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Pointwise limits of subsequences and Σ_2^1 sets

by

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Abstract. The following representation theorem for Σ_2^1 subsets of the space $C[0,1]$ is proved, and some applications of it are given. For any Σ_2^1 set $S \subset C[0,1]$, there exists a sequence $\langle f_i \rangle$ of continuous functions such that S is the set of all continuous pointwise limits of subsequences of $\langle f_i \rangle$.

§ 1. **The Main Theorem.** A *Polish space* is a topological space homeomorphic to a separable complete metric space. In this paper all spaces are Polish. For any space X, let X^ω denote the topological product of countably many copies of X. Let C be the space $C[0,1]$ of continuous real-valued functions on the unit interval, with the uniform metric. This paper is mainly concerned with the two spaces C and C^ω. The elements of C^ω are sequences of functions; our notation for these sequences is $\langle f_i \rangle$, $\langle g_i \rangle$, ...

A pointset is Σ_1^1 if it is the projection of a Borel set (in some product space). A Π_n^1 set is the complement of a Σ_n^1 set and a Σ_{n+1}^1 set is the projection of a Π_n^1 set (in some product space). A set is Δ_n^1 if it is both Σ_n^1 and Π_n^1. This is the logicians' notation — the classical names for Σ_1^1, Π_1^1, Σ_2^1, Π_2^1, Σ_3^1, ... are A (analytic), CA (coanalytic), PCA, CPCA, PCPCA, ... Any two uncountable Polish spaces are Borel isomorphic, and these classes are all preserved under Borel isomorphism, so as far as the abstract theory of Σ_n^1 sets is concerned, there is only one space. Hence *descriptive set theory*, the study of *pointclasses* such as Σ_n^1, is frequently presented in the context of one fixed space, ω^ω (Baire space), where ω is the natural numbers with the discrete topology. A good reference for descriptive set theory is Moschovakis [12], whose notation and terminology will be used in this paper.

1.1. Definition. Let $\langle f_i \rangle \in C^\omega$. Then $A_{\langle f_i \rangle}$ denotes the following subset of C:

$\{h \in C$: there is a subsequence of $\langle f_i \rangle$ which converges pointwise to $h\}$.

Note that for any $\langle f_i \rangle$, the pointset $A_{\langle f_i \rangle}$ is Σ_2^1, uniformly. (This is proved by the methods of [12, 1C and 1E].) The main theorem of this paper is the converse — every Σ_2^1 set can be represented in this manner.

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1.2. THEOREM. *For any $S \subset C$, if S is Σ_2^1 then there exists an $\langle f_i \rangle$ in C^ω such that $S = A_{\langle f_i \rangle}$.*

Several representation theorems for Σ_1^1 sets have appeared in the literature. One example of such a theorem is due to Poprougénko [14], who showed that every Σ_1^1 set of real numbers is the range of a derivative. Others can be found in the following references: Bagemihl–Mc Millan [1], Kaufman [5], [6], Lorentz-Zeller [10], Nishiura [13]. I would like to thank Alexander Kechris for bringing these theorems to my attention. There are also several theorems which give representations of Σ_n^1 sets in terms either of concepts from the abstract theory of Σ_n^1 sets, or concepts from mathematical logic (see Moschovakis [12]). Theorem 1.2 is the only example I know of which gives a representation of Σ_2^1 sets in terms of concepts from analysis or topology. I know of no examples for Σ_n^1 when $n \geqslant 3$.

This paper is organized as follows. § 2 contains some corollaries to the theorem. It also contains a number of miscellaneous remarks and questions. The rest of the paper is devoted to proving Theorem 1.2. In § 3 we prove 1.2 for Π_1^1 sets, that is, we show that for any Π_1^1 set $P \subset C$, there is an $\langle f_i \rangle$ such that $P = A_{\langle f_i \rangle}$. In § 4 we show how to transfer this result from Π_1^1 to Σ_2^1, and thereby complete the proof.

§ 2. Corollaries, examples, remarks, questions, etc.

2.1. Let D be any dense subset of C. It is obvious that given any $\langle g_i \rangle \in C^\omega$, there is an $\langle h_i \rangle \in D^\omega$ such that $A_{\langle g_i \rangle} = A_{\langle h_i \rangle}$.

Hence Theorem 1.2 can be strengthened by requiring the f_i's to be in D. For example, the f_i's can always be taken to be polynomials, or to be piecewise-linear.

2.2. Most subsets of C (in the sense of cardinality) are not Σ_2^1, but as a practical matter, virtually every pointset that would ever occur in ordinary mathematics is. To cite one familiar example, the set of differentiable functions is Π_1^1, hence Σ_2^1. (The set of differentiable functions is not Borel, by a theorem of Mazurkiewicz [11] — see also Kechris -Woodin [8].) The simplest example known to me of a set which is not Σ_2^1, and hence to which the Main Theorem is not applicable, is the set of functions satisfying the Mean Value Theorem, that is,

$$\Big\{f \in C\colon \text{for any } a,\ b, \text{ if } 0 \leqslant a < b \leqslant 1 \text{ then there is a } c \text{ such that}$$
$$a < c < b,\ f \text{ is differentiable at } c, \text{ and } f'(c) = \frac{f(b)-f(a)}{b-a}\Big\}.$$

This set is clearly Π_2^1; that it is not Σ_2^1 is an unpublished theorem of Woodin. (Incidentally, the set of functions satisfying Rolle's Theorem is Σ_1^1.)

2.3. Let $A \subset C^\omega \times C$ be the set $\{(\langle f_i \rangle, h)\colon h \in A_{\langle f_i \rangle}\}$. Then A is a *universal set* for $\Sigma_2^1 \restriction C$, that is, A is Σ_2^1 and every Σ_2^1 subset of C is a vertical section of A. (It is a classical theorem that for any n and any uncountable Polish spaces X and Y, there is a $U \subset X \times Y$ which is a universal set for $\Sigma_n^1 \restriction Y$, and similarly for Π_n^1 [12, 1D.1]. While the abstract theory tells us that universal sets exist, it leaves open the question of whether there are any examples of this phenomenon which arise naturally in analysis.) The representation theorems for Σ_1^1 mentioned in § 1 also give natural examples of universal sets for Σ_1^1.

2.4. The Main Theorem holds *uniformly*. This means that if $U \subset \omega^\omega \times C$ is the canonical universal set for $\Sigma_2^1 \restriction C$ and $U_x = \{h \in C\colon (x, h) \in U\}$, then there is a recursive (hence continuous) function $G\colon \omega^\omega \to C^\omega$ such that for all $x \in \omega^\omega$, if we let $\langle f_i^x \rangle$ denote the sequence $G(x)$, then $U_x = A_{\langle f_i^x \rangle}$. It follows that for *any* Σ_2^1 set $S \subset X \times C$, there is a continuous function $G_S\colon X \to C^\omega$ with the above property. For details, see [12, 3H]. The proof that the Main Theorem holds uniformly is implicit in the proof of the Main Theorem to be given in this paper.

2.5. Define two subsets E and F of C^ω as follows:

$$E = \{\langle f_i \rangle \in C^\omega\colon \text{for every } h \in C,\ h \in A_{\langle f_i \rangle}\}.$$
$$F = \{\langle f_i \rangle \in C^\omega\colon \text{there exists an } h \in C \text{ such that } h \in A_{\langle f_i \rangle}\}.$$

Clearly E is Π_3^1 and F is Σ_2^1. It follows from 2.4 that E is complete Π_3^1, hence not Σ_3^1, and F is complete Σ_2^1, hence not Π_2^1. Let G be the set

$$\{\langle f_i \rangle \in C^\omega\colon \langle f_i \rangle \text{ converges pointwise and } \lim_{i\to\infty} f_i \text{ is continuous}\}.$$

G is Π_1^1. G is not Σ_1^1, since if it was then F would also be Σ_1^1. (Assuming Π_1^1-determinacy, any set which is Π_1^1 and not Σ_1^1 is complete Π_1^1 [12, 7D.3]]; as one would expect, it is provable in ZFC that G is complete Π_1^1.)

2.6. For $\langle f_i \rangle \in C^\omega$, let

$$\hat{A}_{\langle f_i \rangle} = \{h \in C\colon \text{there is a recursive-in-}h \text{ function } \alpha\colon \omega \to \omega,\ \alpha \text{ strictly increasing, such that the subsequence } \langle f_{\alpha(i)} \rangle_{i\in\omega} \text{ converges pointwise to } h\}.$$

For any $\langle f_i \rangle$, the pointset $\hat{A}_{\langle f_i \rangle}$ is Π_1^1, uniformly. It is implicit in the proof given in § 3 that for any $P \subset C$, if P is Π_1^1 then there exists an $\langle f_i \rangle$ in C^ω such that $P = A_{\langle f_i \rangle} = \hat{A}_{\langle f_i \rangle}$. Therefore if $\hat{A} \subset C^\omega \times C$ is defined to be $\{(\langle f_i \rangle, h)\colon h \in \hat{A}_{\langle f_i \rangle}\}$, then $\hat{A}$ is a universal set for $\Pi_1^1 \restriction C$.

2.7. It is consistent with ZF+DC that the binary relation

$$B = \{(\langle f_i \rangle, h)\colon h \notin A_{\langle f_i \rangle}\} \subset C^\omega \times C$$

cannot be uniformized, that is, there is no choice function which assigns to each $\langle f_i \rangle$ in the domain of B, an h such that $(\langle f_i \rangle, h) \in B$. It is consistent with ZFC that B has no uniformization ordinal-definable from a real parameter. (Throughout Remark 2.7, a "real" is a subset of ω.)

The first consistency result follows from the second by standard methods. Toward proving the second, let $R \subset 2^\omega \times 2^\omega$ be the binary relation $\{(x, y)\colon y \notin L(x)\}$, where $L(x)$ is the constructible universe relativized to the real x, and consider the following proposition:

(*) R has no uniformization which is ordinal-definable from a real.

Since R is $\boldsymbol{\Pi}^1_2$, it follows from (*), the fact that C is Borel isomorphic to 2^ω, and the uniform version of the Main Theorem (2.4), that B has no uniformization ordinal-definable from a real. The consistency of (*) with ZFC is part of the folklore of set theory; however I have not been able to find it in print. Using some well-known facts about forcing, it is not hard to show that (*) holds in the model obtained by adding $\aleph_1$ Cohen reals to L. For information on forcing and consistency proofs, see Jech [4]. The consistency results mentioned here are all variants of a theorem of Levy [9].

Of course it is also consistent with ZFC that B does have an ordinal-definable uniformization. This is the case if $V = L$.

2.8. There are various types of pathological pointsets (e.g., nonmeasurable) which can be produced using the axiom of choice. Assuming $V = L$, these pathologies all exist at the second level of the projective hierarchy (see [12, 5A.8]); hence by 1.2 these pathologies can be sets of the form $A_{\langle f_i \rangle}$. Thus it is consistent with ZFC that there exists an $\langle f_i \rangle \in C^\omega$ such that all of the following hold:

(a) $A_{\langle f_i \rangle}$ is uncountable but has no perfect subset.

(b) $A_{\langle f_i \rangle}$ does not have the property of Baire.

(c) $A_{\langle f_i \rangle}$ is not measurable with respect to any nonatomic σ-finite Borel measure on C.

On the other hand, it is also consistent that no Σ^1_2 set can exhibit such pathologies (see [4, p. 534 and p. 548]).

2.9. Let $E \subset C^\omega$ be as in 2.5, and let

$$E_1 = \{\langle f_i \rangle \in C^\omega \colon \text{there is exactly one } h \in C \text{ such that } h \notin A_{\langle f_i \rangle}\}.$$

By combining the proof that E is not Σ^1_3 (2.5) with the method of Becker [2], we obtain the following theorem. Assuming Δ^1_2-determinacy, E and E_1 are a pair of disjoint $\boldsymbol{\Pi}^1_3$ sets which cannot be separated by any Δ^1_3 set. (If $V = L$, then every pair of disjoint $\boldsymbol{\Pi}^1_3$ sets can be separated by a Δ^1_3 set [12, 5A.3].)

2.10. Any $\boldsymbol{\Pi}^1_1$ set admits a $\boldsymbol{\Pi}^1_1$-norm. (See [12] for definitions and details.) It is the thesis of Kechris–Woodin [8] that *natural* examples of $\boldsymbol{\Pi}^1_1$ sets will have *natural* $\boldsymbol{\Pi}^1_1$-norms; they give an example of such a norm on the set of differentiable functions. For another example, consider the space $K(X)$ of compact subsets of X, with the Hausdorff metric, and the pointset in $K(X)$ of all countable compact sets. This is $\boldsymbol{\Pi}^1_1$ (and by a theorem of Hurewicz [3], is complete $\boldsymbol{\Pi}^1_1$). The Cantor–Bendixson rank is a $\boldsymbol{\Pi}^1_1$-norm on this set. An example more closely related to the topic of this paper, a natural $\boldsymbol{\Pi}^1_1$-norm on the $\boldsymbol{\Pi}^1_1$ set of pointwise convergent sequences, was discovered by Zalcwasser [17].

We wish to consider the same question for $\boldsymbol{\Pi}^1_3$. Assuming Δ^1_2-determinacy, every $\boldsymbol{\Pi}^1_3$ set admits a $\boldsymbol{\Pi}^1_3$-norm. (Assuming $V = L$, this is not so.) Consider the $\boldsymbol{\Pi}^1_3$ set $E \subset C^\omega$ of 2.5, and for the rest of Remark 2.10, assume Δ^1_2-determinacy. Does E have any *natural* $\boldsymbol{\Pi}^1_3$-norm? A positive answer to this (rather vague) open question would be of interest for two reasons. First, such a norm would constitute a hierarchical structure on the set E, and this structure might give useful information about E or be used to prove theorems by transfinite induction. Second, since E is a *complete* $\boldsymbol{\Pi}^1_3$ set (2.5), the length of any $\boldsymbol{\Pi}^1_3$-norm on E must be the ordinal δ^1_3 [12, 4C.14]; hence a natural norm would give a description of this ordinal in terms of real analysis rather than of logic.

2.11. A *Baire*-1 function is, by definition, a pointwise limit of continuous functions. The set of Baire-1 functions does not form a Polish space in any natural way. However we can encode Baire-1 functions by elements of the space C^ω: $\langle f_i \rangle$ encodes the pointwise limit of $\langle f_i \rangle$, if it exists. The set of codes is $\boldsymbol{\Pi}^1_1$, and the induced equivalence relation on the codes is also $\boldsymbol{\Pi}^1_1$. Sets of Baire-1 functions correspond to *invariant* subsets of C^ω, and we say that a set of Baire-1 functions is $\boldsymbol{\Sigma}^1_2$ if the corresponding subset of C^ω is Σ^1_2.

For $\langle f_i \rangle \in C^\omega$, let $B_{\langle f_i \rangle}$ denote the following set of Baire-1 functions (cf. 1.1):

$$\{h \colon \text{there is a subsequence of } \langle f_i \rangle \text{ which converges pointwise to } h\}.$$

Note that for any $\langle f_i \rangle$, $B_{\langle f_i \rangle}$ is Σ^1_2, uniformly. Define $\hat{E}$ as follows (cf. 2.5):

$$\{\langle f_i \rangle \in C^\omega \colon \text{for every Baire-1 function } h,\ h \in B_{\langle f_i \rangle}\}.$$

Clearly $\hat{E}$ is $\boldsymbol{\Pi}^1_3$. We have four questions about Baire-1 functions.

Question 1. Is it true that for any set S of Baire-1 functions, if S is Σ^1_2 then there exists an $\langle f_i \rangle$ in C^ω such that $S = B_{\langle f_i \rangle}$?

Question 2. Does there exist an $\langle f_i \rangle \in C^\omega$ such that $B_{\langle f_i \rangle}$ is the set of discontinuous Baire-1 functions?

Question 3. Is $\hat{E}$ complete $\boldsymbol{\Pi}^1_3$?

Question 4. Is $\hat{E}$ not Σ^1_3?

An affirmative answer to Question n implies an affirmative answer to Question $n+1$ ($1 \leqslant n \leqslant 3$). For $n = 2$ we give an explanation, the other cases being trivial. Let $\langle g_i \rangle$ be a fixed sequence such that $B_{\langle g_i \rangle}$ is the discontinuous functions. For any $\langle f_i \rangle \in C^\omega$, let $\langle \hat{f}_i \rangle$ be the sequence: $\hat{f}_{2i} = f_i$, $\hat{f}_{2i+1} = g_i$. Then $B_{\langle \hat{f}_i \rangle}$ is the union of $A_{\langle f_i \rangle}$ and the set of discontinuous Baire-1 functions. So the map $\langle f_i \rangle \mapsto \langle \hat{f}_i \rangle$ reduces the set E of 2.5 to $\hat{E}$. Since E is complete, so is $\hat{E}$.

We have been able to prove the following weak version of a positive answer to Question 1. For $\langle f_i \rangle \in (C[0,2])^\omega$, let $\tilde{B}_{\langle f_i \rangle}$ denote the following set of Baire-1 functions on $[0,1]$:

$$\{h \colon \text{there is a subsequence } \langle f_{n_i} \rangle \text{ of } \langle f_i \rangle \text{ such that } \langle f_{n_i} \rangle \text{ converges pointwise (on all of } [0,2]) \text{ and for } x \in [0,1],\ \lim_{i \to \infty} f_{n_i}(x) = h(x)\}.$$

Then for every Σ^1_2 set S of Baire-1 functions, there is an $\langle f_i \rangle$ such that $S = \tilde{B}_{\langle f_i \rangle}$. This is enough to imply, as in 2.5, that

$$\hat{F} = \{\langle f_i \rangle \in C^\omega \colon \text{Some subsequence of } \langle f_i \rangle \text{ is pointwise convergent}\}$$

is complete Σ^1_2, hence not $\boldsymbol{\Pi}^1_2$. (It follows from this, as in 2.5, that the $\boldsymbol{\Pi}^1_1$ set $\hat{G} \subset C^\omega$ of pointwise convergent sequences is not Σ^1_1. However there is a simple direct proof that $\hat{G}$ is complete $\boldsymbol{\Pi}^1_1$ — see [2]).

2.12. For any $\langle f_i\rangle \in C^\omega$, let $K_{\langle f_i\rangle}$ denote the following subset of C:

$\{h \in C$: there is a subsequence $\langle f_{n_i}\rangle$ of $\langle f_i\rangle$ such that $\langle f_{n_i}\rangle$ is uniformly bounded and $\langle f_{n_i}\rangle$ converges pointwise to $h\}$.

This notion of convergence is *weak convergence* with respect to the Banach space $C[0,1]$. Again, for any $\langle f_i\rangle$, $K_{\langle f_i\rangle}$ is Σ^1_2, uniformly.

QUESTION (Kechris). *Is it true that for any $S \subset C$, if S is Σ^1_2 then there exists an $\langle f_i\rangle \in C^\omega$ such that $S = K_{\langle f_i\rangle}$?*

§ 3. **Proof for Π^1_1 sets.** In this section we prove the Main Theorem for the special case of Π^1_1 sets (Lemma 3.6, below).

We first establish some notation, following Moschovakis [12]. $X^{<\omega}$ is the set of all finite sequences from X. Finite sequences are denoted by Greek letters $\sigma, \tau, \ldots$. $\sigma \prec \tau$ means that σ is an initial segment of τ, and similarly if x is an infinite sequence, $\sigma \prec x$ means that σ is an initial segment of x. A *tree* T on X is a subset of $X^{<\omega}$ such that if $\sigma \prec \tau$ and $\tau \in T$ then $\sigma \in T$. The proof will involve trees on $\omega \times 2$; we identify a sequence in $(\omega\times 2)^{<\omega}$ of length n with two length n sequences, one each from $\omega^{<\omega}$ and $2^{<\omega}$, and similarly for infinite sequences. Let T be a tree on $\omega \times 2$. The *body* of the tree T, denoted $[T]$, is

$\{(x, y) \in \omega^\omega \times 2^\omega$: for all n, $(x \restriction n, y \restriction n) \in T\}$,

i.e., the set of all infinite branches through T.

Convergence of a sequence of functions always means pointwise convergence.

Fix two families, $\{I^\sigma_\tau\}$ and $\{J^\sigma_\tau\}$, of closed subintervals of $[0,1]$, indexed by $(\sigma, \tau) \in (\omega\times 2)^{<\omega}$ and satisfying the following properties.

3.1. (a) $J^\sigma_\tau \subset \text{Interior}\,(I^\sigma_\tau)$.

(b) If (σ, τ) is a proper initial segment of (σ', τ') then $I^{\sigma'}_{\tau'} \subset J^\sigma_\tau$.

(c) If (σ, τ) and (σ', τ') are incompatible then $I^\sigma_\tau \cap I^{\sigma'}_{\tau'} = \emptyset$.

(d) If $n = \text{length}(\sigma)$ ($= \text{length}\,(\tau)$) then the length of the interval I^σ_τ is at most $\frac{1}{n+1}$.

For any $(y, z) \in (\omega^\omega \times 2^\omega)$, there is exactly one real number in

$$\bigcap \{I^\sigma_\tau\colon (\sigma, \tau) \prec (y, z)\};$$

let $r(y, z)$ denote the number.

3.2. For $n \in \omega$, define the set $Z_n \subset (\omega\times 2)^{<\omega}$ as follows. $(\sigma, \tau) \in Z_n$ if there exists a $k \in \omega$ such that:

(a) $\sigma = (i_0, i_1, \ldots, i_{k-1}) \in \omega^k$,

(b) $\tau = (j_0, j_1, \ldots, j_{k-1}) \in 2^k$,

(c) $i_0 < i_1 < i_2 < \ldots < i_{k-1}$,

(d) $i_{k-1} = n$,

(e) $j_{k-1} = 1$.

Note that Z_n is finite and that the intervals $\{I^\sigma_\tau\colon (\sigma, \tau) \in Z_n\}$ are pairwise disjoint.

3.3. For any function $g \in C$, any $n \in \omega$, and any tree T on $\omega \times 2$, let $f^{(g,n,T)}$ be the unique continuous function from $[0,1]$ into $\boldsymbol{R}$ satisfying the following three conditions.

(a) If $x \notin \bigcup \{I^\sigma_\tau\colon (\sigma, \tau) \in Z_n \cap T\}$ then $f^{(g,n,T)}(x) = g(x)$.

(b) If $x \in \bigcup \{J^\sigma_\tau\colon (\sigma, \tau) \in Z_n \cap T\}$, then $f^{(g,n,T)}(x) = n$.

(c) On each interval of $\bigcup \{I^\sigma_\tau \setminus J^\sigma_\tau\colon (\sigma, \tau) \in Z_n \cap T\}$, $f^{(g,n,T)}$ is linear.

The first lemma examines a sequence of functions of the above form, and considers questions of convergence. (This lemma deals with a fixed sequence and does not consider convergence of its subsequences; however, it will later be applied to an arbitrary subsequence of a given sequence.) Let SI be the closed subspace of ω^ω consisting of strictly increasing functions.

3.4. LEMMA. *Let $\langle g_i\rangle \in C^\omega$, let $w = (m_0, m_1, m_2, \ldots) \in \mathrm{SI}$, and let T be a tree on $\omega\times 2$. Consider the sequence of functions:*

$$\langle f^{(g_i, m_i, T)}\rangle_{i\in\omega}.$$

(a) *If $\langle f^{(g_i, m_i, T)}\rangle$ converges then $\langle g_i\rangle$ converges and $\lim_{i\to\infty} f^{(g_i, m_i, T)} = \lim_{i\to\infty} g_i$.*

(b) *If $\langle f^{(g_i, m_i, T)}\rangle$ does not converge and $\langle g_i\rangle$ does converge, then there is a $y \in \omega^\omega$ and a $z \in 2^\omega$ such that:*

(i) $(y, z) \in [T]$,

(ii) *w and y have a common subsequence, i.e., there exist $v, \alpha, \beta \in \mathrm{SI}$ such that for all $j \in \omega$, $v(j) = w(\alpha(j)) = y(\beta(j))$,*

(iii) *$\{j\colon z(j) = 1\}$ is infinite.*

(c) *If there exists a $z \in 2^\omega$ such that*

(i) $(w, z) \in [T]$,

(ii) *$\{j\colon z(j) = 1\}$ is infinite,*

then $\langle f^{(g_i, m_i, T)}\rangle$ diverges.

Proof. Let $R = \{r(y, z)\colon y \in \omega^\omega,\ z \in 2^\omega\}$. Two facts follow directly from the definitions (3.1 and 3.3).

(1) If $x \in [0,1]$ is not in R, then for all but finitely many $i \in \omega$, $f^{(g_i, m_i, T)}(x) = g_i(x)$.

(2) If $x \in R$, then for any i, either $f^{(g_i, m_i, T)}(x) = g_i(x)$ or $f^{(g_i, m_i, T)}(x) = m_i$.

(a) If $f^{(g_i, m_i, T)}(x) = m_i$ for infinitely many i, then $\langle f^{(g_i, m_i, T)}\rangle$ diverges. So (a) follows immediately from (1) and (2).

(b) Suppose that $\langle g_i\rangle$ converges and $\langle f^{(g_i, m_i, T)}\rangle$ does not. Then by (1) and (2), there exist $x \in [0,1]$, $y \in \omega^\omega$, and $z \in 2^\omega$ such that $x = r(y, z)$ and for infinitely many i,

$$f^{(g_i, m_i, T)}(x) = m_i \neq g_i(x).$$

These infinitely many i's give us the subsequence. Formally, let $v \in \mathrm{SI}$ be a subsequence of w, say $v(j) = w(\alpha(j))$, such that for all $j \in \omega$,

$$f^{(g_{\alpha(j)}, m_{\alpha(j)}, T)}(x) = m_{\alpha(j)} = w(\alpha(j)) = v(j) \neq g_{\alpha(j)}(x).$$

By 3.3 (a), for any j there must be a $(\sigma^j, \tau^j) \in Z_{v(j)} \cap T$ such that $x \in I_{\tau^j}^{\sigma^j}$. Since $x = r(y, z)$, clearly $(\sigma^j, \tau^j) \prec (y, z)$.

Let $k^j = \text{length}(\sigma^j)$. By definition of $Z_{v(j)}$ (3.2), we have the following:

(i′) $(\sigma^j, \tau^j) \in T$,
(ii′) $\sigma^j(k^j-1) = v(j)$,
(iii′) $\tau^j(k^j-1) = 1$,
(iv′) $\sigma^j(0) < \sigma^j(1) < \sigma^j(2) < \ldots < \sigma^j(k^j-1)$.

Since each σ^j is an initial segment of y, clearly for any $j, j' \in \omega$, σ^j and $\sigma^{j'}$ must be compatible. From this, (ii′), and (iv′), plus the fact that $v \in \mathrm{SI}$, it follows that k^j is a strictly increasing function of j; that is, the inital segments (σ^j, τ^j) of (y, z) keep getting longer. Hence (i′) implies (i) of part (b), (ii′) implies (ii), and (iii′) implies (iii).

(c) Assume the hypothesis of (c). Let $x = r(w, z)$. Consider a $k \in \omega$ such that $z(k-1) = 1$; there are infinitely many of these k's. Let $n = w(k-1) = m_{k-1}$, let $\sigma = w \restriction k$ and let $\tau = z \restriction k$. By definition of Z_n (3.2), $(\sigma, \tau) \in Z_n$. Then

$$f^{(g_{k-1}, m_{k-1}, T)}(x) = m_{k-1},$$

by definition of the function (see 3.3 (b)). Since $w \in \mathrm{SI}$, $f^{(g_i, m_i, T)}(x)$ can be made arbitrarily large by choosing a large enough i such that $z(i) = 1$. So the sequence of reals $\langle f^{(g_i, m_i, T)}(x) \rangle$ diverges, hence the sequence of functions $\langle f^{(g_i, m_i, T)} \rangle$ diverges. ∎

3.5. LEMMA. *Let $Q \subset \omega^\omega$ be Σ_1^1. There is a tree T on $\omega \times 2$ such that for all $y \in \omega^\omega$,*

$y \in Q \Leftrightarrow$ *there exists a z in 2^ω such that $(y, z) \in [T]$ and $\{j\colon z(j) = 1\}$ is infinite.*

Proof. The usual representation for Σ_1^1 subsets of ω^ω [12, 2B] gives a tree U on $\omega \times \omega$ such that Q is the projection of $[U]$ onto the first coordinate. Encode elements of ω^ω by elements of 2^ω in the following way: $z \in 2^\omega$ is a code iff infinitely many of its coordinates are 1, and it encodes $\hat{z} \in \omega^\omega$ where

$$\hat{z}(n) = \text{the number of 0's between the } n\text{th and the } (n+1)^{\text{st}} \text{ 1 in } z.$$

Let T be the tree on $\omega \times 2$ that corresponds to U under this coding; that is, $(\sigma, \tau) \in T$ iff there exists a $y \in \omega^\omega$ and a $z \in 2^\omega$ such that $(\sigma, \tau) \prec (y, z)$ and z is a code and $(y, \hat{z}) \in [U]$. It is easy to see that T satisfies the lemma. ∎

(There is a reason for representing Σ_1^1 sets via trees on $\omega \times 2$, rather than using the customary trees on $\omega \times \omega$. It is necessary to work with 2 rather than ω in defining the sets Z_n in 3.2, since if ω is used then $Z_n \cap T$ may be infinite. If $Z_n \cap T$ is infinite, then in defining the function $f^{(g, n, T)}$, in 3.3, there will be infinitely many intervals, hence there will be limit points, hence $f^{(g, n, T)}$ will not be continuous.)

3.6. LEMMA. *For any $P \subset C$, if P is Π_1^1 then there exists an $\langle f_i \rangle$ in C^ω such that $P = A_{\langle f_i \rangle}$.*

Proof. Let $p_0, p_1, p_2, \ldots$ be a fixed sequence of functions which is dense in C, e.g., the polynomials with rational coefficients. Define $Q \subset \omega^\omega$ as follows:

$$y \in Q \Leftrightarrow [y \in \mathrm{SI} \;\&\; (\exists h \in C)\ (h \notin P \;\&\; \text{for a.e. } x \in [0, 1],\ \lim_{i \to \infty} p_{y(i)}(x) = h(x))].$$

The pointset Q is Σ_1^1. This is proved by the methods of [12, 1C and 1E], i.e., quantifier-counting. That is, the complement of P is Σ_1^1 and the pointclass of Σ_1^1 sets is closed under all the operations used to define Q, so Q is Σ_1^1. We are using one nontrivial closure property of Σ_1^1 (in addition to the trivial ones), namely the fact that Σ_1^1 is closed under quantification of the form "for a.e. $x \in [0, 1]$", where *almost every* refers to Lebesgue measure on $[0, 1]$; this is a theorem of Tanaka [16] — see also Sacks [15] and Kechris [7]. (The motivation for this definition is that we would really like to define Q to consist of those y in SI such that the subsequence $\langle p_{y(i)} \rangle$ of $\langle p_i \rangle$ converges pointwise to an element of $C \setminus P$; the problem is that this is not Σ_1^1, and we need a Σ_1^1 set. The only reason for introducing measure at all is to get around this problem.)

Since Q is Σ_1^1, by Lemma 3.5, there is a tree T on $\omega \times 2$ such that for all $y \in \omega^\omega$,

3.7. $y \in Q \Leftrightarrow$ there exists a z in 2^ω such that $(y, z) \in [T]$ and $\{j\colon z(j) = 1\}$ is infinite.

Fix such a T. For all $n \in \omega$, let $f_n = f^{(p_n, n, T)}$ (see 3.3 for definition). We will prove below that $P = A_{\langle f_i \rangle}$.

$P \subset A_{\langle f_i \rangle}$. Let $g \in P$. Let $\langle p_{n_i} \rangle$ be a subsequence of $\langle p_i \rangle$ which converges to g. We show that $\langle f_{n_i} \rangle$ converges to g. Assume that this is not so. Then by Lemma 3.4 (a), $\langle f_{n_i} \rangle$ does not converge. So by 3.4 (b), there is a $y \in \omega^\omega$ and a $z \in 2^\omega$ such that:

(i) $(y, z) \in [T]$,
(ii) $\langle n_i \rangle$ and y have a common subsequence,
(iii) $\{j\colon z(j) = 1\}$ is infinite.

From (i), (iii) and 3.7 we conclude that $y \in Q$. By definition of Q, there is an $h \in C$ such that $h \notin P$ and for a.e. x, $\lim_{i \to \infty} p_{y(i)}(x) = h(x)$. By (ii), the sequences $\langle p_{n_i} \rangle$ and $\langle p_{y(i)} \rangle$ have a common subsequence, hence for any point $x \in [0, 1]$, if $\langle p_{n_i} \rangle$ and $\langle p_{y(i)} \rangle$ both converge at x, then they must both converge to the same value. So for a.e. x, $g(x) = h(x)$. But g and h are both continuous functions, so $g = h$. This contradicts the fact that $g \in P$ and $h \notin P$.

$A_{\langle f_i \rangle} \subset P$. Let $\langle f_{n_i} \rangle$ be a subsequence of $\langle f_i \rangle$, let $h \in C$, and suppose that $\langle f_{n_i} \rangle$ converges to h. We must show that $h \in P$. By Lemma 3.4 (a), $\langle p_{n_i} \rangle$ converges to h. Let $w = (n_0, n_1, n_2, \ldots)$. Now suppose that $h \notin P$. Then by definition of Q, $w \in Q$. So by 3.7, there exists a $z \in 2^\omega$ such that $(w, z) \in [T]$ and z has infinitely many 1's. That is, w satisfies the hypothesis of Lemma 3.4 (c); therefore $\langle f_{n_i} \rangle$ diverges, contrary to assumption. ∎

§ 4. Proof for Σ_2^1 sets.

Lemma 3.6 and its proof can be generalized from C to a large class of spaces. In this section of the paper we give the generalization, and from it derive the Main Theorem (1.2).

Consider two real numbers a and b, with $a<b$, and a nonempty closed set F in $C[a,b]$. Then F (topologized as a subspace of $C[a,b]$) and F^ω are Polish spaces. Call F *suitable* if there exist two reals c and d, with $a<c<d<b$, satisfying the following property: For any functions f and g in $C[a,b]$, if $g\in F$ and for every x in $([a,b]\setminus[c,d])$, $f(x)=g(x)$, then $f\in F$. For $\langle f_i\rangle\in F^\omega$, let $A^F_{\langle f_i\rangle}$ denote the following subset of F:

$\{h\in F$: there is a subsequence of $\langle f_i\rangle$ which converges pointwise to $h\}$.

This definition is, of course, the analog of 1.1 for the space F.

4.1. LEMMA. *Let $a,b\in \mathbf{R}$, with $a<b$, and let $F\subset C[a,b]$ be a suitable closed set. For any $P\subset F$, if P is Π^1_1 then there exists an $\langle f_i\rangle$ in F^ω such that $P=A^F_{\langle f_i\rangle}$.*

The proof of Lemma 4.1 is essentially the same as the proof of Lemma 3.6 given in § 3. There is one extra detail: Make sure that the intervals I^σ_τ of 3.1 all lie within $[c,d]$, where c and d are as in the definition of *suitable*. (Thus the coding up of the tree into the functions all takes place inside $[c,d]$.) The only obstacle to generalizing §3 to *arbitrary* closed subsets of $C[a,b]$ is that the new functions $f^{(g,n,T)}$ constructed in 3.3 may not be in that closed set; suitability guarantees that this problem will not occur.

Now we fix an F. Let F be the following subset of $C[0,2]$:

$\{f\in C[0,2]$: $f(2)\in[0,1]$ and $f\upharpoonright[1,2]$ is linear$\}$.

Note that this F is closed and suitable, so Lemma 4.1 is applicable to it.

Next consider the spaces $D=(C[0,1]\times[0,1])$ and D^ω. We say that a sequence $\langle g_i,y_i\rangle\in D^\omega$ *converges* to (h,y) if $\langle g_i\rangle$ converges pointwise to h and $\langle y_i\rangle$ converges to y. For any $\langle g_i,y_i\rangle\in D^\omega$, let $B_{\langle g_i,y_i\rangle}$ denote the following subset of D:

$\{(h,y)\in D$: there is a subsequence of $\langle g_i,y_i\rangle$ which converges to $(h,y)\}$.

4.2. LEMMA. *For any $P\subset D$, if P is Π^1_1 then there exists a $\langle g_i,y_i\rangle\in D^\omega$ such that $P=B_{\langle g_i,y_i\rangle}$.*

Proof. Let $P\subset D$ be Π^1_1. Let P^* be the subset of F defined as follows.

$$P^*=\{f\in F\colon (f\upharpoonright[0,1],f(2))\in P\}.$$

There is a homeomorphism from D onto F which maps P to P^*, hence P^* is also Π^1_1. So by Lemma 4.1, there is an $\langle f_i\rangle\in F^\omega$ such that $P^*=A^F_{\langle f_i\rangle}$. Let $g_i=f_i\upharpoonright[0,1]$ and let $y_i=f_i(2)$. Then it is easy to see that $B_{\langle g_i,y_i\rangle}=P$. ■

We can now lift the representation theorem from Π^1_1 sets to Σ^1_2 sets.

Proof of Theorem 1.2. Let $S\subset C$ be a Σ^1_2 set. Then there is a Π^1_1 set $P\subset(C\times[0,1])=D$ such that S is the projection of P, that is, for all $h\in C$:

$h\in S\Leftrightarrow$ there exists a $y\in[0,1]$ such that $(h,y)\in P$.

(The official definition of Σ^1_2 given in this paper is that a subset of X is Σ^1_2 if it is the projection of a Π^1_1 set in $X\times Y$ for some Y. But all uncountable Polish spaces are Borel isomorphic, so without loss of generality, the space Y may be taken to be $[0,1]$. Moschovakis [12] defines a subset of X to be Σ^1_2 if it is the projection of a Π^1_1 set in $X\times\omega^\omega$; the same remark applies to this definition.)

By Lemma 4.2, there is a sequence $\langle f_i,y_i\rangle\in D^\omega$ such that $P=B_{\langle f_i,y_i\rangle}$. We show below that $S=A_{\langle f_i\rangle}$, and thus prove Theorem 1.2.

$S\subset A_{\langle f_i\rangle}$. Let $h\in S$. Then there is a $y\in[0,1]$ such that $(h,y)\in P$. Since P is $B_{\langle f_i,y_i\rangle}$ there is a subsequence $\langle f_{n_i},y_{n_i}\rangle$ of $\langle f_i,y_i\rangle$ which converges to (h,y). Therefore $\langle f_{n_i}\rangle$ converges to h, and hence, $h\in A_{\langle f_i\rangle}$.

$A_{\langle f_i\rangle}\subset S$. Suppose that $h\in A_{\langle f_i\rangle}$. Then by definition of $A_{\langle f_i\rangle}$, there is a subsequence $\langle f_{n_i}\rangle$ of $\langle f_i\rangle$ which converges to h. Consider the corresponding sequence $\langle y_{n_i}\rangle$ of points in $[0,1]$. By compactness it has a convergent subsequence — call it $\langle y_{m_i}\rangle$ — and let y be the limit of $\langle y_{m_i}\rangle$. Now (h,y) is a point in $C\times[0,1]=D$ and the sequence $\langle f_{m_i},y_{m_i}\rangle\in D^\omega$ converges to (h,y); so by definition of $B_{\langle f_i,y_i\rangle}$, $(h,y)\in B_{\langle f_i,y_i\rangle}$. Since $B_{\langle f_i,y_i\rangle}=P$ and S is the projection of P, $h\in S$. ■

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Solution of Kuratowski's problem on function having the Baire property, I.

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Abstract. In this paper it is proved: ZFC + "there is measurable cardinal" is equiconsistent with ZFC + "there is a Baire metric space X, a metric space Y, and a function $f\colon X\to Y$ having the Baire property such that there is no meager set $F\subseteq X$ for which $f|X\setminus F$ is continuous".

In 1935 K. Kuratowski [11] posed the following problem: whether a function $f\colon X\to Y$ having the Baire property, where X is completely metrizable and Y is metrizable, is continuous apart from a meager set (cf. P. 6 [12]).

In this paper it will be proved:

THEOREM. *The following theories are equiconsistent*:

(1) ZFC + $\exists$ *measurable cardinal*;

(2) ZFC + *there is a complete metric space X, a metric space Y, and a function $f\colon X\to Y$ having the Baire property such that there is no meager set $F\subseteq X$ for which $f|X\setminus F$ is continuous*;

(3) ZFC + *there is a Baire metric space X, a metric space Y, and a function $f\colon X\to Y$ having the Baire property such that there is no meager set $F\subseteq X$ for which $f|X\setminus F$ is continuous.*

1. Definitions and the basic facts. Let X be a topological space, and $A\subseteq X$. The set A is said to have the *Baire property* if

$$A=(G\setminus P_1)\cup P_2\,,$$

where G is open and P_1, P_2 are meager sets (for basic facts see Kuratowski [10]). A map $f\colon X\to Y$ has the *Baire property* iff for each open set $V\subseteq Y$, $f^{-1}(V)$ has the Baire property.

1.1 In [4] the equivalence of the following statements has been proved: Let X, Y be metric

(i) for each subspace $X^*=G\setminus F$ of X, where G is a nonempty open set and F is a meager set and for each partition $\mathscr{F}$ of X^* into meager sets, there is a family $\mathscr{F}'\subseteq\mathscr{F}$ such that $\mathscr{F}'$ does not have the Baire property.