ACTA ARITHMETICA
XLIX (1987)

On exceptions to Szegedy's theorem

by

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*Thanks once more to Professor Erdős
for so many things he did for us*

1. Introduction and main result. Finally the attempts [1], [3], [5], [6], [7] to answer a question of R. L. Graham on

$$\max_{\substack{i \neq j \\ 1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant n}} \frac{a_i}{(a_i, a_j)}$$

ended in the following result of M. Szegedy:

If $n \geqslant n_0$, where n_0 is an explicitly computable constant, then for any n distinct positive integers $a_1, a_2, \ldots, a_n$:

$$\max_{i,j} \frac{a_i}{(a_i, a_j)} \geqslant n. \tag{00}$$

This beautiful result being asymptotic; the way is not closed for combinatorially minded researchers to look for possible exceptions or to look for a proof that no exceptions exist.

For some particular cases this has been done in the papers cited above.

The main result of this paper is formulated in the following theorem:

THEOREM 1. *Let $a_1 < a_2 < \ldots < a_n$ be natural integers, $n \geqslant 2$. If s is the smallest number of primes such that each a_i, $i = 1, 2, \ldots, n$ is a product of powers of those primes and $s \leqslant 5$ then*

$$\max_{i,j} \frac{a_i}{(a_i, a_j)} \geqslant n.$$

The methods of this paper also provide an easy proof of the following theorem, a weaker version of Szegedy's result.

THEOREM 2. *Let $\{a_i\}_{i=1}^n$ and s be as in Theorem 1 but without the condition $s \leqslant 5$. Then for sufficiently large n*

$$\max \frac{a_i}{(a_i, a_1)} \geqslant n;$$

i.e., for fixed s there is a n_0 such that the conclusion holds for $n \geqslant n_0$.

2. Definitions, notation. Let $a_1 < a_2 < \ldots < a_n$ be natural integers, $n \geqslant 2$ and put $A = \{a_1, a_2, \ldots, a_n\}$. Denote the quotient $a_i/(a_i, a_j)$ by g_{ij}. We shall also use $g(a, b) = a/(a, b)$. Let the least common multiple of the members of A have the decomposition into different primes $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s}$. Then each a_i has the form

$$a_i = p_1^{\alpha_{1i}} p_2^{\alpha_{2i}} \ldots p_s^{\alpha_{si}} \quad \text{with} \quad 0 \leqslant \alpha_{mi} \leqslant \alpha_m \tag{0}$$

and therefore $g_{ij} = p_1^{\gamma_1} p_2^{\gamma_2} \ldots p_s^{\gamma_s}$ with $0 \leqslant \gamma_m \leqslant \alpha_{mi}$.

A set A will be called *good* respectively *bad* according as it satisfies (00) or not.

Since in our consideration only the set $\{g_{ij}\}$ matters we can and shall assume that the largest common divisor of the members in A is 1. We shall refer to this as Assumption A.

We shall also assume that the primes occurring in (0) are the first primes, i.e., $p_1 = 2$, $p_2 = 3$, $p_3 = 5, \ldots$ This is justified by observing that on replacing any s primes by the first s primes the corresponding g_{ij}'s will not increase.

A well-known [5] observation is that if M is the least common multiple of the members of A then the set $\{g_{ij}\}$, $\{i, j\} \subset \{1, 2, \ldots, n\}$ is the same as the set $\{g'_{ij}\}$ corresponding to $A' = \left\{\frac{M}{a_1}, \frac{M}{a_2}, \ldots, \frac{M}{a_n}\right\}$. We shall refer to this as the Symmetry property.

3. Some basic propositions and consequences.

PROPOSITION 1. *Let A, $\{p_i\}_{i=1}^s$ and $\{\alpha_i\}_{i=1}^n$ be as above. If for some m and i*

$$p_m^{\alpha_{mi}} \geqslant n \tag{1}$$

then A is good.

Proof. By Assumption A there is some j such that a_j is not a multiple of p_m; then $g_{ij} \geqslant p_m^{\alpha_{mi}} \geqslant n$.

COROLLARY 1. *If A is bad then for $m = 1, 2, \ldots, s$ and $i = 1, 2, \ldots, n$*

$$0 \leqslant \alpha_{mi} < \frac{\log n}{\log p_m}. \tag{2}$$

DEFINITION 1. For given n denote the set of integers of the form

$$b = p_1^{\beta_1} p_2^{\beta_2} \ldots p_s^{\beta_s} \quad \text{with} \quad 0 \leqslant \beta_m < \frac{\log n}{\log p_m}$$

by $\boldsymbol{B}_s(n)$. Further denote the number of members of $\boldsymbol{B}_s(n)$ by $N_s(n)$.

OBSERVATION 1.

$$N_s(n) = \prod_{m=1}^{s} \left\lceil \frac{\log n}{\log p_m} \right\rceil.$$

COROLLARY 2. *If A is bad then*

$$A \subset \boldsymbol{B}_s(n) \quad \text{and} \quad n \leqslant \prod_{m=1}^{s} \left\lceil \frac{\log n}{\log p_m} \right\rceil.$$

It is convenient sometimes to use

COROLLARY 3. *If $n > N_s(n)$ then A is good.*

PROPOSITION 2. *There is an integer n_1 depending only on s such that*

$$n \geqslant n_1 \quad \text{implies} \quad \frac{N_s(n)}{n} < 1. \tag{3}$$

Proof. This can be seen by elementary calculus, using

$$N_s(n) \leqslant \left\lceil \frac{\log n}{\log 2} \right\rceil^s.$$

DEFINITION 2. An integer n' will be called *good* if (3) holds for $n' = n_1$.

COROLLARY 4. *If $|A| = n$ and $n \geqslant n'$ and n' is good then A is good.*

PROPOSITION 3. *If $\boldsymbol{B}_s(n)$ contains two members b_1 and b_2 such that one of $g(b_1, b_2)$ and $g(b_2, b_1)$ is at least n then if A is bad it does not contain both.*

PROPOSITION 4. *If $\boldsymbol{B}_s(n)$ contains k pairwise disjoint pairs b_i, b_j as in Proposition 3 then $n > N_s(n) - k$ implies A is good.*

PROPOSITION 5. *Let A be $\{a_1, a_2, \ldots, a_n\}$, and as before $a_1 < a_2 < \ldots < a_n$. If there is no bad subset of A containing a_i then there is no such subset containing a_{n-i+1}.*

Proof. This follows from the Symmetry property.

Proof of Theorem 2. For any $n \geqslant n_1$, where n_1 is good, the assumption of Corollary 3 is fulfilled; therefore A is good, i.e., n_0 in Theorem 2 can be taken to be n_1.

Remark 1. Since "2 is good for every s" would mean that there are no exceptions to Szegedy's theorem, it is clearly of interest to determine good values as small as we can. We shall do that in Section 4.1.

4. Proof of Theorem 1. The proof is based on facts from Section 3 and on numerical results obtained by computation. In particular the theorem will follow straight away from Proposition 6 in 4.2 and Proposition 7 in 4.3.

4.1. An algorithm. It is not too difficult to decide for a fixed s whether a set N_s of N_s numbers contains a bad set of size n_s or less if N_s and n_s are not too large. We could formulate and program on a computer such an algorithm, based essentially on trial and error and have the answer in reasonable time if $N_s \leqslant 100000$ with $n_s \leqslant 150$, while for $N_s \leqslant 15000$ even with $n_s \leqslant 7500$. These bounds are sufficient for our purposes and are not sharp. In this way the first step in proving Theorem 1 is made.

4.2. A procedure for obtaining smallest good integers.

Procedure 1.

Suppose $n^{(1)}$ is good; put $N_s(n^{(1)}) = l_1$ and $n^{(2)} = l_1+1$. Then $N_s(n^{(2)}) \leqslant N_s(n^{(1)}) < n^{(2)}$, and $n^{(2)}$ is a good value smaller than $n^{(1)}$, provided $l_1+1 < n_1$; notice $l_1+1 \leqslant n_1$ holds always.

Repeating the above procedure one defines a sequence

$$n^{(1)} > n^{(2)} > \ldots > n^{(k)} = n^{(k+1)}$$

and the value $n^{(k)}$ cannot be improved in the same way.

DEFINITION 3. The smallest good value which may be obtained by Procedure 1 will be denoted by $n^{(0)}$ and $n^0(s)$ emphasizing s.

EXAMPLE 1. Put $s = 2$, i.e., $p_1 = 2$, $p_2 = 3$, then 64 is a good value, since

$$24 = 6\cdot 4 = \left\lceil \frac{\log 64}{\log 2} \right\rceil \left\lceil \frac{\log 64}{\log 3} \right\rceil < 64.$$

Put

$l_1 = 24$, $n^{(1)} = 64$, then $n^{(2)} = 25$.

$$l_2 = \left\lceil \frac{\log 25}{\log 2} \right\rceil \left\lceil \frac{\log 25}{\log 3} \right\rceil = 5\cdot 3 = 15, \text{ then } n^{(3)} = 16.$$

$$l_3 = \left\lceil \frac{\log 16}{\log 2} \right\rceil \left\lceil \frac{\log 16}{\log 3} \right\rceil = 4\cdot 3 = 12, \text{ then } n^{(4)} = 13.$$

$$l_4 = \left\lceil \frac{\log 13}{\log 2} \right\rceil \left\lceil \frac{\log 13}{\log 3} \right\rceil = 4\cdot 3 = 12, \text{ therefore } n^{(0)} = 13.$$

PROPOSITION 6. $n^0(2) = 13$, $n^0(3) = 160$, $n^0(4) = 1540$, $n^0(5) = 33600$.

Proof. The above values can be obtained by computation using Procedure 1.

Remark 2. Proposition 6 shows that smallest good values are not good enough to prove that for fixed s Szegedy's theorem is true with no exceptions.

DEFINITION 4. An integer n'' is *bad* if for every bad A it follows that $n \leqslant n''$. Notice that here (3) is not used.

COROLLARY 5. $\bar{n}_1 = n^{(0)}-1$ *is bad.*

Remark 3. "1 is bad" would mean that there are no exceptions, so it is of interest to find small bad values.

4.3. A procedure for obtaining small bad values.

Procedure 2.

Consider $B_s(n_s^{(0)}-1)$; determine as many as possible pairs b_i, b_j as in Proposition 3. Suppose there are k_1 such pairs. Put $\bar{n}_2 = N(\bar{n}_1)-k_1$. If $k_1 > 0$ then this value is bad and certainly smaller than $\bar{n}_1$. Repeating this procedure one defines a sequence $\bar{n}_1 > \bar{n}_2 > \ldots > \bar{n}_k = \bar{n}_k+1$ where $\bar{n}_k$ cannot be improved by this procedure.

DEFINITION 5. Write $\bar{n}$, and emphasizing s, $\bar{n}(s)$, for the smallest bad value which can be obtained in this way.

EXAMPLE 2. Let $s = 2$. Then $\bar{n}_1(2) = n_2^{(0)}-1 = 12$.

$$B_2(12) = \begin{Bmatrix} 1 & 2 & 4 & 8 \\ 3 & 6 & 12 & 24 \\ 9 & 18 & 36 & 72 \end{Bmatrix}.$$

Observe that the 4 pairs $\{1, 12\}$, $\{2, 24\}$, $\{3, 36\}$, $\{6, 72\}$ are as required. Therefore $\bar{n}_2 = 12-4 = 8$; then consider

$$B_2(8) = \begin{Bmatrix} 1 & 2 & 4 & 8 \\ 3 & 6 & 12 & 24 \end{Bmatrix}$$

containing the pairs $\{3, 24\}$, $\{1, 8\}$, $\bar{n}_3 = 8-2 = 6$,

$$B_2(6) = \begin{Bmatrix} 1 & 2 & 4 \\ 3 & 6 & 12 \end{Bmatrix}$$

with pairs $\{2, 12\}$, $\{1, 6\}$, $\bar{n}_4 = 6-2 = 4$, $B_2(4) = \{1, 2, 3, 4\}$ containing $\{1, 4\}$, $\bar{n}_5 = 4-1 = 3$, $B_2(3) = \{1, 2, 3\}$ containing $\{1, 3\}$, $\bar{n}_6 = 2$, $B_2(2) = \{1, 2\}$, $\bar{n}_7 = 1$ so $\bar{n} = 1$.

PROPOSITION 7. $\bar{n}(2) = 1$, $\bar{n}(3) = 44$, $\bar{n}(4) = 759$, $\bar{n}(5) = 7350$.

Proof. The above values are obtained by Procedure 2.

Final remark. As already mentioned the numerical results in Sections 4 are not sharp. To improve Theorem 1, i.e., to prove its validity for values of s larger than 5, one should apply more powerful tools than those used in Section 4.

Another way to improve our results would be to replace the inequality in Corollary 2 by a stronger inequality, namely

$$n \leqslant c_s \prod_{m=1}^{s} \left[\frac{\log n}{\log p_m}\right],$$

where c_s depends only on s and decreases rapidly with it. Some heuristic evidence, based on geometric considerations and on a refinement of the Symmetry property, indicates the existence of such a constant; this could lead to prove Szegedy's theorem with no exceptions.

Finally, this would also follow from the following conjecture in the spirit of [3].

Conjecture (Schönheim). If $a_1 < a_2 < \ldots < a_n$, $n \geqslant 2$ are natural integers and $a_n/a_1 < n$ then

$$\left|\left\{\frac{a_i}{(a_i, a_j)}\right\}\right| \geqslant n.$$

References

[1] R. D. Boyle, *On a problem of R. L. Graham*, Acta Arith. 34 (1968), pp. 163–177.

[2] R. L. Graham, *Unsolved problem 5749*, Amer. Math. Monthly 77 (1970), p. 775.

[3] J. Marica and J. Schönheim, *Differences of sets and a problem of Graham*, Canad. Math. Bull. 12 (1969), pp. 635–637.

[4] M. Szegedy, *The solution of Graham's GCD problem* (to appear).

[5] W. Y. Vélez, *Some remarks on a number theoretic problem of Graham*, Acta. Arith. 32 (1977), pp. 233–238.

[6] G. Weinstein, *On a conjecture of Graham concerning greatest common divisors*, Proc. Amer. Math. Soc. 63 (1977), pp. 33–38.

[7] R. Winterle, *A problem of R. L. Graham*; in: *Combinatorial Number Theory Proceedings*, Louisiana Conference on Combinatorics, Baton Rouge 1970, pp. 357–361.

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Received on 28.2.1986 (1596)