[3] G. Chrystal, *Textbook of Algebra, Part Two*, Dover, New York 1961.
[4] M. Herzog and J. Schönheim, *Research problem No. 9*, Canad. Math. Bull. 17 (1974), 150.
[5] J. J. Rotman, *The Theory of Groups*, Allyn and Bacon, Boston 1973.

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Forcing smooth square roots and integration

by

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Abstract. This paper is concerned with models of Synthetic Differential Geometry (SDG, cf. Introduction).

We give affirmative answers to the following questions:

1) Is the existence of square roots of nonnegative (smooth) reals compatible with the axioms of SDG?

2) Does the integration axioms ("every functions from $[0, 1]$ into R has a unique primitive vanishing at 0") hold in the generic (local) Archemedean C^{∞}-ring?

Introduction. This paper is a contribution to the study of models of Synthetic Differential Geometry (SDG). The aim of this theory is to give an intrinsic, naïve axiomatization of Differential Geometry as a foundation for the synthetic reasoning used by people like Darboux, Lie, Cartan (as well as physicists and engineers) in this field. Its basic notions are those of a commutative ring with 1, R ("the (smooth) reals") and its subset D of elements of square 0 ("infinitesimals of first order"). The basic assumption, the Kock-Lawvere axiom, asserts that D is large enough to make the map $\alpha\colon R\times R \to R^D$ invertible, where $\alpha(a, b)(h) = a+bh$, $\forall h \in D$. ("In the infinitely small, any curve is a line").

Since this axiom is incompatible with classical logic, no set-theoretical models exist for this theory. On the other hand, several topos-theoretical models have been constructed, showing the compatibility of SDG with intuitionistic logic. Many of these will be described in this paper.

Further developments of SDG require, naturally, more axioms on R. We shall assume that R is a local ring equipped with order relations $<$ and $\leqslant$ which are compatible with the ring structure and with each other i.e. we assume the following

Axioms (*):

$\neg 0 = 1$,

$\forall x \in R(x \text{ invertible} \vee (1-x) \text{ invertible})$,

$0 < 1$,

$\forall x, y \in R(0 < x \wedge 0 < y \to 0 < x+y \wedge 0 < x\cdot y)$,

$0 \leqslant 1$,

$\forall x, y \in R(0 \leqslant x \wedge 0 \leqslant y \to 0 \leqslant x+y \wedge 0 \leqslant x \cdot y)$,
$\forall x \in R(0 < x \to 0 \leqslant x)$,
$\forall x \in R(0 < x \to \neg x \leqslant 0)$,
$\forall x \in R(0 < x \leftrightarrow \exists y\,(y \text{ invertible} \wedge x = y^2))$,
$\forall x \in R(x \text{ invertible} \leftrightarrow x < 0 \vee x > 0)$.

(The first two axioms say that R is a local ring). For further information on SDG see Kock [1981], as well as a forthcoming monograph Moerdijk–Reyes [to appear]. For further information on topos theory (in particular generic structures and classifying toposes), see Johnstone [1977] or Makkai–Reyes [1977].

It is well known that for a smooth nonnegative function $f\colon R \to R$, the function $x \mapsto \sqrt{f(x)}$ need not be smooth (it need not even be C^2). This implies that the generic C^∞-ring R in the classifying topos for C^∞-rings (see section 2) does not satisfy the axiom

(1) $$\forall x \in R(x \geqslant 0 \to \exists y \in R(x = y^2)).$$

In fact, this axiom also fails for the generic local C^∞-ring, the generic local Archimedean C^∞-ring, and for the models of synthetic differential geometry (SDG) that have recently been studied, such as Dubuc's topos $\mathscr{G}$ (Dubuc (1981), Moerdijk & Reyes (1984)) or the smooth Zariski topos $\mathscr{Z}$ (Moerdijk & Reyes (1983)). All these toposes will be described below, in Section 2.

Thus it can be asked whether it is consistent to add the axiom (1) to SDG, or more specifically, whether there is a model of SDG in which (1) holds, and moreover the category of manifolds is fully and faithfully embedded (really, one would want the stronger condition that the category of finitely presented C^∞-rings is fully and faithfully embedded in the model).

A second, apparently unrelated question is whether the integration axiom of SDG (first discussed in Kock & Reyes (1981)),

(2) $$\forall f \in R^{[0,1]} \exists!\, g \in R^{[0,1]}(g(0) = 0 \wedge g' \equiv f)$$

is true of the generic (local) (Archimedean) C^∞-ring R in the corresponding classifying topos.

The aim of this note is to point out that both questions are easily answered affirmatively, as a consequence of the following two results.

Lemma 1 (Whitney's lemma on even functions). *Let $f(x,t)\colon R^n \times R \to R$ be a smooth function such that $f(x,t) \equiv f(x,-t)$. Then there is a smooth $g(x,t)\colon R^n \times R \to R$ such that $f(x,t) \equiv g(x,t^2)$.*

This lemma is a well-known consequence of the Malgrange preparation theorem, but there is a simple direct proof, which we will give below for the sake of completeness. The following lemma seems to be new. (Here $C^\infty(M)$ denotes the ring of smooth functions on M, and for $\varphi \in C^\infty(M)$, $(\varphi) \subset C^\infty(M)$ denotes the ideal generated by φ.)

Lemma 2. *Let M be a manifold, and $f, g \in C^\infty(M)$. Then*
$$f(x) \in (g(x) - t^2) \subset C^\infty(M \times R)$$
iff for some $\varrho \in C^\infty(R)$ which vanishes on $R_{\geqslant 0}$, $f(x) \in (\varrho(g(x))) \subset C^\infty(M)$.

1. Proof of the two lemmas. First of all, we point out that Whitney's lemma follows easily from Borel's theorem, which states that the Taylor-expansion at 0 in the t-coordinate
$$C^\infty(R^m \times R) \xrightarrow{T_0} C^\infty(R^m)[[t]]$$
is a surjective ring-homomorphism (see e.g. Martinet (1982), VI 7.4). To see this, suppose $f(x,t) = f(x,-t)$. Then $T_0(f)$ must be of the form $\sum_{n\geqslant 0}\varphi_n(x)t^{2n}$. By Borel's theorem, there exists an $h \in C^\infty(R^m \times R)$ such that $T_0(h) = \sum_{n\geqslant 0}\varphi_n(x)t^n$. Then $k(x,t) = h(x,t^2) - f(x,t)$ is flat (i.e., all partial derivatives vanish) at $(x,0)$ for every x, and from this it follows easily that the function $l(x,t) = k(x,\sqrt{|t|})$ is smooth. Moreover, $f(x,t) = h(x,t^2) - l(x,t^2)$. This proves Whitney's lemma.

The proof of lemma 2 uses a result on flat functions which we state below as lemma 3. First, some notation. If X is a closed subset of R^n, we let L_X^∞ be the set of locally bounded functions f on R^n which are flat in the sense that
$$\forall n \in N\ \forall \varepsilon > 0\ \exists \delta > 0\big(d(x,X) < \delta \to |f(x)|\, d(x,X)^{-n} < \varepsilon\big).$$

Lemma 3. *Let X, Y be closed subsets of R^n, R^m respectively, and let $\{f_p\}_p$ be a sequence of functions in $L^\infty_{X\times Y}$. Then there are nonnegative (smooth) functions $\varphi \in C^\infty(R^n)$, $\psi \in C^\infty(R^m)$ vanishing only on X and Y respectively and such that*
$$\forall p \in N\ \forall q \in N\ \frac{f_p(x,y)}{(\varphi(x)+\psi(y))^q} \to 0$$
whenever $d((x,y), X\times Y) \to 0$.

Proof (in sketch). By considering the sequence $\{f_p^{1/q}\}_p$, which is still in $L^\infty_{X\times Y}$, we may assume that $q = 1$.

We distinguish two cases: if X, Y are both compact, this is proved in Quê–Reyes [1982, Lemma 2].

For the general case, let $\{U_i\}_i$, $\{V_j\}_j$ be locally finite bounded open covers of R^n, R^m respectively, and let $\{\varrho_i\}_i$, $\{\nu_j\}_j$ be partitions of unity subordinated to them. We further let, for each i and j,
$$X_i = X \cap \operatorname{support}(\varrho_i), \qquad Y_j = Y \cap \operatorname{support}(\nu_j).$$
Since X_i, X_j are compact, we can find (by the first case) functions $\varphi_i \in C^\infty(R^n)$, $\psi_j \in C^\infty(R^m)$ satisfying the conclusion.

We define $\varphi(x) = \sum_i \varrho_i(x)\varphi_i(x)$, $\psi(y) = \sum_j \nu_j(y)\psi_j(y)$ and check that
$$\frac{f_p(x,y)}{\varphi(x)+\psi(y)} \to 0, \quad \text{whenever } d((x,y), X\times Y) \to 0. \ \blacksquare$$

Next, we prove the second lemma stated above. It suffices to consider the case $M = \mathbf{R}^m$. Note that $\Leftarrow$ is obvious, since if $\varrho(x)\colon \mathbf{R}\to\mathbf{R}$ vanishes on $\mathbf{R}_{\geqslant 0}$, then $\varrho(x)\in(x-t^2)\subset C^\infty(\mathbf{R}^2)$. For $\Rightarrow$, suppose $f, g\colon \mathbf{R}^m\to\mathbf{R}$ are smooth, and $f(x) = A(x,t)(g(x)-t^2)$ for some smooth $A\colon \mathbf{R}^m\times\mathbf{R}\to\mathbf{R}$. We first prove that

(3) $$f(x)\in(g^n(x))\subset C^\infty(\mathbf{R}^m\times\mathbf{R}), \quad \text{for all } n,$$

by defining by induction functions $B_n(x,t)$ such that

$(4)_n$ $$g^n(x)B_n(x,t^2)(g(x)-t^2) = f(x).$$

For $n=0$, note that $f(x) = \frac{1}{2}(A(x,t)+A(x,-t))(g(x)-t^2)$. By Whitney's lemma, $\frac{1}{2}(A(x,t)+A(x,-t)) = B_0(x,t^2)$ for some B_0. If B_n is given such that $(4)_n$ holds, then by putting $t=0$, we find $g^n(x)B_n(x,t^2)(g(x)-t^2) = g^{n+1}(x)B_n(x,0)$. Writing $B_n(x,s) = B_n(x,0)+sB_{n+1}(x,s)$, we get $g^{n+1}(x)B_{n+1}(x,t^2) = g^n(x)B_n(x,t^2)$, so by $(4)_n$ we conclude that $f(x) = g^{n+1}(x)B_{n+1}(x)(g(x)-t^2)$, i.e. $(4)_{n+1}$ holds.

Clearly, from (3) we derive by induction on $|\alpha|$ that

(5) $$D^\alpha f\in(g^n)$$

for all n, and all multi-indices $\alpha = (\alpha_1,\ldots,\alpha_m)$. Let $F = \{x|\ g(x)\geqslant 0\}$. Since $D^\alpha f(x) = 0$ if $g(x)=0$ by (5), while $D^\alpha f(x) = 0$ if $g(x)>0$ (since by assumption, f vanishes on $\{x|\ g(x)>0\}$), f is flat on F.

Now define for $N\in\mathbf{N}-\{0\}$, $\alpha = (\alpha_1,\ldots,\alpha_m)$, and $t\geqslant 0$,

$$\varphi_N^\alpha(t) = \sup\{|D^\alpha f(x)|\colon |x|\leqslant N,\ g(x)\geqslant -t\},$$

and put $\varphi_N^\alpha(t) = 0$ for $t\leqslant 0$. We claim that for any $q>0$,

(6) $$\frac{\varphi_N^\alpha(t)}{|t^q|}\to 0 \quad \text{whenever } t\to 0.$$

Indeed for arbitrary fixed $t>0$ there is (by compactness) an x_t with $|x_t|\leqslant N$ and $-t\leqslant g(x_t)\leqslant 0$ such that $\varphi_N^\alpha(t) = |D^\alpha f(x_t)|$, and

$$\frac{|D^\alpha f(x_t)|}{|t^q|}\leqslant\frac{|D^\alpha f(x_t)|}{|g^q(x_t)|} = |g(x_t)\cdot A_{q+1}(x_t,t)|\leqslant|t|\cdot|A_{q+1}(x_t,t)|,$$

where A_{q+1} is the function witnessing $D^\alpha f\in(g^{q+1})$, see (5). Since x_t is bounded, $|A_{q+1}(x_t,t)|$ does not depend on t, so (6) follows.

By Lemma 3 there is a smooth function $\varphi\colon\mathbf{R}\to\mathbf{R}$, flat and vanishing only on $\mathbf{R}_{\leqslant 0}$, such that for all α, N and q,

(7) $$\frac{\varphi_N^\alpha(t)}{\varphi(t)^q}\to 0 \quad \text{whenever } t\to 0^+.$$

The function $\varrho(t) = \varphi(-t)$ is the one we need to prove the lemma. The function

$$k(x) = \begin{cases} f(x)/\varphi(-g(x)) & \text{if } g(x)<0,\\ 0 & \text{if } g(x)\geqslant 0\end{cases}$$

is smooth on $\mathbf{R}^m$, for if $\{x_n\}$ is a sequence of points converging to $x\in F$, then without loss $|x_n|\leqslant N$ for all n, so

$$|D^\alpha f(x_n)|\leqslant\sup\{|D^\alpha f(y)|\colon |y|\leqslant N,\ g(y)\geqslant g(x_n)\} = \varphi_N^\alpha(-g(x_n)),$$

and hence

$$\frac{|D^\alpha f(x_n)|}{|\varphi(-g(x_n))|^q}\leqslant\frac{|\varphi_N^\alpha(-g(x_n))|}{|\varphi(-g(x_n))|^q}\to 0 \quad \text{if } n\to\infty$$

by (7).

This completes the proof of the lemma.

Remark. The same proof gives the following generalization of Lemma 2 for a symmetric neighborhood V of 0 (i.e., $t\in V\Rightarrow -t\in V$) instead of $\mathbf{R}$ (since only symmetry is needed to apply Whitney's lemma).

LEMMA 2′. *Let M be a manifold, and $f, g\in C^\infty(M)$. Then*

$$f(x)\in(g(x)-t^2)\subset C^\infty(M\times V)$$

iff for some $\varrho\in C^\infty(\mathbf{R})$ which vanishes on $\mathbf{R}_{\geqslant 0}$, $f(x)\in(\varrho(g(x)))\subset C^\infty(M)$.

2. Description of some smooth toposes. We briefly recall the definition of some relevant toposes. References are given at the end of this section.

The category $\mathbf{L}$ of *loci* or *formal C^∞-varieties* has as objects formal duals $\overline{A}$ of finitely generated C^∞-rings A, i.e. rings A (isomorphic to ones) of the form $C^\infty(\mathbf{R}^n)/I$, where $n\geqslant 0$ and I is any ideal. Morphisms in $\mathbf{L}$ from one such dual $\overline{C^\infty(\mathbf{R}^n)/I}$ to another $\overline{C^\infty(\mathbf{R}^m)/J}$ are equivalence classes of smooth functions $\mathbf{R}^n\xrightarrow{\varphi}\mathbf{R}^m$ with the property that $f\in J\Rightarrow f\circ\varphi\in I$, where two such functions φ and φ' are equivalent if $\pi_i\circ\varphi-\pi_i\circ\varphi'\in I$, $i = 1,\ldots,m$. $\mathbf{F}$ is the full subcategory of $\mathbf{L}$ whose objects are (up to isomorphism) of the form $\overline{C^\infty(\mathbf{R}^n)/I}$ where I is a *closed* ideal (in the Fréchet topology on $C^\infty(\mathbf{R}^n)$ of uniform convergence of functions and their derivatives on compacta). $\mathbf{G}$ is the full subcategory of $\mathbf{L}$ whose objects are (up to isomorphism) of the form $\overline{C^\infty(\mathbf{R}^n)/I}$ where I is a *germ-determined* ideal (i.e. $f\in I\Leftrightarrow\Leftrightarrow\forall x\in\mathbf{R}^n\,\exists g\in I\ g|_x = f|_x$, $(-)|_x$ being the germ at x). Furthermore, let $\mathbf{C}$ be the full subcategory of $\mathbf{L}$ whose objects are of the form $\overline{C^\infty(\mathbf{R}^n)/I}$, where I is a finitely generated ideal. We have $\mathbf{F}\subset\mathbf{G}$ and $\mathbf{C}\subset\mathbf{G}$.

Each of the categories $\mathbf{C}$, $\mathbf{G}$, $\mathbf{F}$, $\mathbf{L}$ results in a presheaf topos $\mathrm{Sets}^{\mathbf{C}^{\mathrm{op}}}$, $\mathrm{Sets}^{\mathbf{G}^{\mathrm{op}}}$, $\mathrm{Sets}^{\mathbf{F}^{\mathrm{op}}}$, $\mathrm{Sets}^{\mathbf{L}^{\mathrm{op}}}$. $\mathrm{Sets}^{\mathbf{C}^{\mathrm{op}}}$ is the *classifying topos* for the algebraic theory of C^∞-rings, and is also denoted by $\mathscr{S}[C^\infty]$.

Each of these four categories can be equipped with the *finite open cover topology.* For $\boldsymbol{L}$, this is the Groethendieck topology generated by the basic covering families isomorphic to ones of the form

$$(8)\qquad \{\overline{C^\infty(U_i)|(I|U_i)} \hookrightarrow \overline{C^\infty(\boldsymbol{R}^n)/I}\}_{i=1}^k$$

where $U_1 \cup \ldots \cup U_k \supseteq Z(J)$ for some finitely generated ideal $J \subseteq I$ (U_i are open subsets of $\boldsymbol{R}^n$); $Z(J) = \{x \in \boldsymbol{R}^n |\ \forall f \in J\, f(x) = 0\}$. (Note that $\overline{C^\infty(U_i)/I|U_i}$ is isomorphic to an object of $\boldsymbol{L}$). The category of sheaves for this topology on $\boldsymbol{L}$ is the *smooth Zariski topos* Z. For $\boldsymbol{C}$, this Grothendieck topology can be described by taking as basic covering families those of the form (8), but now we (can) require $Z(I) \subseteq U_1 \cup \cup \ldots \cup U_k$. The category of sheaves on $\boldsymbol{C}$ for this topology is the classifying topos for *local C^∞-rings*, and is denoted by $\mathscr{S}[C^\infty_{\mathrm{loc}}]$. For $\boldsymbol{G}$, the covering families for the finite open topology are of the form

$$(9)\qquad \{\overline{C^\infty(U_i)/(I|U_i)^\sim} \hookrightarrow \overline{C^\infty(\boldsymbol{R}^n)/I}\}_{i=1}^k$$

where $Z(I) \subseteq U_1 \cup \ldots \cup U_k$, and $(I|U_i)^\sim$ is the smallest germ-determined ideal in $C^\infty(U_i)$ which contains $(I|U_i)$. The resulting topos of sheaves is the topos $\mathscr{G}_{\mathrm{fin}}$. For $\boldsymbol{F}$, we take the covering families as for $\boldsymbol{G}$, but with $(I/U_i)^\sim$ replaced by the closure $\overline{(I|U_i)}$ in the Fréchet topology. The resulting topos is denoted by $\mathscr{F}_{\mathrm{fin}}$. On each of $\boldsymbol{C}$, $\boldsymbol{G}$, $\boldsymbol{F}$, $\boldsymbol{L}$, the finite open cover topology is subcanonical.

Analogously, one can define the *open cover topology* on $\boldsymbol{C}$, $\boldsymbol{F}$, $\boldsymbol{G}$, where basic covers of an object $\overline{C^\infty(\boldsymbol{R}^n)/I}$ come from (countable) covers $\{U_\alpha\}_{\alpha \in \mathscr{A}}$ of $Z(I) \subset \boldsymbol{R}^n$. This results in the following toposes of sheaves: for $\boldsymbol{C}$, it gives the classifying topos for *Archimedean local C^∞-rings* $\mathscr{S}[C^\infty_{\mathrm{Ar\,loc}}]$, for $\boldsymbol{G}$ it gives the topos $\mathscr{G}$, and for $\boldsymbol{F}$ the topos $\mathscr{F}$. On each of $\boldsymbol{C}$, $\boldsymbol{F}$, $\boldsymbol{G}$ this open cover topology is subcanonical. Not so for $\boldsymbol{L}$, however, and consequently we will not consider the corresponding topos.

Some references: for $\boldsymbol{L}$, $\mathrm{Sets}^{\boldsymbol{L}^{\mathrm{op}}}$ and $\mathscr{Z}$: Quê and Reyes (1982), Reyes (1983), Moerdijk and Reyes (1983). For $\boldsymbol{G}$ and $\mathscr{G}$: Dubuc (1981), Moerdijk & Reyes (1984). For $\mathscr{G}_{\mathrm{fin}}$: Moerdijk & Reyes (1984). For $\mathscr{F}$, Bélair (1981), Kock (1981). All the toposes mentioned above are also extensively discussed in Moerdijk & Reyes (to appear). In these references, the toposes are studied as models for SDG. The classifying toposes have hardly been studied in the literature. (One reason may be that the validity of the integration axiom had not been established so far, so that their interest as models for SDG was not clear.)

3. Forcing smooth square roots. We will now point out how the first problem mentioned in the introduction, namely whether (1) is consistent with SDG, is solved by Lemmas 1 and 2. What we need is the following theorem. As usual, we write R for $\overline{C^\infty(\boldsymbol{R})}$ and $R_{\geqslant 0}$ for $\overline{C^\infty(\boldsymbol{R})/m^\infty_{[0,\to)}}$, $m^\infty_{[0,\to)}$ being the ideal of functions which vanish on $[0, \to) \subseteq \boldsymbol{R}$. The inclusion $R_{\geqslant 0} \hookrightarrow R$ in $\boldsymbol{L}$ (or $\boldsymbol{F}$, or $\boldsymbol{G}$) represents the preorder $x \geqslant 0$ in any of the models $\mathscr{Z}$, $\mathscr{F}$, $\mathscr{G}$, $\mathscr{F}_{\mathrm{fin}}$, $\mathscr{G}_{\mathrm{fin}}$ of SDG that were mentioned in section 2.

THEOREM 3. *The map $R \to R_{\geqslant 0}$ in $\boldsymbol{L}$ which is induced by $t \mapsto t^2$ is a stable effective epimorphism.*

Proof. Let $\bar{A} \xrightarrow{g} R_{\geqslant 0}$ be any map in $\boldsymbol{L}$, where $A = C^\infty(\boldsymbol{R}^n)/I$, say. So by definition of $\boldsymbol{L}$, $\varrho \circ g \in I$ for every $\varrho \in m^\infty_{[0,\to)}$. Consider the pullback square

$$\begin{tikzcd}
\bar{A} \arrow[r, "g"] & R_{\geqslant 0} \\
\bar{B} \arrow[u, "\pi_x"] \arrow[r, "\pi_t"'] & R \arrow[u, "t^2"']
\end{tikzcd}$$

where $B = C^\infty(\boldsymbol{R}^n \times \boldsymbol{R})/(I(x), g(x) - t^2)$. We have to show that for any given $\bar{C} \in \boldsymbol{L}$ and $\bar{B} \xrightarrow{f} \bar{C}$, if $f \circ p_1 = f \circ p_2$ then there exists a unique h such that $h \circ \pi_x = f$:

$$\begin{tikzcd}
\bar{B} \times_{\bar{A}} \bar{B} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[dr] & \bar{B} \arrow[r, "\pi_x"] \arrow[d, "f"] & \bar{A} \arrow[dl, dashed, "h"] \\
& \bar{C} &
\end{tikzcd}$$

It is easy to see that it suffices to prove the special case where $\bar{C} = R$. Since $\bar{B} \times_{\bar{A}} \bar{B}$ is the dual of $C^\infty(\boldsymbol{R}^n \times \boldsymbol{R} \times \boldsymbol{R})/(I(x), g(x) - t^2, t^2 - s^2)$, the *existence* of h can be formulated as

$$(10)\qquad f(x, t) - f(x, s) \in (I(x), g(x) - t^2, t^2 - s^2) \Rightarrow \exists h(x)\ \ f(x, t) - h(x) \in (I(x), g(x) - t^2)\,.$$

The *uniqueness* of h comes down to

$$(11)\qquad k(x) \in (I(x), g(x) - t^2) \Rightarrow k \in I\,.$$

Now (10) follows easily from Lemma 1: write $f(x, t) = \frac{1}{2}(f(x, t) + f(x, -t)) + \frac{1}{2}(f(x, t) - f(x, -t))$. By Lemma 1, $\frac{1}{2}(f(x, t) + f(x, -t)) = \tilde{f}(x, t^2)$ for some $\tilde{f}$, so $f(x, t) - \tilde{f}(x, t^2) \in (I(x), g(x) - t^2)$, and we can put $h(x) = \tilde{f}(x, g(x))$.

Lemma 2 was formulated precisely to prove (11). Indeed, suppose $k(x) \in (I(x), g(x) - t^2)$, say $k(x) = \sum_{i=1}^n A_i(x, t)\varphi_i(x) + B(x, t)(g(x) - t^2)$, where $\varphi_i \in I$. Using Lemma 1,

$$k(x) = \sum \tfrac{1}{2}(A_i(x, t) + A_i(x, -t))\varphi_i(x) + \tfrac{1}{2}(B(x, t) + B(x, -t))(g(x) - t^2)$$
$$= \sum \tilde{A}_i(x, t^2)\varphi_i(x) + \tilde{B}(x, t^2)(g(x) - t^2)\,.$$

Since $A_i(x, g(x)) - A_i(x, t^2) \in (g(x) - t^2)$, we can write $k(x) = \sum \tilde{A}_i(x, g(x))\varphi_i(x) + C(x, t^2)(g(x) - t^2)$. By Lemma 2, $C(x, t^2)(g(x) - t^2) \in (\varrho \circ g(x))$ for some $\varrho \in m^\infty_{[0,\to)}$. But $\varrho \circ g \in I$, and therefore $k(x) \in I$.

This completes the proof.

THEOREM 4. *The map* $R \xrightarrow{t^2} R_{\geq 0}$ *is also a stable effective epimorphism in* $\boldsymbol{F}$ *and in* $\boldsymbol{G}$.

Proof. Let us first consider the case of $\boldsymbol{F}$. Reasoning as in the proof of Theorem 3, *existence* now means that for a given *closed* ideal I,

$$(10') \quad f(x,t)-f(x,s) \in \mathrm{Cl}(I(x), g(x)-t^2, t^2-s^2) \Rightarrow$$
$$\Rightarrow \exists h(x) \;\; f(x,t)-h(x) \in \mathrm{Cl}(I(x), g(x)-t^2)$$

where Cl denotes the closure in the Fréchet topology), while uniqueness comes down to

$$(11') \qquad k(x) \in \mathrm{Cl}(I(x), g(x)-t^2) \Rightarrow k(x) \in I.$$

(10′) is proved just as (10), but using the following:

SUBSTITUTION LEMMA. *Let* J *be a closed ideal in* $C^\infty(\boldsymbol{R}^m)$, *and let* $\varphi(x,y)$, $\psi(x,y)$: $\boldsymbol{R}^m \times \boldsymbol{R}^k \to \boldsymbol{R}$, $u(x)$: $\boldsymbol{R}^m \to \boldsymbol{R}^k$ *be smooth functions. If*

$$\varphi(x,y) \in \mathrm{Cl}(J(x), \psi(x,y))$$

in $C^\infty(\boldsymbol{R}^{m+k})$, *then*

$$\varphi(x, u(x)) \in \mathrm{Cl}(J(x), \psi(x, u(x)))$$

in $C^\infty(\boldsymbol{R}^m)$.

This lemma is obvious from the fact that substitution is continuous for the Fréchet topology. For (11′), we use the fact that any ideal in $C^\infty(\boldsymbol{R}^m)$ which is generated by finitely many analytic functions is closed (see Malgrange (1966), chapter VI: or Tougeron (1972)). So to show (11′), suppose $k(x) \in \mathrm{Cl}(I(x), g(x)-t^2)$. Using the notation of the proof of Theorem 3, and writing

$$B' = C^\infty(\boldsymbol{R}^{n+1})/\mathrm{Cl}(I(x), g(x)-t^2),$$

$\bar{B}' \overset{i}{\hookrightarrow} \bar{B}$, we have that $k \circ \pi_x \circ i = 0$ in the diagram below. To show that $k \in I$, i.e. that $k = 0$ as a map $\bar{A} \to R$, it suffices to show (since I is closed, see Reyes (1981) or Kock (1981)) that for any Weil algebra $\bar{W}$ and any $\bar{W} \xrightarrow{\varphi} \bar{A}$, $k\varphi = 0$. Let $\bar{C} = \bar{W} \times_{\bar{A}} \bar{B}$. If $W = C^\infty(\boldsymbol{R}^m)/J$, we may assume J is generated by finitely many polynomials, and $g\varphi$ is represented by a polynomial $\boldsymbol{R}^m \xrightarrow{p(y)} \boldsymbol{R}$:

$$\begin{tikzcd}
& \bar{B}' \arrow[d, hook, "i"] & \\
\bar{C} \arrow[ur, dashed] \arrow[r, "r"] \arrow[d, "q"] & \bar{B} \arrow[r, "t"] \arrow[d, "\pi_x"] & R \arrow[d, "t^2"] \\
\bar{W} \arrow[r, "\varphi"] & \bar{A} \arrow[r, "g"] \arrow[d, "k"] & R_{\geq 0} \\
& R &
\end{tikzcd}$$

Thus $C = C^\infty(R^{m+1})/(J(y), p(y)-t^2)$, and $(J(y), p(y)-t^2)$ is closed. Hence $\bar{C} \xrightarrow{r} \bar{B}$ factors through $\bar{B}'$ as indicated in the diagram. So $k\varphi q = 0$ since $k\pi_x i = 0$. But $\bar{C} \xrightarrow{q} \bar{W}$ is an effective epimorphism by Theorem 3, so $k\varphi = 0$. This proves (11′).

Next, we prove the case of $\boldsymbol{G}$. We have to show that for a given *germ-determined* ideal $I \subset C(\boldsymbol{R}^n)$,

$$(10'') \quad f(x,t)-f(x,s) \in (I(x), g(x)-t^2), t^2-s^2)^\sim \Rightarrow$$
$$\Rightarrow \exists h(x) \;\; f(x,t)-h(x) \in (I(x), g(x)-t^2)^\sim;$$

$$(11'') \qquad k(x) \in (I(x), g(x)-t^2)^\sim \Rightarrow k(x) \in I.$$

(10″) is proved as (10) and (10′), now using a substitution lemma as above, but with Cl( ) replaced by $(\;)^\sim$. For (11″), suppose $k(x) \in (I(x), g(x)-t^2)^\sim$. We have to show $k(x) \in I$. Since I is germ-determined, it suffices to show that for $x_0 \in Z(I)$, the germ $k|_{x_0}$ is in $I|_{x_0}$. So choose $x_0 \in Z(I)$, and let $t_0 = \sqrt{g(x_0)}$. Since $k(x) \in (I(x), g(x)-t^2)^\sim$, there are neighbourhoods U_{x_0} and V_{t_0} with $k(x)|U_{x_0} \in (I|U_{x_0}, (g(x)-t^2)|U_{x_0} \times V_{t_0})$ as ideal in $C^\infty(U_{x_0} \times V_{t_0})$. If $t_0 > 0$ it is clear that $k|U_{x_0} \in (I|U_{x_0})$. If $t_0 = 0$, we apply Lemma 2′ with $M = U_{x_0}$ and $V = V_{t_0}$, and conclude that $k(x)|U_{x_0} \in (I|U_{x_0})$ as in the proof of (11).

This proves Theorem 4.

COROLLARY 5. *The Grothendieck topology on* $\boldsymbol{L}$ (*or* $\boldsymbol{G}$, *or* $\boldsymbol{F}$) *generated by the finite open covers as in section* 2 *and the singleton cover* $\{R \xrightarrow{t^2} R_{\geq 0}\}$ *is subcanonical. Similarly, the Grothendieck topology on* $\boldsymbol{G}$ (*or* $\boldsymbol{F}$) *generated by arbitrary open covers and* $\{R \xrightarrow{t^2} R_{\geq 0}\}$ *is subcanonical.*

Proof. Since the open cover topologies are subcanonical (section 2), this is clear from Theorem 3 and Theorem 4.

COROLLARY 6. *The condition*

$$(1) \qquad \forall x \in R (x \geqslant 0 \to \exists y \in R \;\; x = y^2)$$

is consistent with SDG.

More precisely, in the toposes of sheaves on the sites described in Corollary 5, (1) holds. Moreover, since these sites are subcanonical, the usual proofs for the SDG-axioms in $\mathscr{L}$, $\mathscr{F}$, $\mathscr{G}$, $\mathscr{F}_{\mathrm{fin}}$, $\mathscr{G}_{\mathrm{fin}}$ (such as the Kock-Lawvere axiom, compatibility of $<$ and $\leqslant$ with the commutative local ring structure of R, and the integration axiom) remain valid. See the references cited for $\boldsymbol{L}$, $\boldsymbol{F}$, and $\boldsymbol{G}$ in section 2. For the integration axiom, see also the next section.

Furthermore, the category of manifolds is fully and faithfully embedded in each of the toposes corresponding to the sites of Corollary 5.

One of the typical properties of the models of SDG that have been considered so far is the existence of "the amazing right-adjoint". In all the toposes mentioned in section 2, the exponentiation functor $(-)^D$ $(D = \overline{C^\infty(R)/(x^2)})$ does not only have a left-adjoint $(-)\times D$, but also a right-adjoint $(-)_D$,

$$(-)\times D \dashv (-)^D \dashv (-)_D .$$

This fails, however, in the models described by the sites in Corollary 5. Indeed, surjectivity of $R \xrightarrow{t^2} R_{\geqslant 0}$ is inconsistent with $(-)^D$ having a right-adjoint. For $(-)^D$ sends $R \xrightarrow{t^2} R_{\geqslant 0}$ to the map "compose with t^2": $R^D \to (R_{\geqslant 0})^D$, which corresponds by the Kock–Lawvere axiom to the map

$$R\times R \to R_{\geqslant 0}\times R\,, \qquad (a, b) \mapsto (a^2, 2ab)$$

and this cannot be a surjection.

Going back to Theorem 3, it is natural to ask the following question.

QUESTION 7. *Is the map $R \to R$ induced by t^n (n odd) a stable effective epimorphism in* $\boldsymbol{L}$ *(in* $\boldsymbol{F}$, *in* $\boldsymbol{G}$*)? And is the map $R \to R_{\geqslant 0}$ induced by t^n (n even) one?*

4. Integration in smooth toposes. We will now show how Theorem 3 and Theorem 4 imply that the integration axiom

$$(2) \qquad \forall f \in R^{[0,1]}\, \exists!\, g \in R^{[0,1]} (g(0) = 0 \wedge g' \equiv f)$$

is valid in all the models described in section 2. For the toposes of sheaves over the sites with underlying categories $\boldsymbol{L}$, $\boldsymbol{G}$ or $\boldsymbol{F}$ this is known (see the references in section 2, in particular Quê–Reyes), but the argument does not apply to the classifying toposes with the smaller underlying category $\boldsymbol{C}$, in which $[0, 1]$ is not representable. The argument below applies uniformly to all toposes.

We will use the following "synthetic" argument of Reyes (1981). It essentially reduces the verification of the integration axiom to the density of the squares in $R_{\geqslant 0}$, in the sense of (iii) below. In Reyes (1981), (iii) was verified in the "cahiers-topos" of Dubuc (1979), as well as in a generalization of it. (These two earlier models do not occur in section 2.)

Let R be "the line" in a model of SDG. So R is a commutative ring satisfying the Kock–Lawvere axiom $R^D \cong R\times R$ by which differentiation is defined, and R is equipped with order relations $<$ and $\leqslant$ which are compatible with each other and with the ring-structure of R. (I.e., R satisfies axioms $(*)$ of the Introduction). Now consider the following axioms:

(i) $\forall x \in R (x^2 \geqslant 0)$;
(ii) $\forall f \in R^R \big(f(x) \equiv f(-x) \to \exists F \in R^R (f(x) \equiv F(x^2))\big)$;
(iii) $\forall f \in R^{R_{\geqslant 0}} (\forall x\, f(x^2) = 0 \to f \equiv 0)$;
(iv) $\forall f \in R^R\ \exists!\ g \in R^R (g(0) = 0 \wedge g' \equiv f)$.

LEMMA 8 (synthetic). *Suppose* (i)–(iv) *hold for a ring R as above. Then the integration axiom* (2) *holds.*

Proof. (a) We first prove an integration axiom for $R_{\geqslant 0} = [0, \to)$,

$$(2') \qquad \forall f \in R^{R_{\geqslant 0}}\ \exists!\, g \in R^{R_{\geqslant 0}} (g(0) = 0 \wedge g' \equiv f)\,.$$

Indeed, given f: $R_{\geqslant 0} \to R$, (i)–(iii) imply that f can be extended to $F \in R^R$. (Let $R \xrightarrow{g} R$ be defined by $g(x) = f(x^2)$. Then $g(x) \equiv g(-x)$, so $g(x) \equiv F(x^2)$ for some F; by (iii), $\forall x \geqslant 0\ F(x) = f(x)$, so F extends f.) By (iv), there is a G: $R \to R$ with $G(0) = 0$, $G' \equiv F$. Then $g = G|R_{\geqslant 0}$ shows the existence part of (2′). For uniqueness in (2′), suppose $g \in R^{R_{\geqslant 0}}$ is such that $g(0) = 0$ and $g' \equiv 0$. Let $h(x) = g(x^2)$, h: $R \to R$. Then $h' \equiv 0$. So by (iv), $h \equiv 0$, and hence $g \equiv 0$ by (iii).

(b) Now we prove (2). For existence, suppose f: $[0, 1] \to R$. By using the isomorphisms $[0, 1) \to [0, \to)$ given by $x \mapsto \dfrac{x}{1-x}$ and $[0, 1) \xrightarrow{\sim} (0, 1]$ given by $x \mapsto 1-x$, (2′) implies that there are unique functions g: $[0, 1) \to R$ with $g(0) = 0$, $g' \equiv f'|[0, 1)$, and h: $(0, 1] \to R$ with $h(1) = 0$, $h' \equiv f'|(0, 1]$. Since R is a local ring, $[0, 1) \cup (0, 1] = [0, 1]$, and therefore

$$F(t) = \begin{cases} g(t) & \text{if } t \in [0, 1)\,, \\ h(t) + g(\tfrac{1}{2}) - h(\tfrac{1}{2}) & \text{for } t \in (0, 1] \end{cases}$$

defines the function required in (2). Uniqueness of F is obvious from uniqueness of g and h, again since $[0, 1) \cup (0, 1] = [0, 1]$.

LEMMA 9. *Axioms $(*)$ (cf. Introduction) as well as* (i)–(iv) *hold in the following models of the Kock–Lawvere axiom:* $\mathscr{Z}$, $\mathscr{G}$, $\mathscr{G}_{\mathrm{fin}}$, $\mathscr{F}$, $\mathscr{F}_{\mathrm{fin}}$, $\mathscr{S}[C^\infty_{\mathrm{loc}}]$, $\mathscr{S}[C^\infty_{\mathrm{As\,loc}}]$.

Proof. As far as $(*)$ see Kock [1981] and Quê–Reyes [1982]. (i) is clear. (ii) is a direct translation of Lemma 1, (iv) is trivial. The only problem really is (iii). In those toposes where $[0, \to) = R_{\geqslant 0}$ is representable, (iii) follows from Theorem 3 and Theorem 4. For the classifying toposes, whose sites $\boldsymbol{C}$ are too small to contain $R_{\geqslant 0}$, we need a little argument to show that $R_{\geqslant 0}$ "acts as a representable". So let $\tau \in R^{R_{\geqslant 0}}(\bar{A})$ where $\bar{A} \in \boldsymbol{C}$, i.e. τ is a natural transformation

$$\boldsymbol{C}(-, \bar{A}) \times \boldsymbol{L}(i(-), R_{\geqslant 0}) \to \boldsymbol{C}(-, R)$$

where i: $\boldsymbol{C} \hookrightarrow \boldsymbol{L}$ is the inclusion-functor. We claim that τ is induced by a morphism $\bar{A}\times R_{\geqslant 0} \xrightarrow{f} R$ in $\boldsymbol{L}$ (which would complete the proof, since then Theorem 3 is applicable again). To see this, apply τ to the pair $(\pi_{\bar{A}}, t^2)$ at stage $\bar{A}\times R \in \boldsymbol{C}$, and write $g = \tau_{\bar{A}\times R}(\pi_{\bar{A}}, t^2)$: $\bar{A}\times R \to R$. Now consider the diagram

$$\begin{tikzcd} A\times(R\times_{R_{\geqslant 0}} R) \arrow[r, shift left, "\pi_{12}"] \arrow[r, shift right, "\pi_{13}"'] & A\times R \arrow[r, "A\times t^2"] \arrow[dr, "g"'] & A\times R_{\geqslant 0} \arrow[d, "f"] \\ & & R \end{tikzcd}$$

where π_{12}, π_{13} is the kernel-pair of $\bar{A}\times t^2$. (Note that $R\times_{R_{\geqslant 0}} R$ is an object of $\boldsymbol{C}$, although $R_{\geqslant 0}$ is not.) By naturality of τ, $g\circ\pi_{12} = g\circ\pi_{13}$, so by Theorem 3 there is a unique map f in $\boldsymbol{L}$ making the triangle commute. Now τ comes from composing with f. Indeed, suppose $\bar{B} \xrightarrow{(h,x)} \bar{A} \times R_{\geq 0}$ is a map in $\boldsymbol{L}$, where $\bar{B} \in \boldsymbol{C}$. To show that $\tau_{\bar{B}}(h, x) = f \circ (h, x)$, it suffices by Theorem 3 to prove that $\tau_{\bar{B}}(h, x) \circ p_1 = f \circ (h, x) \circ p_1$, where

$$\begin{tikzcd} \bar{B} \arrow[r, "{(h,x)}"] & \bar{A} \times R_{\geq 0} \arrow[r, "f"] & R \\ \bar{C} \arrow[u, "p_1"] \arrow[r, "p_2"'] & \bar{A} \times R \arrow[u, "\bar{A} \times t^2"'] \arrow[ur, "g"'] & \end{tikzcd}$$

this square is a pullback. But this is clear from the definition of g and the naturality of τ.

Corollary 10. *The integration axiom* (2) *holds in all the models described in section* 2.

The models referred to in Corollary 10 are, besides those mentioned in Proposition 9, $\mathrm{Sets}^{L^{\mathrm{op}}}$, $\mathrm{Sets}^{C^{\mathrm{op}}}$, $\mathscr{S}[C^\infty]$. The integration axiom (2) is seen to hold in $\mathrm{Sets}^{L^{\mathrm{op}}}$, say, from the validity of (2) in the corresponding model of Proposition 9 (in this case $\mathscr{Z} \hookrightarrow \mathrm{Sets}^{L^{\mathrm{op}}}$) since the Zariski topology on $\boldsymbol{L}$ which defines $\mathscr{Z}$ is subcanonical.

Finally, we wish to point out that for the case of toposes over $\boldsymbol{F}$, there is a very simple proof of the validity of the integration axiom along the lines of Lemma 8, since Lemma 9 (or more precisely, the validity of (iii) there) is much easier for these toposes (in particular does not depend on Theorems 1, 2, 3, 4), as we will now explain. For definiteness, let us treat the case of $\mathscr{F}$.

Lemma 11. *Let $X \xrightarrow{\alpha} Y$ be a morphism in the topos $\mathscr{F}$ of sheaves on* $\boldsymbol{F}$ *(see Section 2). Then $R^\alpha\colon R^Y \to R^X$ is a monomorphism in $\mathscr{F}$ iff $\Gamma(R^\alpha)\colon \mathscr{F}(Y, R) \to \mathscr{F}(X, R)$ is a monomorphism in Sets, where Γ is the global sections functor.*

Proof. $\Rightarrow$ is clear, since the global sections functor Γ has a left-adjoint. For $\Leftarrow$ suppose $\Gamma(R^\alpha)$ is mono, and take $\bar{A} \in \mathscr{F}$. Suppose $u, v \in R^Y(\bar{A})$ and $R^\alpha_{\bar{A}}(u) = R^\alpha_{\bar{A}}(v)$, i.e. $u, v\colon \bar{A} \times Y \to R$ and the composites

$$\bar{A} \times X \xrightarrow{\bar{A} \times \alpha} \bar{A} \times Y \underset{v}{\overset{u}{\rightrightarrows}} R$$

are equal. We show that $u = v$. Take an element (f, y) of $\bar{A} \times Y$ at stage $\bar{B} \in \boldsymbol{F}$, i.e. $\bar{B} \xrightarrow{(f,y)} \bar{A} \times Y$. $\bar{B}$ corresponds to a closed ideal, so to show that $u \circ (f, y) = v \circ (f, y)$ it suffices to prove that $u \circ (f, y) \circ \varphi = v \circ (f, y) \circ \varphi$ for every $\bar{W} \xrightarrow{\varphi} \bar{B}$ in $\boldsymbol{F}$, where W is a Weil algebra. By assumption, $u \circ (f\varphi \times Y) \circ (\bar{W} \times \alpha) = v \circ (f\varphi \times Y) \circ (\bar{W} \times \alpha)$,

$$\begin{tikzcd} \bar{A} \times X \arrow[r, "\bar{A} \times \alpha"] & \bar{A} \times Y \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & R \\ \bar{W} \times X \arrow[u, "f\varphi \times X"] \arrow[r, "\bar{W} \times \alpha"'] & \bar{W} \times Y \arrow[u, "f\varphi \times Y"'] & \end{tikzcd}$$

so by exponential adjointness, $u_0 \circ \alpha = v_0 \circ \alpha$,

$$X \xrightarrow{\alpha} Y \underset{v_0}{\overset{u_0}{\rightrightarrows}} R^{\bar{W}}$$

where u_0 is the transposed of $\bar{W} \times Y \xrightarrow{f\varphi \times Y} \bar{A} \times Y \xrightarrow{u} R$, and similarly for v_0. But $R^{\bar{W}} \cong R$ for some n, so by composing with each of the n projections $R^{\bar{W}} \to R$ and using that $\Gamma(R^\alpha)\colon \mathscr{F}(Y, R) \rightarrowtail \mathscr{F}(X, R)$, we conclude that $u_0 = v_0$. Hence $u \circ (f\varphi \times Y) = v \circ (f\varphi \times Y)$, and therefore $u \circ (f, y) \circ \varphi = v \circ (f, y) \circ \varphi$, which was to be shown.

Corollary 12. *Let $M \xrightarrow{\varphi} N$ be a smooth map of manifolds. Then $\varphi(M)$ is dense in N iff $R^{s(\varphi)}\colon R^{s(N)} \to R^{s(M)}$ is a monomorphism in $\mathscr{F}$.*

Corollary 13. *In $\mathscr{F}$, the following are valid.*

(i) $\forall f \in R^{R_{\geq 0}}(\forall t \in R(f(t^{2n}) = 0) \to f \equiv 0)$

(ii) $\forall f \in R^R(\forall t \in R(f(t^{2n+1}) = 0) \to f \equiv 0)$.

In particular, Corollary 13(i) takes care of the difficult part of Lemma 8, namely validity of (iii), in the case of $\mathscr{F}$ (and similarly in the case of $\mathrm{Sets}^{F^{\mathrm{op}}}$, $\mathscr{F}_{\mathrm{fin}}$).

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References

L. Bélair, *Calcul Infinitésimal en Géométrie Différentielle Synthétique*, M. Thesis, Univ. de Montréal, 1981.

E. J. Dubuc, *Sur les modèles de la Géométrie Différentielle Synthétique*, Cahiers de Top. Géom. Différentielle 20 (1979), 231–297.

— *C^∞-schemes*, Amer. J. Math. 103 (4) (1981), 683–690.

P. T. Johnstone, *Topos Theory*, Academic Press, 1977.

A. Kock, *Synthetic Differential Geometry*, Cambridge University Press, 1981.

A. Kock and G. E. Reyes, *Models for synthetic integration theory*, Math. Scand. 48 (1981), 145–152.

A. Makkai and G. E. Reyes, *First Order Categorical Logic*, Lecture Notes in Mathematics 611, Springer-Verlag, 1977.

B. Malgrange, *Ideals of Differentiable Functions*, Oxford University Press 1966.

J. Martinet, *Singularities of Smooth Functions and Maps*, Cambridge University Press, 1982.

I. Moerdijk and G. E. Reyes, *Smooth spaces versus continuous spaces in models of synthetic differential geometry*, Journal of Pure and Applied Algebra 32 (1984), 143–176.

— — *A smooth version of the Zariski topos*, Amsterdam report 83–24, to appear in Adv. in Math.

— — *Models for Smooth Infinitesimal Analysis*, to appear in Springer-Verlag.

N. van Quê and G. E. Reyes, *Smooth functors and synthetic calculus*, in the L. E. J. Brouwer Centenary Symposium, A. S. Troelstra and D. van Dalen (eds.), North-Holland, 1982, 377–395.

G. E. Reyes, *Synthetic reasoning and variable sets* (*Models for synthetic differential geometry*), in *Categories in Continuum Physics*, Lecture Notes in Math. 1174, Springer-Verlag, 1986, 69–82.

— *Analyse dans les topos lisses*, Cahiers de Top. et de Géom. Differentielle 22–2 (1981).

J.-C. Tougeron (1972), *Idéaux de fonctions différentiables*, Springer, Berlin 1972.

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Pointwise limits of subsequences and Σ_2^1 sets

by

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Abstract. The following representation theorem for Σ_2^1 subsets of the space $C[0, 1]$ is proved, and some applications of it are given. For any Σ_2^1 set $S \subset C[0, 1]$, there exists a sequence $\langle f_i \rangle$ of continuous functions such that S is the set of all continuous pointwise limits of subsequences of $\langle f_i \rangle$.

§ 1. **The Main Theorem.** A *Polish space* is a topological space homeomorphic to a separable complete metric space. In this paper all spaces are Polish. For any space X, let X^ω denote the topological product of countably many copies of X. Let C be the space $C[0, 1]$ of continuous real-valued functions on the unit interval, with the uniform metric. This paper is mainly concerned with the two spaces C and C^ω. The elements of C^ω are sequences of functions; our notation for these sequences is $\langle f_i \rangle$, $\langle g_i \rangle$, ...

A pointset is Σ_1^1 if it is the projection of a Borel set (in some product space). A Π_n^1 set is the complement of a Σ_n^1 set and a Σ_{n+1}^1 set is the projection of a Π_n^1 set (in some product space). A set is Δ_n^1 if it is both Σ_n^1 and Π_n^1. This is the logicians' notation — the classical names for Σ_1^1, Π_1^1, Σ_2^1, Π_2^1, Σ_3^1, ... are A (analytic), CA (coanalytic), PCA, CPCA, PCPCA, ... Any two uncountable Polish spaces are Borel isomorphic, and these classes are all preserved under Borel isomorphism, so as far as the abstract theory of Σ_n^1 sets is concerned, there is only one space. Hence *descriptive set theory*, the study of *pointclasses* such as Σ_n^1, is frequently presented in the context of one fixed space, ω^ω (Baire space), where ω is the natural numbers with the discrete topology. A good reference for descriptive set theory is Moschovakis [12], whose notation and terminology will be used in this paper.

1.1. Definition. Let $\langle f_i \rangle \in C^\omega$. Then $A_{\langle f_i \rangle}$ denotes the following subset of C:

$\{h \in C$: there is a subsequence of $\langle f_i \rangle$ which converges pointwise to $h\}$.

Note that for any $\langle f_i \rangle$, the pointset $A_{\langle f_i \rangle}$ is Σ_2^1, uniformly. (This is proved by the methods of [12, 1C and 1E].) The main theorem of this paper is the converse — every Σ_2^1 set can be represented in this manner.

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