e_g$$

for all n. In fact, suppose that (3) is violated for infinitely many indices n. Split (a_n) into two sequences (b_n) and (c_n) such that, say,

$$n_{p_1}(b_n)/e_1 = \ldots = n_{p_f}(b_n)/e_f < n_{p_{f+1}}(b_n)/e_{f+1} \leqslant \ldots \leqslant n_{p_g}(b_n)/e_g, \quad n \geqslant 1$$

and (c_n) is rationally independent of a in Q_J^* for every $J \subseteq H$. Put

$$\underline{a} = \prod_{j=1}^{f} p_j^{e_j}.$$

Setting $E_1 = Z(\underline{a}^{\infty}) \cap \prod_{n=1}^{\infty} \langle b_n\rangle E$, we easily see that E_1 is infinite. It certainly suffices to show that $\overline{\prod_{n=1}^{\infty} \langle c_n \rangle E_1} = T$ and so we may assume, replacing E by E_1, that $E \subseteq Z(\underline{g}^{\infty})$. Repeating the process we arrive, within a finite number of steps, at a situation where (3) holds for all but finitely many indices n. Disposing of the exceptional n's, we may assume that (3) holds for all n.

From now on we shall use only the fact that (a_n) is rationally independent of a in Q_G^*. Let $e = \text{g.c.d.}(e_1, e_2, \ldots, e_g)$. By (3) we see that the sequence $(n_{p_1}(a_n)/e_1)_{n=1}^{\infty}$ takes on at most e distinct values modulo 1. Splitting (a_n) into e sequences, and replacing it by one of these, which is still rationally independent of a in Q_G^*, we may assume that

$$n_{p_j}(a_n)/e_j = r_n + d/e, \quad 1 \leqslant j \leqslant g,\ n \in N \tag{4}$$

where (r_n) is a sequence of non-negative integers and $0 \leqslant d < e$.

Let us now show that it may be assumed that $d = 0$ in (4). Set $t_n = a_n/a^{r_n}$. For each $1 \leqslant j \leqslant g$ the sequence (t_n) lies in $p_j^{de_j/e} Z_{p_j}^*$. Assume first that there exists a j for which $D_j(a) \cap p_j^{de_j/e} Z_{p_j}^* = \emptyset$. Passing to a subsequence, we may assume that $t_n \xrightarrow[n\to\infty]{} t$ in Z_{p_j}. Define (a_n') by $a_n' = \prod_{i=e(n-1)+1}^{en} a_i$. Obviously, $n_{p_j}(a_n')/e_j$ is an integer for each n, and we have $a_n'/a^{er_n+d} \xrightarrow[n\to\infty]{} t^e/a^d$. Since t^e/a^d lies in $Z_{p_j}^*$ and is not a root of unity, (a_n') is rationally independent of a in Q_G^*. Substituting (a_n') for (a_n) we may assume therefore in this case that $d = 0$. Now suppose that

$$D_j(a) \cap p_j^{de_j/e} Z_{p_j}^* \neq \emptyset \quad \text{for each } j.$$

Let $U_j = 1 + p_j Z_j$. ($U_j = 1 + 4Z_2$ if $p_j = 2$.) Define sequences $(\eta_{nj})_{n=1}^{\infty}$ in $D_j(a)$, $1 \leqslant j \leqslant g$, by the condition

$$a_n/\eta_{nj} \in U_j, \quad n = 1, 2, 3, \ldots$$

Put

$$t_{nj} = a_n/\eta_{nj} - 1, \qquad n = 1, 2, 3, \ldots, \quad 1 \leqslant j \leqslant g,$$
$$t_n = (t_{n1}, t_{n2}, \ldots, t_{ng}), \qquad n = 1, 2, 3, \ldots$$

Since (a_n) is rationally independent of a in Q_G^*, we do not have $t_n \xrightarrow[n\to\infty]{R} 0$ in Q_G. For each n take a number $k = k(n)$, $1 \leqslant k \leqslant g$, for which

$$n_{p_k}(t_{nk})/e_k \leqslant n_{p_j}(t_{nj})/e_j, \quad 1 \leqslant j \leqslant g.$$

For $1 \leqslant k \leqslant g$ denote by $(t_n^{(k)})$ the subsequence of (t_n) consisting of those terms t_n with $k(n) = k$. One of the sequences $(t_n^{(k)})$, $1 \leqslant k \leqslant g$, say $(t_n^{(1)})$, still does not converge rapidly to 0 in Q_G. Replacing (t_n) by $(t_n^{(1)})$ we may thus

assume that

$$n_{p_1}(t_{n1})/e_1 \leqslant n_{p_j}(t_{nj})/e_j, \quad 1 \leqslant j \leqslant g. \tag{5}$$

Write $e = p_1^r e'$, with $(e', p_1) = 1$, and

$$t_{n1} = t_{n10} p_1^{l_n} + t_{n11} p_1^{l_n+1} + \ldots + t_{n1r} p_1^{l_n+r} + \mu_n p_1^{l_n+r+1}, \quad n \in N \tag{6}$$

where $1 \leqslant t_{n10} \leqslant p_1 - 1$, $0 \leqslant t_{n1i} \leqslant p_1 - 1$ for $1 \leqslant i \leqslant r$ and $\mu_n \in Z_{p_1}$. Split (a_n) into $(p_1 - 1) p_1^r$ sequences, for each of which $(t_{n10}, t_{n11}, \ldots, t_{n1r})$ is independent of n. Replace (a_n) by one of these sequences, which is still rationally independent of a in Q_G^*. For each l denote by N_l the number (perhaps infinite) of n's for which we have $l_n = l$ in (6). Our assumptions imply that the sequence (N_l) is unbounded above. Reordering (a_n) and passing to a subsequence thereof, we may assume that (l_n) is non-decreasing and that N_l is a multiple of e for each l. Replace (a_n) by (a_n'), where $a_n' = \prod_{i=e(n-1)+1}^{en} a_i$. It is evident that (a_n') is rationally independent of a in Q_G^*, so that we may again assume that $d = 0$.

Since $d = 0$ the sequence (a_n/a^{r_n}) lies in Z_G^*. For each j we have $D_j(a) \cap Z_{p_j}^* = \Omega_j$, and therefore the notations introduced in the preceding paragraph are still applicable. Write $\eta_{nj} = a^{r_n}\omega_{nj}$, where $\omega_{nj} \in \Omega_j$ for $n \in N$, $1 \leqslant j \leqslant g$. Passing to a suitable subsequence of (a_n), we may assume that each of the sequences $(\omega_{nj})_{n=1}^{\infty}$, $1 \leqslant j \leqslant g$, is constant. Replacing (a_n) as before by a sequence (a_n'), each term in the latter of which being a product of several terms from the former, we may assume that $\omega_{nj} = 1$ for every n and j. We note that, due to all the modifications performed in (a_n), it is possible that (5) is not valid any more. Yet it is easy to see that there exists a constant C such that

$$n_{p_1}(t_{n1})/e_1 \leqslant n_{p_j}(t_{nj})/e_j + C, \quad 1 \leqslant j \leqslant g.$$

Now let m be an arbitrary fixed positive integer. Write

$$t_{nj} = t_{nj0} p_j^{e_j l_{nj} + s_{nj}} + t_{nj1} p_j^{e_j l_{nj} + s_{nj} + 1} + \ldots$$
$$\ldots + t_{nj,e_j m-1} p_j^{e_j l_{nj} + s_{nj} + e_j m - 1} + \mu_{nj} p_j^{e_j l_{nj} + s_{nj} + e_j m}$$

with $1 \leqslant t_{nj0} \leqslant p_j - 1$, $0 \leqslant t_{nji} \leqslant p_j - 1$ for $1 \leqslant i \leqslant e_j m - 1$ and $\mu_{nj} \in Z_{p_j}$. Additional modifications of (a_n), similar to those performed earlier, enable us to assume that:

(1) $s_{nj} = 0$ for $n \in N$, $1 \leqslant j \leqslant g$.
(2) $l_{nj} \geqslant l_{n1} + m$ for $n \in N$, $1 \leqslant j \leqslant g$.
(3) The sequence $(t_{n10}, t_{n11}, \ldots, t_{n1,e_1 m-1})_{n=1}^{\infty}$ is constant, say

$$(\bar{t}_{10}, \bar{t}_{11}, \ldots, \bar{t}_{1,e_1 m-1}).$$

Reordering (a_n) we may assume that in $\boldsymbol{Z}_{p_1}$ we have

$$a_n = a^{r_n}(1+\bar{t}_{10}p_1^{e_1 l}+\bar{t}_{11}p_1^{e_1 l+1}+\ldots+\bar{t}_{1,e_1 m-1}p_1^{e_1(l+m)-1}+\mu_{n1}p_1^{e_1(l+m)}),\quad n=1,2,\ldots,p_1^{e_1 m}$$

while in $\boldsymbol{Z}_{p_j}$, $j\geqslant 2$, we have

$$a_n = a^{r_n}(1+\mu_{nj}p_j^{e_j(l+m)}),\quad n=1,2,\ldots,p_1^{e_1 m}$$

where $\mu_{nj}\in \boldsymbol{Z}_{p_j}$ for $1\leqslant n\leqslant p_1^{e_1 m}$, $1\leqslant j\leqslant g$. For every $1\leqslant n\leqslant p_1^{e_1 m}$ consider the action of $\prod_{i=1}^{n} a_i\in\Delta$ on $\{\xi/a^{l+m+\sum_{i=1}^{n} r_i}\}\in E$:

$$a_1 a_2\ldots a_n\{\xi/a^{l+m+r_1+\ldots+r_n}\} = \{\xi/a^{l+m}\}$$
$$+\{[\prod_{i=1}^{n}(1+\bar{t}_{10}p_1^{e_1 l}+\ldots+\bar{t}_{1,e_1 m-1}p_1^{e_1(l+m)-1}+\mu_{i1}p_1^{e_1(l+m)})-1]\cdot\xi_1/a^{l+m}\}$$
$$+\sum_{j=2}^{g}\{[\prod_{i=1}^{n}(1+\mu_{ij}p_j^{e_j(l+m)})-1]\cdot\xi_j/a^{l+m}\}.$$

The first term on the right-hand side does not depend on n. Since $1\leqslant\bar{t}_{10}\leqslant p_1-1$, as n varies from 1 to $p_1^{e_1 m}$ the second term assumes all the values $k/p_1^{e_1 m}$, $0\leqslant k\leqslant p_1^{e_1 m}-1$. The third term obviously vanishes.

It has thus been proved that, for each m, ΔE contains a translate of $\boldsymbol{T}[p_1^{e_1 m}]$. Consequently $\overline{\Delta E}=\boldsymbol{T}$, which proves the lemma.

LEMMA II.7. *If (a_n) is rationally independent of a in $\boldsymbol{Q}_J^*$ for every non-empty $J\subseteq H$, then it splits into infinitely many sequences, each having the same property.*

The proof is routine.

Theorem II.1 follows immediately from Proposition II.2, Lemma II.6 and Lemma II.7.

Chapter III. Topological dynamics techniques

1. The main theorems; IP-systems. In Theorem II.2 we saw that if S is a sufficiently thin subset of N then $\Delta=\langle a\rangle^\infty\prod_{n\in S}\langle a^n+1\rangle$ is not a DD set. In this chapter we shall deal with sets Δ of similar forms, and show that if S is sufficiently large then the ensuing sets are DD sets.

THEOREM III.1. *Let u and v be non-zero rationals, $a\geqslant 2$ an integer and $S\subseteq N$ a syndetic set. Suppose that ua^n+v is an integer for every $n\in S$. Then*

$$\Delta=\prod_{n\in S}\langle ua^n+v\rangle \tag{1}$$

is a DD set.

THEOREM III.2. *Let $a\geqslant 2$, $l\in N$, $(d_i)_{i=0}^{l}$ integers with $d_0\neq 0$ and $d_l=\pm 1$, and $S\subseteq N$ a syndetic set. Then*

$$\Delta=\langle a\rangle^\infty\prod_{n\in S}\langle d_0 a^{ln}+d_1 a^{(l-1)n}+\ldots+d_l\rangle \tag{2}$$

is a DD set.

In the course of the proofs we shall repeatedly make use of some terminology and results related to IP-sets, which we now review (for a comprehensive exposition see [4. Ch. 8]). Given a sequence of positive integers (n_i), the set of all numbers of the form $n_{i_1}+n_{i_2}+\ldots+n_{i_k}$ with $i_1<i_2<\ldots<i_k$ will be denoted by $\sum_{i=1}^{\infty}\{0,n_i\}$. A set $S\subseteq N$ is an *IP-set* if $S=\sum_{i=1}^{\infty}\{0,n_i\}$ for a certain sequence (n_i). If $N=\bigcup_{i=1}^{d}S_i$, then at least one of the sets S_i contains an IP-set [4, Th. 8.11], whence any syndetic set contains a translate of an IP-set. Denote by $\mathscr{F}$ the set of all finite non-empty subsets of N. A *homomorphism* $\varphi:\mathscr{F}\to\mathscr{F}$ is a map such that $\varphi(f_1\cup f_2)=\varphi(f_1)\cup\varphi(f_2)$ and $f_1\cap f_2=\emptyset\Rightarrow\varphi(f_1)\cap\varphi(f_2)=\emptyset$. Such a homomorphism φ is determined by $\varphi\{i\}$ for all i, which can be any disjoint finite sets f_i. Then

$$\varphi\{i_1,i_2,\ldots,i_k\}=f_{i_1}\cup f_{i_2}\cup\ldots\cup f_{i_k}.$$

An *IP-subset of* $\mathscr{F}$ is the image $\varphi(\mathscr{F})$ of a homomorphism $\varphi:\mathscr{F}\to\mathscr{F}$. An *$\mathscr{F}$-sequence* of elements in an arbitrary space X is a sequence $(x_f)_{f\in\mathscr{F}}$, indexed by elements of $\mathscr{F}$. If X is a semigroup, we say that an $\mathscr{F}$-sequence defines an *IP-system* if $x_{\{i_1,i_2,\ldots,i_k\}}=x_{i_1}x_{i_2}\ldots x_{i_k}$ for $i_1<i_2<\ldots<i_k$. For $f_1,f_2\in\mathscr{F}$ we denote $f_1<f_2$ if all the elements of f_1 are smaller than all those of f_2. An *$\mathscr{F}$-subsequence* of an $\mathscr{F}$-sequence $(x_f)_{f\in\mathscr{F}}$ is an $\mathscr{F}$-sequence of the form $(x_{\varphi(f)})_{f\in\mathscr{F}}$, where $\varphi:\mathscr{F}\to\mathscr{F}$ is a homomorphism. Given an $\mathscr{F}$-sequence (x_f) and a point x in a topological space X, we say that $x_f\to x$ *as an $\mathscr{F}$-sequence* if for every neighborhood V of x there is some $f_V\in\mathscr{F}$ such that $x_f\in V$ for all $f>f_V$. For any $\mathscr{F}$-sequence (x_f), taking values in a compact metric space, there exists an $\mathscr{F}$-subsequence $(x_{\varphi(f)})$ which converges as an $\mathscr{F}$-sequence [4, Th. 8.14]. Now let $(T^f)_{f\in\mathscr{F}}$ be an IP-system of continuous transformations of a compact metric space X, i.e., a system of maps T^f: $X\to X$ indexed by $f\in\mathscr{F}$, with $T^{f_1\cup f_2}=T^{f_1}T^{f_2}$ for $f_1<f_2$. For any $x\in X$ there exist a subsequence $(T^{\varphi(f)})$ and a $y\in X$ such that $T^{\varphi(f)}x\to y$ and $T^{\varphi(f)}y\to y$ as $\mathscr{F}$-sequences [4, Lemma 8.15].

2. Proof of Theorem III.1. Throughout this section Δ will be as in (1), but the assumptions regarding u, v and S will vary. Notice that a may be assumed not to be a non-trivial power of another integer.

LEMMA III.1. *Suppose that u and a are rationally independent and that S is syndetic. Then Δ is a DD set.*

Proof. Assume first that $u > 0$. Let $S = \{n_1, n_2, \ldots\}$, with (n_k) monotone increasing, and write $a_k = ua^{n_k}+v$ for $k \in \mathbf{N}$. Fix a prime p for which a, u, $v \in \mathbf{Z}_p^*$. The set $R_p = \{n\colon p \mid ua^n+v\}$ forms an arithmetic progression, whose density can be made arbitrarily small by taking p sufficiently large. We may assume therefore that $S \setminus R_p$ is syndetic and, substituting $S \setminus R_p$ for S, that (a_k) is a PR sequence. It is clear that (a_k) is an SM sequence and that $a^m \in \operatorname{Lim} \bar{a}/\bar{a}$ for an appropriately chosen $m \in \mathbf{N}$. Since a and u are rationally independent and $\log_a(ua^n+v) \xrightarrow[n\to\infty]{} \log_a u \pmod 1$, (a_k) is rationally independent of a^m. In view of Theorem I.1, Δ is a DD set.

If $u < 0$, write $S = S_1 \cup S_2$, with S_1 syndetic and S_2 infinite. Set

$$\Delta_i = \prod_{n\in S_i} \langle ua^n+v\rangle \quad \text{for} \quad i = 1, 2.$$

Let $\alpha \in \mathbf{T}$ be an irrational. By the preceding part of the proof we have $\overline{\Delta_1\alpha} \cup -\overline{\Delta_1\alpha} = \mathbf{T}$ so that $\overline{\Delta_1\alpha}$ has a non-empty interior. Hence $\overline{\Delta\alpha} \supseteq \Delta_2\overline{\Delta_1\alpha} = \mathbf{T}$. This completes the proof.

PROPOSITION III.1. *Let r, s and t be non-zero integers with r and s rationally independent, α an irrational and β arbitrary. Then $\{r^m s^n\alpha + t^{m+n}\beta\colon m, n \geqslant 0\}$ is dense modulo 1.*

Proof. It suffices to show that for any $\varepsilon > 0$ there exists a positive integer M such that the set $\{r^m s^{M-m}\alpha\colon 0 \leqslant m \leqslant M\}$ forms an ε-net modulo 1. We may assume that $r > s \geqslant 2$. Choose a real number x for which $\{(r/s)^n x\colon n \geqslant 0\}$ is dense modulo 1. We can find a positive integer N and a neighborhood I of x such that $\{(r/s)^n y\colon 0 \leqslant n \leqslant N\}$ is an ε-net modulo 1 for every $y \in I$. Put $I_1 = I/s^N$. Pick k and l such that $r^k s^l \alpha = d+\theta$ with $d \in \mathbf{Z}$, $\theta \in I_1$. We claim that $\{r^{k+j}s^{l+N-j}\alpha\colon 0 \leqslant j \leqslant N\}$ forms an ε-net modulo 1. In fact, modulo 1 this set is just $\{(r/s)^j s^N\theta\colon 0 \leqslant j \leqslant N\}$, which is an ε-net since $s^N\theta \in I$. Thus, putting $M = k+l+N$, we see that $\{r^m s^{M-m}\alpha\colon 0 \leqslant m \leqslant M\}$ forms an ε-net modulo 1. This proves the proposition.

PROPOSITION III.2. *Assume that u and v are non-zero integers and S forms a translate of an IP-set. If $\overline{\langle a\rangle^\infty\alpha}$ contains no torsion elements of $\mathbf{T}$, then $\overline{\Delta\alpha} = \mathbf{T}$.*

Proof. We may assume that $S = k+S_1$, where k is a positive integer with $|ua^k+v| \geqslant 2$ and S_1 is an IP-set. From [9, Th. 3] we infer that there exists an $l \in S_1$ such that ua^k+v and $ua^{k+l}+v$ are rationally independent. We may assume that $S_1 = \{0, l\}+S_2$, where $S_2 = \sum_{i=1}^{\infty} \{0, m_i\}$. Let $(T^f)_{f\in\mathcal{F}}$ be the IP-system of endomorphisms of $\mathbf{T}$ given by $T^{\{i_1,i_2,\ldots,i_r\}} = T_a^m$, where $m = \sum_{j=1}^{r} m_{i_j}$. Passing to an IP-subset of S_2 we may assume that $T^f\alpha \to \alpha_0$ as an $\mathcal{F}$-sequence for some α_0. Our assumptions guarantee that α_0 is an irrational. For arbitrary fixed non-negative integers r and s we now have

$$\prod_{i=t}^{t+r-1} (ua^{k+m_i}+v) \prod_{i=t+r}^{t+r+s-1} (ua^{k+l+m_i}+v)\alpha \xrightarrow[t\to\infty]{} (ua^k+v)^r(ua^{k+l}+v)^s\alpha_0 + v^{r+s}(\alpha-\alpha_0).$$

It follows that $(ua^k+v)^r(ua^{k+l}+v)^s\alpha_0 + v^{r+s}(\alpha-\alpha_0) \in \overline{\Delta\alpha}$ for every r, $s \geqslant 0$. By Proposition III.1, this implies that $\overline{\Delta\alpha} = \mathbf{T}$. This proves the proposition.

LEMMA III.2. *Suppose that a and v are rationally independent and that S is syndetic. If $\overline{\langle a\rangle^\infty\alpha}$ contains a torsion element, then $\overline{\Delta\alpha} = \mathbf{T}$.*

Proof. By Lemma III.1 we may assume that $u = \pm a^{\pm k}$ for some k. Replacing S by a translate thereof, we may therefore assume that $u = \pm 1$. Write $S = S_1 \cup S_2 \cup S_3$, with S_1 and S_2 syndetic and S_3 infinite, and put

$$\Delta_i = \prod_{n\in S_i} \langle ua^n+v\rangle \quad \text{for} \quad 1 \leqslant i \leqslant 3.$$

Assume first that $0 \in \overline{\langle a\rangle^\infty\alpha}$. Select a sequence (n_i) such that $a^{n_i}\alpha \xrightarrow[i\to\infty]{} 0$. Since S_1 is syndetic we may assume that $n_i \in S_1$ for each i. Passing to a subsequence we may assume that, putting $N_k = \sum_{i=1}^{k} n_i$, we have

$$a^{N_k}\|a^{n_{k+1}}\alpha\| \xrightarrow[k\to\infty]{} 0.$$

For every fixed r we then get

$$\prod_{i=t}^{t+r-1} (ua^{n_i}+v)\alpha \xrightarrow[t\to\infty]{} v^r\alpha.$$

It follows that $\overline{\Delta_1\Delta_2\alpha} \supseteq \overline{\langle v\rangle^\infty\Delta_2\alpha}$. Similarly to the proof of Lemma III.1, we may assume that $\langle v\rangle^\infty\Delta_2$ is a PR set. Theorem I.3 now implies that $\langle v\rangle^\infty\Delta_2$ is a DD set, so that $\overline{\Delta_1\Delta_2\alpha} = \mathbf{T}$.

In the general case the foregoing considerations show that $\overline{l\Delta_1\Delta_2\alpha} = \mathbf{T}$ for some $l \in \mathbf{N}$, and consequently $\overline{\Delta_1\Delta_2\alpha}$ has a non-empty interior. It follows that $\overline{\Delta\alpha} \supseteq \Delta_3\overline{\Delta_1\Delta_2\alpha} = \mathbf{T}$, which proves the lemma.

LEMMA III.3. *Suppose that S is syndetic. If $\overline{\langle a\rangle^\infty\alpha}$ contains a torsion element, then $\overline{\Delta\alpha} = \mathbf{T}$.*

Proof. As in the preceding lemma, we may assume that $0 \in \overline{\langle a\rangle^\infty\alpha}$. By Lemmas III.1 and III.2 we have to deal only with the case where $u = \pm 1$

and $v = \pm a^r$ for a certain non-negative integer r. Let $E = \overline{\langle a\rangle^{\infty}\alpha}$. It is easily seen that, since $0 \in \overline{\langle a\rangle^{\infty}\alpha}$, E contains an infinite subset of $\boldsymbol{Z}(a^{\infty})$. Let $\xi \in \boldsymbol{Z}_a$ be as in Lemma II.5 and d an arbitrary fixed positive integer.

Assume first that $r = 0$. Since S is syndetic we can choose a sequence (n_i) in S such that $a^{n_i}\alpha \xrightarrow[i\to\infty]{} \{\xi/a^e\}$, where $e \geqslant d$. Put $N_k = \sum_{i=1}^{k} n_i$. Passing to a subsequence, we may assume that $a^{N_k}\|a^{n_1+n_{k+1}}\alpha\| \xrightarrow[k\to\infty]{} 0$. For each k we then have

$$\prod_{i=t}^{t+2k-1} (ua^{n_i}+v)\alpha \xrightarrow[t\to\infty]{} v^{2k}\alpha + 2kv^{2k-1}u\{\xi/a^e\} = \alpha + k\{2w\xi/a^e\}$$

where $w = \pm 1$. Thus, $\alpha + k\{2\xi/a^e\} \in \overline{\Delta\alpha}$ for each k. Since the order of $\{2\xi/a^e\}$ is at least $p^e/2$, p being the smallest prime divisor of a, this means that $\overline{\Delta\alpha}$ contains translates of finite subgroups of $\boldsymbol{T}$ having arbitrarily high orders. This implies $\overline{\Delta\alpha} = \boldsymbol{T}$.

Now to the case $r > 0$. As in the proof of Lemma III.2, we can show that

$$\overline{\Delta\alpha} \supseteq \langle v\rangle^{\infty} \prod_{n\in S_1} \langle ua^n+v\rangle\alpha \quad \text{for a certain syndetic } S_1.$$

It suffices therefore to show that

$$\overline{\langle a\rangle^{\infty}\prod_{n\in S} \langle ua^n+v\rangle\alpha} = \boldsymbol{T}.$$

Let k be any positive integer. We can find a sequence (n_i) in S such that

$$a^{n_i}\alpha \xrightarrow[i\to\infty]{} \{\xi/a^{e+(2k-1)r}\} \quad \text{for some } e \geqslant d.$$

Passing to a subsequence we get, similarly to the former part,

$$\prod_{i=t}^{t+2k-1} (ua^{n_i}+v)\alpha \xrightarrow[t\to\infty]{} v^{2k}\alpha + 2kv^{2k-1}u\{\xi/a^{e+(2k-1)r}\} = a^{2kr}\alpha + k\{2w\xi/a^e\}$$

with $w = \pm 1$. Inasmuch as α can be replaced by any element of the form $a^m\alpha$, we may assume that $a^{2kr}\alpha$ is arbitrarily close to 0. It follows that $k\{2w\xi/a^e\} \in \overline{\langle a\rangle^{\infty}\prod_{n\in S}\langle ua^n+v\rangle\alpha}$, and hence the subgroup of $\boldsymbol{T}$ generated by $\{\xi/a^d\}$ is contained in $\overline{\langle a\rangle^{\infty}\prod_{n\in S}\langle ua^n+v\rangle\alpha}$. This shows that the latter set is $\boldsymbol{T}$ itself, and thereby the proof is complete.

Theorem III.1 follows directly from Lemma III.1, Proposition III.2 and Lemma III.3.

3. Proof of Theorem III.2. In the course of the proof we shall distinguish between two types of irrationals $\alpha \in \boldsymbol{T}$, those for which $\overline{\langle a\rangle^{\infty}\alpha}$ contains an infinite a-minimal set and those for which it does not. It will turn out that, if we deal only with points of one of these types, then the assumptions of Theorem III.2 can be relaxed.

PROPOSITION III.3. *Suppose that Δ is as in* (2), *where S is a translate of an IP-set and $(d_i)_{i=0}^{l}$ are integers, at least two of which are non-zero. If $\overline{\langle a\rangle^{\infty}\alpha}$ contains an infinite a-minimal set, then $\overline{\Delta\alpha} = \boldsymbol{T}$.*

Proof. Let $E = \overline{\langle a\rangle^{\infty}\alpha}$. We may assume E to be a-minimal. Let

$$S = k + S_1, \quad \text{where} \quad S_1 = \sum_{i=1}^{\infty} \{0, m_i\}.$$

If n is sufficiently large then a and $\sum_{i=0}^{l} d_i a^{(l-i)n}$ are rationally independent, so that a and $b = \sum_{i=0}^{l} d_i a^{(l-i)k}$ may be assumed to be rationally independent. For $1 \leqslant j \leqslant l$ we denote by $(T^{f,j})_{f\in\mathcal{F}}$ the IP-system of endomorphisms of $\boldsymbol{T}$ given by $T^{\{i_1,i_2,\ldots,i_r\},j} = T_a^{jm}$, where $m = \sum_{s=1}^{r} m_{i_s}$. We claim that there exist a homomorphism $\varphi\colon \mathcal{F} \to \mathcal{F}$ and a point $\alpha_1 \in E$ such that $T^{\varphi(f),j}\alpha_1 \to \alpha_1$ as an $\mathcal{F}$-sequence for every $1 \leqslant j \leqslant l$. In fact, this would follow at once from the multiple recurrence theorem for IP-systems [4, Th. 8.19] if all the transformations $T^{f,j}$ were invertible. Now, since all the transformations in question are powers of T_a, we can pass, as in the proof of Theorem 2.6 in [4], from the system $(E, (T^{f,j})_{f\in\mathcal{F},1\leqslant j\leqslant l})$ to a system $(\tilde{E}, (\tilde{T}^{f,j})_{f\in\mathcal{F},1\leqslant j\leqslant l})$ in which all transformations are invertible. By [4, Th. 8.19] we can find a homomorphism φ and a point $\tilde{\alpha}_1 \in \tilde{E}$ such that $\tilde{T}^{\varphi(f),j}\tilde{\alpha}_1 \to \tilde{\alpha}_1$ as an $\mathcal{F}$-sequence for every $1 \leqslant j \leqslant l$. The image α_1 of $\tilde{\alpha}_1$ in E has the required properties. Replacing S_1 and α by a suitable IP-subset of S_1 and by another point of E, respectively, we may therefore assume that $T^{f,j}\alpha \to \alpha$ as an $\mathcal{F}$-sequence for each j. For $f_1, f_2, \ldots, f_r \in \mathcal{F}$, with $f_1 < f_2 < \ldots < f_r$, denote

$$\varrho_{f_1,j_1,\ldots,f_r,j_r} = \|T^{f_1,j_1} \ldots T^{f_r,j_r}\alpha - \alpha\|.$$

Passing to an IP-subset of S_1, we may assume that $\varrho_{f_1,j_1,\ldots,f_r,j_r} \to 0$ as $\min f_1 \to \infty$. Given any non-negative integers r and s we then have

$$a^r \prod_{i=t}^{t+s-1} \Big(\sum_{j=0}^{l} d_j a^{(l-j)(m_i+k)}\Big)\alpha \xrightarrow[t\to\infty]{} a^r b^s \alpha.$$

Consequently $a^r b^s \alpha \in \overline{\Delta\alpha}$ for every r and s, whence $\overline{\Delta\alpha} = \boldsymbol{T}$. This proves the proposition.

PROPOSITION III.4. *Suppose that $\overline{\langle a\rangle^{\infty}\alpha}$ contains no infinite a-minimal set. Then for every positive integer l there exists a sequence (m_i) and torsion elements $(x_j)_{j=1}^{l}$ in T such that*

$$a^{jm_i}\alpha \xrightarrow[i\to\infty]{} x_j, \qquad 1 \leqslant j \leqslant l.$$

Proof. Employ induction on l. The case $l=1$ is clear. Assume the proposition to be true for $l-1$. Take $(m_i)_{i=1}^{\infty}$ such that each of the sequences $(a^{jm_i}\alpha)_{i=1}^{\infty}$, $1 \leqslant j \leqslant l-1$, converges to a torsion element of T. Replacing α by a suitable integer multiple thereof, we may assume that $a^{jm_i}\alpha \xrightarrow[i\to\infty]{} 0$, $1 \leqslant j \leqslant l-1$. Passing to a subsequence of (m_i) we may assume that the sequence $(a^{lm_i}\alpha)_{i=1}^{\infty}$ is convergent, say $a^{lm_i}\alpha \xrightarrow[i\to\infty]{} \beta$. Take a sequence (n_i) such that $a^{ln_i}\beta \xrightarrow[i\to\infty]{} x$, where x is a torsion element. Substituting for (m_i) a sufficiently rapidly growing subsequence thereof, we may assume that

$$a^{j(m_i+n_i)}\alpha \xrightarrow[i\to\infty]{} 0, \qquad 1 \leqslant j \leqslant l-1,$$
$$a^{l(m_i+n_i)}\alpha \xrightarrow[i\to\infty]{} x.$$

This proves the proposition.

PROPOSITION III.5. *Let*

$$\Delta = \prod_{n\in S} \langle d_0 a^{ln} + d_1 a^{(l-1)n} + \ldots + d_l\rangle$$

where $d_0 \neq 0$, $d_l = \pm 1$ and S is syndetic. Suppose that there exists a sequence (m_i) such that for every j with $d_j \neq 0$ the sequence $(a^{(l-j)m_i}\alpha)_{i=1}^{\infty}$ converges to a torsion element of T. Then $\overline{\Delta\alpha} = T$.

Proof. Set $D = \{j\colon 0 \leqslant j \leqslant l-1,\ d_j \neq 0\}$. It is easy to see that we may assume that

$$a^{(l-j)m_i}\alpha \xrightarrow[i\to\infty]{} 0, \qquad j \in D. \tag{3}$$

Write $D = \{j_1, j_2, \ldots, j_t\}$ with $0 = j_1 < j_2 < \ldots < j_t \leqslant l-1$. Put

$$A = \begin{bmatrix} a^{l-j_1} & & & 0 \\ & a^{l-j_2} & & \\ & & \ddots & \\ 0 & & & a^{l-j_t} \end{bmatrix} \quad \text{and} \quad \bar{\alpha} = (\alpha, \alpha, \ldots, \alpha)^T.$$

Consider A and $\bar{\alpha}$ as an endomorphism of T^t and as a point of T^t, respectively. Set $E = \overline{\{A^n\alpha\colon n \geqslant 0\}}$. From (3) it is easy to conclude that E contains an infinite subset of $Z(a^{\infty})^t$. Similarly to Lemma II.5, this implies the existence of a-adic integers $(\xi_j)_{j\in D}$, not all of which are 0, such that

$$(\{\xi_j/a^{(l-j)n}\})_{j\in D}^{T} \in E \quad \text{for every } n \in N.$$

Take a prime divisor p of a such that the p-adic component of one of the ξ_j's is non-zero. Let D_1 be the subset of D consisting of those j's for which the p-adic component of ξ_j is non-zero. Select s such that at least one out of any s consecutive positive integers belongs to S. Let d be an arbitrary fixed positive integer.

We can find a sequence (n_i) in S such that

$$a^{(l-j)n_i}\alpha \xrightarrow[i\to\infty]{} \{\xi_j/a^{(l-j)M}\}, \qquad j \in D$$

where $M \geqslant d + \binom{l}{2}s + \max_{j\in D_1} n_p(d_j\xi_j)$. Passing to a subsequence of (n_i), we may assume that there exist integers $(r_k)_{k=0}^{\binom{l}{2}}$, with $0 < r_{k+1} - r_k \leqslant s$ for $0 \leqslant k < \binom{l}{2}$, such that $n_i + r_k \in S$ for every $i \in N$ and $0 \leqslant k \leqslant \binom{l}{2}$. For each $0 \leqslant k \leqslant \binom{l}{2}$ consider the numbers $n_p(d_j\xi_j/\alpha^{(l-j)(M-r_k)})$, $j \in D_1$. Obviously, an equality of the form

$$n_p(d_{j_1}\xi_{j_1}/a^{(l-j_1)(M-r_k)}) = n_p(d_{j_2}\xi_{j_2}/a^{(l-j_2)(M-r_k)})$$

for some fixed $j_1, j_2 \in D_1$, $j_1 \neq j_2$, can hold for at most a single k. Hence for an appropriately chosen k there exists a $\underline{j} \in D_1$ such that

$$n_p(d_j\xi_j/a^{(l-j)(M-r_k)}) > n_p(d_{\underline{j}}\xi_{\underline{j}}/a^{(l-\underline{j})(M-r_k)}), \qquad j \in D_1 \setminus \{\underline{j}\}.$$

Replace the sequence $(n_i)_{i=1}^{\infty}$ by $(n_i + r_k)_{i=1}^{\infty}$ for that k. We have then

$$a^{(l-j)n_i}\alpha \xrightarrow[i\to\infty]{} \{\xi_j/a^{(l-j)N}\}, \qquad j \in D$$

where $N \geqslant d + \max_{j\in D_1} n_p(d_j\xi_j)$. Putting $N_k = \sum_{i=1}^{k} n_i$ we may assume, passing to a subsequence of (n_i), that

$$a^{lN_k}\|a^{n_1+(l-j)n_{k+1}}\alpha\| \xrightarrow[k\to\infty]{} 0 \quad \text{for every } j \in D.$$

Then for each k

$$\prod_{i=t}^{t+2k-1} \Big(\sum_{j=0}^{l} d_j a^{(l-j)n_i}\Big)\alpha \xrightarrow[t\to\infty]{} d_l^{2k}\alpha + 2kd_l^{2k-1}\Big\{\sum_{j\in D} d_j\xi_j/a^{(l-j)N}\Big\}$$
$$= \alpha + k\Big\{2d_l \sum_{j\in D} d_j\xi_j/a^{(l-j)N}\Big\}.$$

It follows that $\overline{\Delta\alpha}$ contains a translate of a subgroup of T whose order is at least $p^d/2$. Since d is arbitrary, this means that $\overline{\Delta\alpha} = T$. This completes the proof.

Theorem III.2 follows from Propositions III.3–III.5.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
Los Angeles, California 90024, U.S.A.

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