References

[1] Chang and Keisler, *Model Theory*, North Holland, 1973.
[2] K. J. Devlin, *Constructibility*, Springer Verlag, 1984.
[3] Emeryk, R. Frankiewicz and Kulpa, *Remarks on Kuratowski's Theorem on meager sets*, Bull. Acad. Pol. Sci. 27, 6, (1979), 493–498.
[4] Emeryk and R. Frankiewicz, Kulpa, *On function having the Baire property*, Bull. Acad. Pol. Sci. 27, 6, (1979), 489–491.
[5] W. G. Fleissner and K. Kunen, *Barely Baire spaces*, Fund. Math. 101, (1978).
[6] R. Frankiewicz, *Solution of Kuratowski's problem*, II (appendix), preprint.
[7] R. Frankiewicz, Gutek, S. Plewik and Roczniak, *On the theorem on measurable selectors*, Bull. Acad. Pol. Sci. 30, 1—2, (1982), 33–40.
[8] Jech, *Set Theory*, Academic Press, 1976.
[9] Jech and Mitchel, *Some examples of precipitous ideals*, Ann. Pure Appl. Logic 24 (1983), 24–40.
[10] K. Kuratowski, *Topology*, vol. 1, Academic Press, 1976.
[11] — *Quelques problèmes concernant les espaces métriques nonseparables*, Fund. Math. 25 (1935), 534–545.
[12] Rogers, Jayne eds., *Analytic sets*, Academic Press, 1980.

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$P_\varkappa\lambda$ Partition relations

by

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Abstract. We study the partition relations $X \to (I^+)^n$, $X \to (uhf)^n$, and $X \to (uhf, I^+)^n$ where $X \subseteq P_\varkappa\lambda$, $n \geqslant 1$, I is a proper, nonprincipal $\varkappa$-complete ideal on $P_\varkappa\lambda$, and a *uhf* is an unbounded homogeneous function (see 1.3, 2.1 below).

THEOREM. *If* $\lambda^{<\varkappa} = \lambda$, *then* $\varkappa$ *is* λ-*ineffable iff* $X \to (NS^+_{\varkappa\lambda})^2$ *holds for some* $X \subseteq P_\varkappa\lambda$. (4.2, 4.3).

THEOREM. *If* $X \to (SNS^+_{\varkappa\lambda})^2$ *holds for some* $X \subseteq P_\varkappa\lambda$, *then* $\varkappa$ *is almost* λ-*ineffable*. (1.7).

THEOREM. *If* $\lambda^{<\varkappa} = \lambda$ *and* $\varkappa$ *is almost* λ-*ineffable, then* $X \to (I^+_{\varkappa\lambda})^2$ *holds for every* $X \in NAIn^+_{\varkappa\lambda}$. (4.2).

THEOREM. *If* $\lambda^{<\varkappa} = \lambda$, *then* $\varkappa$ *is mildly* λ-*ineffable iff* $X \to (uhf)^n$ *holds for every* $X \in I^+_{\varkappa\lambda}$ *and* $n \geqslant 2$. (2.4)

THEOREM. *If* $\lambda^{<\varkappa} = \lambda$ *and* $\varkappa$ *has the* λ-*Shelah property, then* $X \to (uhf, NSh^+_{\varkappa\lambda})^2$ *holds for every* $X \in NSh^+_{\varkappa\lambda}$. (5.4).

All of the ideal-theoretic notation is explained in 0.0 and 0.4.

0. Introduction

0.0. Notation and basic facts. Unless we specify otherwise, $\varkappa$ denotes an uncountable regular cardinal and λ a cardinal $\geqslant \varkappa$. For any such pair, $P_\varkappa\lambda$ denotes the set $\{x \subseteq \lambda:\ |x| < \varkappa\}$.

The basic combinatorial notions are defined here for $P_\varkappa\lambda$ as in Jech [12]. For any $x \in P_\varkappa\lambda$, $\hat{x}$ denotes the set $\{y \in P_\varkappa\lambda:\ x \subseteq y\}$. $X \subseteq P_\varkappa\lambda$ is said to be *unbounded* iff $(\forall x \in P_\varkappa\lambda)(X \cap \hat{x} \neq 0)$, and $I_{\varkappa\lambda}$ denotes the *ideal of not unbounded subsets of* $P_\varkappa\lambda$. In the sequel, an "ideal on $P_\varkappa\lambda$" is always a "proper, nonprincipal, $\varkappa$-complete ideal on $P_\varkappa\lambda$ extending $I_{\varkappa\lambda}$" unless we specify otherwise. Further, for any ideal I on $P_\varkappa\lambda$, I^+ denotes the set $\{X \subseteq P_\varkappa\lambda:\ X \notin I\}$, and I^* the filter dual to I; $FSF_{\varkappa\lambda}$ denotes $I_{\varkappa\lambda}{}^*$

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$C \subseteq P_\varkappa\lambda$ is said to be *closed* in $P_\varkappa\lambda$ iff $(\forall X \subseteq C)$ (X is a $\subseteq$-chain of length $< \varkappa \to \bigcup X \in C$), and is called a *cub* iff it is both closed and unbounded. Further $S \subseteq P_\varkappa\lambda$ is said to be *stationary* in $P_\varkappa\lambda$ iff $S \cap C \neq 0$ for every cub $C \subseteq P_\varkappa\lambda$. Finally, $NS_{\varkappa\lambda}$ denotes the nonstationary ideal on $P_\varkappa\lambda$, and $CF_{\varkappa\lambda}$ its dual.

The *diagonal union* $\nabla(X_\alpha\colon \alpha < \lambda)$ of a λ-sequence $(X_\alpha\colon \alpha < \lambda)$ of subsets of $P_\varkappa\lambda$ is defined by $\nabla(X_\alpha\colon \alpha<\lambda) = \{x \in P_\varkappa\lambda\colon (\exists\alpha \in x)(x \in X_\alpha)\}$, and for any ideal I on $P_\varkappa\lambda$, ∇I denotes the set $\{X \subseteq P_\varkappa\lambda\colon (\exists (X_\alpha\colon \alpha<\lambda) \in^\lambda I)(X = \nabla(X_\alpha\colon \alpha<\lambda))\}$. It is easy to see that ∇I is a (not necessarily proper) ideal on $P_\varkappa\lambda$ extending I.

An ideal I is said to be *normal* iff $\nabla I = I$, equivalently iff every function $f\colon P_\varkappa\lambda \to \lambda$ which is regressive on a set in I^+ (i.e. with the property that $\{x \in P_\varkappa\lambda\colon f(x) \in x\} \in I^+$) is constant on a set in I^+. Jech proved in [12] that $NS_{\varkappa\lambda}$ is normal, and we proved in [3] that it is the *smallest* normal ideal on $P_\varkappa\lambda$ extending $I_{\varkappa\lambda}$.

In [3], we defined $C \subseteq P_\varkappa\lambda$ to be a *strong cub* iff $(\forall X \subseteq C)(|X| < \varkappa \to \bigcup X \in C)$. It is easy to see that the family of strong cubs generates a $\varkappa$-complete filter on $P_\varkappa\lambda$. We denote this filter by $SCF_{\varkappa\lambda}$ (the *strong cub filter* on $P_\varkappa\lambda$) and its dual by $SNS_{\varkappa\lambda}$. In [3] we used some results of Menas [15] to show that $\nabla I_{\varkappa\lambda} = SNS_{\varkappa\lambda} \subsetneqq NS_{\varkappa\lambda} = \nabla\nabla I_{\varkappa\lambda}$.

0.1. In [5], [7] we studied mild λ-ineffability and the λ-Shelah property as natural $P_\varkappa\lambda$ generalizations of weak compactness. The former notion, which is due to Di Prisco and Zwicker [11] is ideal-theoretically weak (see 0.4 below). The latter notion is due to us [4], [5], and is ideal-theoretically strong (see 0.4). Our definition of this notion was inspired by Shelah's work in [17].

In [7] we provided characterizations of these two notions in terms of suitable $P_\varkappa\lambda$ generalizations of the tree property and in terms of $P_\varkappa\lambda$ filter extension properties. We also provided a characterization of the λ-Shelah property in terms of a $P_\varkappa\lambda$ Π^1_1 — indescribability property suggested by Baumgartner in [2]. In this paper, we provide partition-theoretic characterizations of mild λ-ineffability (Theorem 2.4) and λ-ineffability (Theorem 4.3), a partition-theoretic condition for almost λ-ineffability (Theorem 1.7), and partition-theoretic consequences of almost λ-ineffability (Theorem 4.2) and the λ-Shelah property (Theorem 4.4). For Theorems 2.4, 4.2, 4.3 and 5.4, we assume that $\lambda^{<\varkappa} = \lambda$.

0.2. Jech [11] provided natural $P_\varkappa\lambda$ analogues of $\varkappa \to (\varkappa)^2$ and $\varkappa \to (\varkappa,\text{ stationary set})^2$. He defined $\mathrm{Part}(\varkappa, \lambda)$ to hold *iff* for every $f\colon [P_\varkappa\lambda]^2 \to 2$,

$$(\exists i \in 2)(\exists H \in I^+_{\varkappa\lambda})(\forall x, y \in H)(x \subset y \vee y \subset x \to f(\{x, y\}) = i).$$

Finally, he proved that if $\mathrm{Part}(\varkappa, \lambda)$ holds for some $\lambda \geqslant \varkappa$, then $\varkappa$ is weakly compact.

Magidor [14] proved that $\varkappa$ is supercompact *iff* $\varkappa$ is λ-ineffable for every $\lambda \geqslant \varkappa$, and that if $\mathrm{Part}^*(\varkappa, \lambda)$ holds, then $\varkappa$ is λ-ineffable. Menas [16] showed that if $\varkappa$ is $2^{\lambda^{<\varkappa}}$-supercompact, then $\mathrm{Part}^*(\varkappa, \lambda)$ holds. Thus $\varkappa$ is supercompact iff $\mathrm{Part}^*(\varkappa, \lambda)$ holds for every $\lambda \geqslant \varkappa$. *But we do not know if* $\mathrm{Part}^*(\varkappa, \lambda)$ *follows just from λ-ineffability.*

Di Prisco and Zwicker [11] proved that $\varkappa$ is strongly compact *iff* $\varkappa$ is mildly λ-ineffable for every $\lambda \geqslant \varkappa$. Several individuals (e.g. Baumgartner [2], Carr [5], Di Prisco [10]) have independently shown that if $\mathrm{Part}^3(\varkappa, \lambda)$ holds, then $\varkappa$ is mildly λ-ineffable — or something amounting to this. Thus *if* $\mathrm{Part}^3(\varkappa, \lambda)$ holds for every $\lambda \geqslant \varkappa$, *then* $\varkappa$ is strongly compact. *Does the converse of this hold*; *does* $\mathrm{Part}^3(\varkappa, \lambda)$ *follow from mild λ-ineffability*?

0.3. Repeated efforts to obtain "$\varkappa$ is mildly λ-ineffable $\to \mathrm{Part}^3(\varkappa, \lambda)$", "$\mathrm{Part}^2(\varkappa, \lambda) \to \varkappa$ is mildly λ-ineffable" failed miserably. This led us to wonder if $\mathrm{Part}(\varkappa, \lambda)$ is the "right" $P_\varkappa\lambda$ analogue of $\varkappa \to (\varkappa)^2$. We subsequently found that a notion due to Baumgartner [2] appeared to be more suitable in some respects. The relations studied in sections 2, 5 below are a slightly modified version of this notion.

In section 1 we define the basic partition relation, and establish a partition-theoretic condition for almost λ-ineffability. In section 3, we establish some facts that are needed in the sequel, and which relate to Zwicker's work in [18], [19].

We conclude this section with a brief description of the ideal-theoretic notation used in the sequel.

0.4. For any uncountable regular cardinal $\varkappa$ and any cardinal $\lambda \geqslant \varkappa$,

$$X \subseteq P_\varkappa\lambda \text{ is said to be } \begin{cases} \lambda\text{-}\textit{ineffable} & (1),\\ \textit{almost } \lambda\text{-}\textit{ineffable} & (2),\\ \textit{mildly } \lambda\text{-}\textit{ineffable} & (3),\end{cases}$$

iff for every $(A_x\colon x \in X)$ such that $(\forall x \in X)(A_x \subseteq x)$,

$$\begin{cases} (\exists A \subseteq \lambda)(\{x \in X\colon A_x = A \cap x\} \in NS^+_{\varkappa\lambda}),\\ (\exists A \subseteq \lambda)(\{x \in X\colon A_x = A \cap x\} \in I^+_{\varkappa\lambda}),\\ (\exists A \subseteq \lambda)(\forall x \in P_\varkappa\lambda)(\{y \in X \cap \hat{x}\colon A_y \cap x = A \cap x\} \in I^+_{\varkappa\lambda}).\end{cases}$$

Finally $X \subseteq P_\varkappa\lambda$ is said to have the λ-*Shelah property* (4) iff for every $(f_x\colon x \in X)$ such that $(\forall x \in X)(f_x\colon x \to x)$,

$$(\exists f\colon \lambda \to \lambda)(\forall x \in P_\varkappa\lambda)(\{y \in X \cap \hat{x}\colon f_y|x = f|x\} \in I^+_{\varkappa\lambda}).$$

Thus $\varkappa$ *is λ-ineffable* (almost λ-ineffable, mildly λ-ineffable, has the λ-Shelah property resp.) *iff $P_\varkappa\lambda$ has property* (1) ((2), (3), (4) resp.)

Let $NIn_{\varkappa\lambda}$, $NAI_{\varkappa\lambda}$, $NMI_{\varkappa\lambda}$, $NSh_{\varkappa\lambda}$ resp. denote the sets of all those subsets of $P_\varkappa\lambda$ which do *not* have property (1), (2), (3), (4) resp. We showed in [5], [6] that $\varkappa$ *is λ-Shelah* (almost λ-ineffable, λ-ineffable resp.) *iff $NSh_{\varkappa\lambda}$* ($NAI_{\varkappa\lambda}$, $NIn_{\varkappa\lambda}$ resp.) *is a normal ideal on* $P_\varkappa\lambda$, and that $\varkappa$ *is mildly λ-ineffable iff* $NMI_{\varkappa\lambda} = I_{\varkappa\lambda}$. Further, we showed that $NSh_{\varkappa\lambda} \subseteq NAI_{\varkappa\lambda}$ and that $\varkappa$ *is supercompact iff $\varkappa$ is λ-Shelah for every $\lambda \geqslant \varkappa$ iff $\varkappa$ is almost λ-ineffable for every $\lambda \geqslant \varkappa$.*

1. $P_\varkappa\lambda$ partition relations,

1.1. Definitions. For each $x, y \in P_\varkappa\lambda$ write $x < y$ iff $0 \neq x \subset y$ and $|x| < |y \cap \varkappa|$, and for each $x \in P_\varkappa\lambda$, let $\tilde{x}$ denote the set $\{y \in P_\varkappa\lambda\colon x < y\}$ and $\varkappa_x$ the cardinal $|x \cap \varkappa|$.

1.2. Remark. It is easy to see that $\tilde{x}$ is a strong cub for each $x \in P_\varkappa\lambda$ (see 0.0) and hence that $\{\tilde{x}: x \in P_\varkappa\lambda\}$ $\varkappa$-generates a filter on $P_\varkappa\lambda$. Moreover, this filter is just $FSF_{\varkappa\lambda}$, the dual of $I_{\varkappa\lambda}$.

1.3. Definition. For any finite $n \geqslant 1$ and $X \subseteq P_\varkappa\lambda$, $(X)^n$ denotes the set $\{(x_0, \ldots, x_{n-1}) \in X^n: x_0 < \ldots < x_{n-1}\}$. For any ideal I on $P_\varkappa\lambda$, $X \to (I^+)^n$ denotes the assertion that for every partition $f: (X)^n \to 2$,

$$(\exists i \in 2)(\exists H \subseteq X)\ (H \in I^+ \wedge f''((H)^n) = \{i\}).$$

H is said to be *i-homogeneous* for f.

1.4. Remark. It is easy to see that if $X \to (I^+)^{n+1}$ holds, then so does $X \to (I^+)^n$. Further, routine arguments show that *if $P_\varkappa\lambda \to (I^+)^n$ holds, then*

$$\{X \subseteq P_\varkappa\lambda: X \nrightarrow (I^+)^n\}$$

is an ideal on $P_\varkappa\lambda$ extending I.

1.5. Remark. The relations defined in 1.3 look weaker than the obvious generalizations of Jech's Part$(\varkappa, \lambda)$ and Part*$(\varkappa, \lambda)$; in those generalizations we would use $\subset$ instead of $<$. However, a slight modification of the arguments used in [2], [5], [10] to prove that if Part$^3(\varkappa, \lambda)$ holds, then $\varkappa$ is mildly λ-ineffable yields the same conclusion from $P_\varkappa\lambda \to (I^+_{\varkappa\lambda})^3$ as defined in 1.3. And a slight modification of Magidor's proof in [14] that if Part*$(\varkappa, \lambda)$ holds, then $\varkappa$ is λ-ineffable yields the same conclusion from $P_\varkappa\lambda \to (NS^+_{\varkappa\lambda})^2$. In section 4 we will show that if $\lambda^{<\varkappa} = \lambda$ and $\varkappa$ is λ-ineffable, then $P_\varkappa\lambda \to (NS^+_{\varkappa\lambda})^2$ (Theorem 4.2). Our main reason for using $<$ as defined in 1.1 instead of $\subset$ is that it seems to be what we need to make the proofs of 4.2 and 5.4 work.

The main result of this section is that *if $P_\varkappa\lambda \to (SNS^+_{\varkappa\lambda})^2$ holds, then $\varkappa$ is* almost λ-ineffable; this is an immediate consequence of Theorem 1.7 below. Our proof of 1.7 requires a preliminary (Lemma 1.6) which is proved in [3].

1.6. Lemma. *For any $X \subseteq P_\varkappa\lambda$, $X \in SNS^+_{\varkappa\lambda}$ iff for every regressive function $f: X \to \lambda$,* $(\exists\alpha < \lambda)(f^{-1}(\{\alpha\}) \in I^+_{\varkappa\lambda})$. ∎

An argument inspired by Magidor's proof in [14] that if Part*$(\varkappa, \lambda)$ holds, then $\varkappa$ is λ-ineffable now yields our result:

1.7. Theorem. *For any $X \subseteq P_\varkappa\lambda$, if $X \to (SNS^+_{\varkappa\lambda})^2$ holds, then $X \in NAIn^+_{\varkappa\lambda}$; thus if $X \to (SNS^+_{\varkappa\lambda})^2$ holds for some $X \subseteq P_\varkappa\lambda$, $\varkappa$ is almost λ-ineffable.*

Proof. Let $(A_x: x \in X)$ be such that $(\forall x \in X)(A_x \subseteq x)$, and let $\prec$ denote the lexicographic ordering on $P_\varkappa\lambda$. Define $f: (X)^2 \to 2$ by

$$f(x, y) = \begin{cases} 0 & \text{if } A_x \prec A_y \cap x, \\ 1 & \text{otherwise.} \end{cases}$$

Now let $H \in P(X) \cap SNS^+_{\varkappa\lambda}$ be homogeneous for f.

We define the required $A \subseteq \lambda$ inductively as follows. Pick $\alpha < \lambda$ and suppose that we've defined $A \cap \alpha$ so that $\{x \in H: A_x \cap \alpha = A \cap x \cap \alpha\} \in I^+_{\varkappa\lambda}$.

If H is 0-homogeneous, put $\alpha \in A$ iff

(i) $(\exists x \in H \cap \{\tilde{\alpha}\})(\alpha \in A_x \wedge A_x \cap \alpha = A \cap x \cap \alpha)$.

If H is 1-homogeneous, put $\alpha \notin A$ iff

(ii) $(\exists x \in H \cap \{\tilde{\alpha}\})(\alpha \notin A_x \wedge A_x \cap \alpha = A \cap x \cap \alpha)$.

Suppose by way of contradiction that this doesn't work, i.e. that

$$\{x \in X: A_x = A \cap x\} \in I_{\varkappa\lambda}.$$

Pick $z \in P_\varkappa\lambda$ such that $(\forall x \in X \cap \tilde{z})(A_x \neq A \cap x)$, and notice that $H \cap \tilde{z} \in SNS^+_{\varkappa\lambda}$. For each $x \in H \cap \tilde{z}$, let α_x be the least ordinal in the symmetric difference $A_x \triangle A \cap x$, and then let $\alpha < \lambda$ be such that $Y = \{y \in H \cap z: \alpha_y = \alpha\} \in I^+_{\varkappa\lambda}$; such an α exists by 1.6 above. We derive the required contradiction by showing that in each of cases (1) and (2) below, neither $\alpha \in A$ nor $\alpha \notin A$ is possible.

Case (1). Suppose that H is 0-homogeneous. Then

(iii) $(\forall x, y \in H)(x < y \to A_x \prec A_y \cap x)$.

First, suppose that $\alpha \notin A$. Then $(\forall y \in Y)(\alpha \in A_y \wedge A_y \cap \alpha = A \cap \alpha)$. But then, each element of Y witnesses (i) above for α thereby contradicting the assumption $\alpha \notin A$.

So now suppose that $\alpha \in A$ and hence that $(\forall y \in Y)(\alpha \notin A_y)$. Now pick $x_\alpha \in H \cap \{\tilde{\alpha}\}$ witnessing (i) above for α, and then pick $y \in Y \cap \tilde{x}_\alpha \subseteq H \cap \tilde{x}_\alpha$. We will show that the least ordinal in $A_{x_\alpha} \triangle A_y \cap x_\alpha$ is α itself; this will be the required contradiction since (iii) above requires that this ordinal be in A_y.

Since $y \in Y$, $A_y \cap \alpha = A \cap y \cap \alpha$, so $A_y \cap x_\alpha \cap \alpha = A \cap x_\alpha \cap \alpha$. Since x_α satisfies (i) above for α, $A \cap x_\alpha \cap \alpha = A_{x_\alpha} \cap \alpha$. Thus $A_y \cap x_\alpha \cap \alpha = A_{x_\alpha} \cap \alpha$, so α is the least ordinal in $A_{x_\alpha} \triangle A_y \cap x_\alpha$ as claimed above.

Case (2). Suppose that H is 1-homogeneous. Then

$$(\forall x, y \in H)(x < y \to A_y \cap x \preccurlyeq A_a) \ldots \text{(iv)}$$

Argue as in case (1) using (ii) and (iv) above in place of (i) and (iii). ∎

1.8. Remark. Notice that an easy modification of the proof of Theorem 1.7 yields the following stronger result: *if $X \to (SNS^+_{\varkappa\lambda})^2$ holds for some $X \subseteq P_\varkappa\lambda$, then for any $(A_x: x \in P_\varkappa\lambda)$ such that* $(\forall x \in P_\varkappa\lambda)(A_x \subseteq x)$,

$$(\exists A \subseteq \lambda)(\{x \in P_\varkappa\lambda: A_x = A \cap x\} \in SNS^+_{\varkappa\lambda}).$$

In 4.4 below, we shall see that the converse of this is true too if $\lambda^{<\varkappa} = \lambda$.

Recall that by our work in [5, 6], $\varkappa$ is supercompact iff $\varkappa$ is almost λ-ineffable for every $\lambda \geqslant \varkappa$, and that by Menas's work in [16], if $\varkappa$ is $2^{\lambda^{<\varkappa}}$-supercompact, then $P_\varkappa\lambda \to (NS^+_{\varkappa\lambda})^2$. These facts together with Theorem 1.7 yield the following result.

1.9. Corollary. *$\varkappa$ is supercompact iff $P_\varkappa\lambda \to (SNS^+_{\varkappa\lambda})^2$ holds for every $\lambda \geqslant \varkappa$.* ∎

2. A partition-theoretic characterization of mild λ-ineffability. In some ways, the following relations, which are based on a notion of Baumgartner [2] seem to be better $P_\varkappa\lambda$ analogues of $\varkappa \to (\varkappa)^2$ and $\varkappa \to (\varkappa, \text{stationary set})^2$.

2.1. Definition. For any finite $n \geq 1$ and $X \subseteq P_\varkappa\lambda$, $X \to (uhf)^n$ denotes the assertion that for every partition $f\colon (X)^n \to 2$ there are an $i \in 2$ and an $h\colon P_\varkappa\lambda \to X$ such that

(1) $(\forall x \in P_\varkappa\lambda)(x < h(x))$, and

(2) $(\forall (x_0, \ldots, x_{n-1}) \in (P_\varkappa\lambda)^n)(h(x_0) < \ldots < h(x_{n-1}) \wedge f(h(x_0), \ldots, h(x_{n-1})) = i)$.

Any such h is called an *unbounded homogeneous function of color* i for f; hence the abbreviation *uhf*.

Further, for any ideal I on $P_\varkappa\lambda$, $X \to (uhf, I^+)^n$ denotes the assertion that for any $f\colon (X)^n \to 2$,

either there is a *uhf* of color 0 for f

or else there is a homogeneous $H \in P(X) \cap I^+$ of color 1 for f.

It is easy to see that for any $n \geq 2$ and any $X \subseteq P_\varkappa\lambda$, if $X \to (uhf)^{n+1}$ holds, then so does $X \to (uhf)^n$, and that if $X \to (I^+_{\varkappa\lambda})^n$ holds, so does $X \to (uhf)^n$. Further, routine arguments show that if $P_\varkappa\lambda \to (uhf)^n$ holds, then $\{X \subseteq P_\varkappa\lambda\colon X \nrightarrow (uhf)^n\}$ is an ideal on $P_\varkappa\lambda$; likewise, if $P_\varkappa\lambda \to (uhf, I^+)^n$ holds, $\{X \subseteq P_\varkappa\lambda\colon X \nrightarrow (uhf)^n\}$ is an ideal extending I. Finally, an argument similar to the one Jech used in [12] to show that if $\mathrm{Part}(\varkappa, \lambda)$ holds for some $\lambda \geq \varkappa$ then $\varkappa$ is weakly compact yields the same conclusion from $P_\varkappa\lambda \to (uhf)^2$.

It is clear that for every $n \geq 1$ and $X \subseteq P_\varkappa\lambda$, *if* $X \to (I^+_{\varkappa\lambda})^n$ holds, *then* so does $X \to (uhf)^n$; likewise, *if* $X \to (I^+_{\varkappa\lambda}, NS^+_{\varkappa\lambda})^n$ holds, then so does $X \to (uhf, NS^+_{\varkappa\lambda})^n$. However, repeated efforts to obtain the converses of these have failed as have efforts to obtain $P_\varkappa\lambda \to (I^+_{\varkappa\lambda})^2$ and $P_\varkappa\lambda \to (I^+_{\varkappa\lambda}, NS^+_{\varkappa\lambda})^2$ from the hypotheses of Theorems 2.4(1) and 5.4 respectively.

A consequence of Baumgartner's work in [2] is that if $\lambda^{<\varkappa} = \lambda$, then $\varkappa$ is mildly λ-ineffable *iff* $P_\varkappa\lambda \to (uhf)^n$ holds for every $n \geq 1$. We will sharpen that result here by proving that if $\lambda^{<\varkappa} = \lambda$, then $\varkappa$ is mildly λ-ineffable *iff* $I^+_{\varkappa\lambda} \to (uhf)^n$ holds for every $n \geq 1$, i.e. $X \to (uhf)^n$ holds for every $X \in I^+_{\varkappa\lambda}$ and every $n \geq 1$ (Theorem 2.4).

The reverse implication (which does not require the assumption $\lambda^{<\varkappa} = \lambda$) is proved by essentially the argument used independently by several individuals to prove the result stated in 0.2 above. The forward implication follows by a $P_\varkappa\lambda$ version of a familiar proof that $\varkappa \to (\varkappa)^n$ holds for every $n \geq 1$ if $\varkappa$ is weakly compact (e.g. see [13]). This requires two preliminaries.

2.2. Lemma. *$\varkappa$ is mildly λ-ineffable iff $NMI^+_{\varkappa\lambda} = I^+_{\varkappa\lambda}$.*

Proof. See [6], Proposition 1.4. ∎

2.3. Lemma. *If $\lambda^{<\varkappa} = \lambda$, then for any $X \in I^+_{\varkappa\lambda}$, X is mildly λ-ineffable iff for any $\varkappa$-complete field B of subsets of $P_\varkappa\lambda$ such that $|B| = \lambda$ and $\{X\} \cup \{\tilde{x}\colon x \in P_\varkappa\lambda\} \subseteq B$, there is a $\varkappa$-complete ultrafilter U in B such that $\{X\} \cup \{\tilde{x}\colon x \in P_\varkappa\lambda\} \subseteq U$.*

Proof. See the proof of Theorem 3.2 in [7]. ∎

2.4. Theorem. (1) *If $\lambda^{<\varkappa} = \lambda$ and $\varkappa$ is mildly λ-ineffable, then $X \to (uhf)^n$ holds for every $X \in I^+_{\varkappa\lambda}$ and every $n \geq 1$, and*

(2) *if $X \to (uhf)^3$ holds for some $X \subseteq P_\varkappa\lambda$, then $\varkappa$ is mildly λ-ineffable.*

Proof. (1) Suppose that $\lambda^{<\varkappa} = \lambda$ and $\varkappa$ is mildly λ-ineffable, and pick $X \in I^+_{\varkappa\lambda} = NMI^+_{\varkappa\lambda}$, $n \geq 2$ and $f\colon (X)^n \to 2$. Further let U be any $\varkappa$-complete ultrafilter containing $\{X\} \cup \{\tilde{x}\colon x \in P_\varkappa\lambda\}$ in the $\varkappa$-complete field B of subsets of $P_\varkappa\lambda$ generated by $\{X\} \cup \{\tilde{x}\colon x \in P_\varkappa\lambda\}$; the assumption $\lambda^{<\varkappa} = \lambda$ guarantees that $|B| = \lambda$.

For each $r < n$, define $f_{n-r}\colon (X)^{n-r} \to 2$ inductively as follows. Set $f_n = f$. Pick $k < n-1$ and assume that we've found f_{n-k+1}. Then define f_{n-k} by $f_{n-k}(x_0, \ldots, x_{n-k-1}) = i$ *iff* $\{y \in X \cap \tilde{x}_{n-k-1}\colon f_{n-k+1}(x_0, \ldots, x_{n-k-1}, y) = i\} \in U$. In this way we eventually obtain $f_1\colon X \to 2$.

Now let $i < 2$ such that $A_i = f_1^{-1}(\{i\}) \in U$. We construct a *uhf* $h\colon P_\varkappa\lambda \to X$ of color i inductively as follows.

Pick $\alpha < \lambda$, set $O_\alpha\lambda = \{x \in P_\varkappa\lambda\colon ot(x) < \alpha\}$ and suppose that we've defined $h\colon O_\alpha\lambda \to A_i$ so that

(i) $(\forall x \in O_\alpha\lambda)(h(x) \in A_i \cap \tilde{x})$ and

(ii) for every m satisfying $1 \leq m \leq n$ and every $z_0 < \ldots < z_{m-1}$ from $O_\alpha\lambda$, $h(z_0) < \ldots < h(z_{m-1})$ and $f_m(h(z_0), \ldots, h(z_{m-1})) = i$.

Now pick $x \in P_\varkappa\lambda$ such that $ot(x) = \alpha$. Notice that since $\varkappa$ is inaccessible, $Z_x = \{(z_0, \ldots, z_{m-1})\colon 1 \leq m \leq n \wedge z_0 < \ldots < z_{m-1} < x\}$ has size $< \varkappa$. Also, notice that for each m satisfying $1 \leq m \leq n$ and each $\bar{z} = (z_0, \ldots, z_{m-1}) \in Z_x$,

$$A_{\bar{z}} = \{y \in A_i \cap \tilde{x}\colon f_{m+1}(h(z_0), \ldots, h(z_{m-1}), y) = i\} \in U$$

since $f_m(h(z_0), \ldots, h(z_{m-1})) = i$. Thus $\bigcap \{A_{\bar{z}}\colon \bar{z} \in Z_x\} \in U$; thus let $h(x)$ be some element of $\bigcap \{A_{\bar{z}}\colon \bar{z} \in Z_x\}$. It is easy to see that the h obtained in this manner is a *uhf* of color i for f.

(2) Suppose that $X \to (uhf)^3$ holds for some $X \subseteq P_\varkappa\lambda$. Then $P_\varkappa\lambda \to (uhf)^3$ holds. Let $(A_x\colon x \in P_\varkappa\lambda)$ be such that $(\forall x \in P_\varkappa\lambda)(A_x \subseteq x)$ and define $f\colon (P_\varkappa\lambda)^3 \to 2$ by

$$f(x, y, z) = \begin{cases} 0 & \text{if } A_y \cap x = A_z \cap x, \\ 1 & \text{otherwise}. \end{cases}$$

Now let $h\colon P_\varkappa\lambda \to P_\varkappa\lambda$ be a *uhf* for f. We will show that h has color 0 and then use this fact to define the required $A \subseteq \lambda$.

Pick $x \in P_\varkappa\lambda$. Since $\varkappa$ is inaccessible and $\tilde{x} \in I^+_{\varkappa\lambda}$,

$$(\exists w \subseteq h(x))(W = \{y \in \tilde{x}\colon A_{h(y)} \cap h(x) = w\} \in I^+_{\varkappa\lambda}).$$

Notice that for any $y, z \in W$ such that $y < z$, $A_{h(y)} \cap h(x) = A_{h(z)} \cap h(x)$; thus $f(h(x), h(y), h(z)) = 0$.

Notice that for each $\alpha < \lambda$ *either* $(\forall y \in \{\tilde{\alpha}\})(\alpha \in A_{h(y)})$ *or else* $(\forall y \in \{\tilde{\alpha}\})(\alpha \notin A_{h(y)})$. Set $A = \{\alpha < \lambda\colon (\forall y \in \{\tilde{\alpha}\})(\alpha \in A_{h(y)})\}$. It is easy to see that this A works.

An immediate consequence of the proof of Theorem 2.4 is

2.5. Corollary. *If $\lambda^{<\varkappa} = \lambda$, then $\varkappa$ is mildly λ-ineffable iff $P_\varkappa\lambda \to (uhf)^3$ holds.*

Although it is clear that if $\lambda^{<\varkappa} = \lambda$ and $\varkappa$ is mildly λ-ineffable then $P_\varkappa\lambda \to (uhf)^2$ holds, repeated efforts to obtain the converse of this have failed.

Since $\varkappa$ is strongly compact iff $\varkappa$ is mildly λ-ineffable for every $\lambda \geqslant \varkappa$ (Di Prisco and Zwicker [11]), we obtain the following corollary, results similar to which have also been obtained by Baumgartner [2] and Di Prisco [10].

2.6. COROLLARY. *$\varkappa$ is strongly compact iff $P_\varkappa\lambda \to (uhf)^3$ holds for every $\lambda \geqslant \varkappa$.* ■

3. More about the λ-Shelah property. Recall that $X \subseteq P_\varkappa\lambda$ has the λ-Shelah property *iff* for every $(f_x\colon x \in X) \in \Pi\{x\colon x \in X\}$,

$$(\exists f\colon \lambda \to \lambda)(\forall x \in P_\varkappa\lambda)(\{y \in X \cap \hat{x}\colon f_y|x = f|x\} \in I^+_{\varkappa\lambda}).$$

Equivalently (see [7]), $X \subseteq P_\varkappa\lambda$ has the λ-Shelah property *iff* for every λ-sequence $(f_\nu\colon \nu < \lambda)$ of regressive functions on $P_\varkappa\lambda$,

$$(\exists f\colon \lambda \to \lambda)(\forall x \in P_\varkappa\lambda)(\{y \in X \cap \hat{x}\colon (\forall \nu \in x)(f_\nu(y) = f(\nu))\} \in I^+_{\varkappa\lambda}).$$

In [5] we proved that if $\lambda^{<\varkappa} = \lambda$ then $\varkappa$ has the λ-Shelah property *iff* for any λ-sequence $(f_\nu\colon \nu < \lambda)$ of regressive functions on $P_\varkappa\lambda$,

$$(\exists f\colon \lambda \to \lambda)(\forall x \in P_\varkappa\lambda)(\{y \in \hat{x}\colon (\forall \nu \in x)(f_\nu(y) = f(\nu))\} \in SNS^+_{\varkappa\lambda})$$

iff for any λ-sequence $(f_\nu\colon \nu < \lambda)$ of regresive functions on $P_\varkappa\lambda$, $(\exists f\colon \lambda \to \lambda)$ $(\forall x \in P_\varkappa\lambda)(\{y \in \hat{x}\colon (\forall \nu \in x)(f_\nu(y) = f(\nu))\} \in NS^+_{\varkappa\lambda})$. We will establish the ideal-theoretic version of this here (Theorem 3.2 below), and then use it to establish some facts that are needed in the sequel.

3.1. LEMMA. *Suppose that $\lambda^{<\varkappa} = \lambda$, pick $X \subseteq P_\varkappa\lambda$ and let $(f_\nu\colon \nu < \lambda)$ be a λ-sequence of regressive functions on $P_\varkappa\lambda$. Then for any ideal I on $P_\varkappa\lambda$, there is a λ-sequence $(g_\alpha\colon \alpha < \lambda)$ of regressive functions on $P_\varkappa\lambda$ such that*

(1) $\{f_\nu\colon \nu < \lambda\} \subseteq \{g_\alpha\colon \alpha < \lambda\}$,

(2) *for any $x \in P_\varkappa\lambda$ and any $h\colon x \to \lambda$ such that*

$$E_{xh} = \{y \in X\colon (\forall \alpha \in x)(g_\alpha(y) = h(\alpha))\} \in \nabla I,$$
$$(\exists \gamma < \lambda)(\forall \alpha < \lambda)(E_{xh} \cap g_\gamma^{-1}(\{\alpha\}) \in I),$$

and

(3) *for any ideal J on $P_\varkappa\lambda$, if there is a $g\colon \lambda \to \lambda$ such that*

$$(\forall x \in P_\varkappa\lambda)(E_{xg} = \{y \in X\colon (\forall \alpha \in x)(g_\alpha(y) = g(\alpha))\} \in J^+),$$

then there is an $f\colon \lambda \to \lambda$ such that

$$(\forall x \in P_\varkappa\lambda)(E_{xf} = \{y \in X\colon (\forall \nu \in x)(f_\nu(y) = f(\nu))\} \in J^+).$$

Proof. We construct a sequence $F_0 \subseteq \ldots \subseteq F_\alpha \subseteq \ldots (\alpha < \lambda)$ of families of regressive functions on $P_\varkappa\lambda$, each of cardinality $\leqslant \lambda$ inductively as follows.

First, set $F_0 = \{f_\nu\colon \nu < \lambda\}$. Then pick $\alpha < \lambda$ and suppose that we've found $F_0 \subseteq \ldots \subseteq F_\xi \subseteq \ldots (\xi < \alpha)$. If $\lim(\alpha)$, then set $F_\alpha = \bigcup\{F_\xi\colon \xi < \alpha\}$. Clearly $|F_\alpha| \leqslant \lambda$.

Now suppose that $\alpha = \beta+1$, and let $(f^\beta_\nu\colon \nu < \lambda)$ be an enumeration of F_β without repetitions. For each $x \in P_\varkappa\lambda$ and each $h\colon x \to \lambda$ such that

$E_{xh^\beta} = \{y \in X\colon (\forall \nu \in x)(f^\beta_\nu(y) = h(\nu))\} \in \nabla I$, let g_{xh} be a regressive function on $P_\varkappa\lambda$ such that $(\forall \gamma < \lambda)(E_{xh^\beta} \cap g_{xh}^{-1}(\{\gamma\}) \in I)$. Now let F_α be the union of F_β and the set consisting of all these g_{xh}'s. The assumption $\lambda^{<\varkappa} = \lambda$ guarantees that $|F_\alpha| = \lambda$.

Now set $G = \{F_\alpha\colon \alpha < \lambda\}$. It is clear that $|G| = \lambda$ and $\{f_\gamma\colon \nu < \lambda\} \subseteq G$. Finally, let $g_\alpha\colon \alpha < \lambda)$ be an enumeration of G without repetitions; we will show that this sequence satisfies (2) and (3).

Let $x \in P_\varkappa\lambda$ and $h\colon x \to \lambda$ be such that $E_{xh} \in \nabla I$ where

$$E_{xh} = \{y \in X\colon (\forall \alpha \in x)(g_\alpha(y) = h(\alpha))\}.$$

Now let $\beta < \lambda$ be such that $\{g_\alpha\colon \alpha \in x\} \subseteq F_\beta$; notice that such a β exists since the assumption $\lambda^{<\varkappa} = \lambda$ guarantees that $cf(\lambda) \geqslant \varkappa$. For each $\alpha \in x$, let $\nu_\alpha < \lambda$ be such that $g_\alpha = f^\beta_{\nu_\alpha}$. Set $x' = \{\nu_\alpha\colon \alpha \in x\}$ and define $h'\colon x' \to \lambda$ by $h'(\nu_\alpha) = h(\alpha)$. Then for each $y \in X$, $y \in E^\beta_{x'h'}$ *iff* $(\forall \alpha \in x)(f^\beta_{\nu_\alpha}(y) = h'(\nu_\alpha))$ *iff* $(\forall \alpha \in x)(g_\alpha(y) = h(\alpha))$ *iff* $y \in E_{xh}$. Thus $E^\beta_{x'h'} \in \nabla I$. Then since $g_{x'h} \in F_{\beta+1} \subseteq G$ is I-small on $E^\beta_{x'h'} = E_{xh'}$, it follows that (2) holds.

Now let J be any ideal on $P_\varkappa\lambda$ and suppose that $g\colon \lambda \to \lambda$ is such that

$$(\forall x \in P_\varkappa\lambda)(E_{xg} = \{y \in X\colon (\forall \alpha \in x)(g_\alpha(y) = g(\alpha))\} \in J^+).$$

For each $\nu < \lambda$, let $\alpha_\nu < \lambda$ be such that $f_\nu = g_{\alpha_\nu}$, and define $f\colon \lambda \to \lambda$ by $f(\nu) = g(\alpha_\nu)$. For each $x \in P_\varkappa\lambda$, set $x' = \{\alpha_\nu\colon \nu \in x\}$ and notice that

$$\begin{aligned} E_{xf} &= \{y \in X\colon (\forall \nu \in x)(f_\nu(y) = f(\nu))\} \\ &= \{y \in X\colon (\forall \nu \in x)(g_{\alpha_\nu}(y) = g(\alpha_\nu))\} = E_{x'g} \in J^+. \ ■ \end{aligned}$$

3.2. THEOREM. *If $\lambda^{<\varkappa} = \lambda$, then for any $X \subseteq P_\varkappa\lambda$, the following are equivalent:*

(1) *X has the λ-Shelah property;*

(2) *for any λ-sequence $(f_\nu\colon \nu < \lambda)$ of regressive functions on $P_\varkappa\lambda$,*

$$(\exists f\colon \lambda \to \lambda)(\forall x \in P_\varkappa\lambda)(E_{xf} = \{y \in X \cap \hat{x}\colon (\forall \nu \in x)(f_\nu(y) = f(\nu))\} \in SNS^+_{\varkappa\lambda}),$$

and

(3) *for any λ-sequence $(f_\nu\colon \nu < \lambda)$ of regressive functions on $P_\varkappa\lambda$,*

$$(\exists f\colon \lambda \to \lambda)(\forall x \in P_\varkappa\lambda)(E_{xf} \in NS^+_{\varkappa\lambda}).$$

Proof. It is clear that (3) → (2) → (1).

(1) → (2). Let $(f_\nu\colon \nu < \lambda)$ be a λ-sequence of regressive functions on $P_\varkappa\lambda$, and let $(g_\alpha\colon \alpha < \lambda)$ be a sequence satisfying the conclusions of the preceding lemma. Further, let $g\colon \lambda \to \lambda$ be such that $(\forall x \in P_\varkappa\lambda)$

$$(E_{xg} = \{y \in X \cap \hat{x}\colon (\forall \alpha \in x)(g_\alpha(y) = g(\alpha))\} \in I^+_{\varkappa\lambda}).$$

We will show that $(\forall x \in P_\varkappa\lambda)(E_{xg} \in SNS^+_{\varkappa\lambda})$; (2) will then follow by 3.1 (3) above.

Suppose by way of contradiction that $(\exists x \in P_\varkappa\lambda)(E_{xg} \in SNS_{\varkappa\lambda} = \nabla I_{\varkappa\lambda})$, and let $\gamma < \lambda$ be such that $(\forall \alpha < \lambda)(E_{xg} \cap g_\gamma^{-1}(\{\alpha\}) \in I_{\varkappa\lambda})$. Set $z = x \cup \{\gamma\}$. Then for any

$y \in X \cap \hat{z}$, $y \in E_{zg}$ iff $(\forall \alpha \in z)(g_\alpha(y) = g(\alpha))$ iff $(\forall \alpha \in x)(g_\alpha(y) = g(\alpha))$ and $g_\gamma(y) = g(\gamma)$ iff $y \in E_{xg} \cap g_\gamma^{-1}(\{g(\gamma)\}) \in I_{\varkappa\lambda}$; thus $E_{zg} \in I_{\varkappa\lambda}$ thereby contradicting (1).

(2) → (3). This follows by an argument similar to the one used to prove (1) → (2) using the fact that $NS_{\varkappa\lambda} = \nabla SNS_{\varkappa\lambda}$. ■

We now use 3.2 to establish some facts (3.4, 3.5, 3.6) that are needed in the sequel.

Recall that for each $x \in P_\varkappa\lambda$, $\varkappa_x$ denotes the cardinal $|x \cap \varkappa|$, and that if $\varkappa$ is a limit cardinal, then $\{x \in P_\varkappa\lambda\colon x \cap \varkappa = \varkappa_x\}$ is cub in $P_\varkappa\lambda$. Further, recall that if $\varkappa$ has the λ-Shelah property, then $\varkappa$ is inaccessible.

3.3. Lemma. *If $\lambda^{<\varkappa} = \lambda$ and $\varkappa$ has the λ-Shelah property, then for any $X \in NSh^+_{\varkappa\lambda}$ and any $f\colon X \to P_\varkappa\lambda$ such that $(\forall x \in X)(f(x) < x)$, $(\exists y \in P_\varkappa\lambda)(f^{-1}(\{y\}) \in NS^+_{\varkappa\lambda})$.*

Proof. Since $\{x \in P_\varkappa\lambda\colon x \cap \varkappa = \varkappa_x\}$ is cub in $P_\varkappa\lambda$ and $NS_{\varkappa\lambda} \subseteq NSh_{\varkappa\lambda}$, we may assume w.l.o.g. that $(\forall x \in X)(x \cap \varkappa = \varkappa_x)$. Define a regressive function $f_1\colon X \to \varkappa$ by $f_1(x) = |f(x)|$, and then let $\mu < \varkappa$ be such that $X_1 = f_1^{-1}(\{\mu\}) \in NSh^+_{\varkappa\lambda}$.

For each $x \in X_1$, fix an enumeration $(\alpha_\nu^x\colon \nu < \mu)$ of $f(x)$, and then for each $\nu < \mu$, define a regressive function $g_\nu\colon P_\varkappa\lambda \to \lambda$ by

$$g_\nu(x) = \begin{cases} \alpha_\nu^x & \text{if } x \in X_1\,, \\ \text{arbitrarily otherwise}\,. \end{cases}$$

Now let $g\colon \lambda \to \lambda$ be such that $E = \{y \in X_1 \cap \hat{\mu}\colon (\forall \nu < \mu)(g_\nu(y) = g(\nu))\} \in NS^+_{\varkappa\lambda}$; such a g exists by 3.2 above. Notice that $(\forall y \in E)(f(y) = g''(\mu))$. ■

3.4. Proposition. *If $\lambda^{<\varkappa} = \lambda$ and $\varkappa$ has the λ-Shelah property, then*

(1) *$\{x \in P_\varkappa\lambda\colon x \cap \varkappa$ is an inaccessible cardinal$\} \in NSh^*_{\varkappa\lambda}$, and*

(2) *for any bijection $\varphi\colon P_\varkappa\lambda \to \lambda$, $\{x \in P_\varkappa\lambda\colon \varphi''(P_{\varkappa_x}x) = x\} \in NSh^*_{\varkappa\lambda}$.*

Proof. (1) It is easy to see that since $\varkappa$ is inaccessible, $C = \{x \in P_\varkappa\lambda\colon x \cap \varkappa$ is a strong limit cardinal$\}$ is cub in $P_\varkappa\lambda$ and hence is in $NSh^*_{\varkappa\lambda}$. Thus it will suffice to prove that $\{x \in C\colon \varkappa_x \text{ is regular}\} \in NSh^*_{\varkappa\lambda}$.

Suppose that the above set *is not* in $NSh^*_{\varkappa\lambda}$ and hence that

$$X = \{x \in C\colon \varkappa_x \text{ is singular}\} \in NSh^+_{\varkappa\lambda}\,.$$

For each $x \in X$, let y_x be a cofinal subset of $\varkappa_x \subseteq x$ of cardinality $< \varkappa_x$, and then use 3.3 to find a y such that $X_1 = \{x \in X\colon y_x = y\} \in NS^+_{\varkappa\lambda}$. This is the required contradiction since $(\forall x \in X_1)(\varkappa_x = \sup(y))$ *but* $(\forall \alpha < \varkappa)(\{x \in P_\varkappa\lambda\colon \varkappa_x = |\alpha|\} \in I_{\varkappa\lambda})$.

(2) Fix a bijection $\varphi\colon P_\varkappa\lambda \to \lambda$. Arguments similar to those used in [5], [15] show that $\{x \in P_\varkappa\lambda\colon (\forall \alpha \in x)(\varphi^{-1}(\alpha) < x)\}$ is cub in $P_\varkappa\lambda$ and hence is in $NSh^*_{\varkappa\lambda}$. So it remains to prove that $\{x \in P_\varkappa\lambda\colon (\forall y < x)(\varphi(y) \in x)\} \in NSh^*_{\varkappa\lambda}$.

Suppose that the above set *is not* in $NSh^*_{\varkappa\lambda}$ and hence that

$$X = \{x \in P_\varkappa\lambda\colon (\exists y < x)(\varphi(y) \notin x)\} \in NSh^+_{\varkappa\lambda}\,.$$

For each $x \in X$, pick $y_x < x$ such that $\varphi(y_x) \notin x$, and then use 3.3 to find a $y \in P_\varkappa\lambda$ such that $X_1 = \{x \in X\colon y_x = y\} \in NS^+_{\varkappa\lambda}$. This is the required contradiction since $(\forall x \in X_1)(\varphi(y) \notin x)$ *but* $(\forall \alpha < \lambda)(\{x \in P_\varkappa\lambda\colon \alpha \notin x\} \in I_{\varkappa\lambda})$. ■

We can use 3.4 (2) to improve the result given in 3.3; we can prove that $NSh^*_{\varkappa\lambda}$ is *strongly normal* in the following sense:

3.5. Theorem. *If $\lambda^{<\varkappa} = \lambda$ and $\varkappa$ has the λ-Shelah property, then for any $X \in NSh^+_{\varkappa\lambda}$ and any $f\colon X \to P_\varkappa\lambda$ such that $(\forall x \in X)(f(x) < x)$,*

$$(\exists y \in P_\varkappa\lambda)(f^{-1}(\{y\}) \in NSh^+_{\varkappa\lambda})\,.$$

Proof. Pick $X \in NSh^+_{\varkappa\lambda}$ and $f\colon X \to P_\varkappa\lambda$ such that $(\forall x \in X)(f(x) < x)$. Further, let $\varphi\colon P_\varkappa\lambda \to \lambda$ be a bijection, set $B = \{x \in P_\varkappa\lambda\colon \varphi''(P_{\varkappa_x}x) = x\} \in NSh^*_{\varkappa\lambda}$ and $X_1 = X \cap B$. Finally, define $g\colon X_1 \to \lambda$ by $g(x) = \varphi(f(x)) \in x$ and then let $\alpha < \lambda$ be such that $g^{-1}(\{\alpha\}) \in NSh^+_{\varkappa\lambda}$. Then $f^{-1}(\{\varphi^{-1}(\alpha)\}) \in NSh^+_{\varkappa\lambda}$. ■

Since $NSh_{\varkappa\lambda} \subseteq NAI_{\varkappa\lambda} \subseteq NI_{\varkappa\lambda}$, the analogues of 3.4 and 3.5 go through under the hypotheses "$\lambda^{<\varkappa} = \lambda$ and $\varkappa$ is almost λ-ineffable" and "$\lambda^{<\varkappa} = \lambda$ and $\varkappa$ is λ-ineffable". Thus we have

3.6. Theorem. *If $\lambda^{<\varkappa} = \lambda$ and $\varkappa$ is λ-ineffable (almost λ-ineffable), then*

(1) *the sets given in 3.4 are both in $NIn^*_{\varkappa\lambda}$ ($NAI^*_{\varkappa\lambda}$), and*

(2) *$NIn_{\varkappa\lambda}$ ($NAI_{\varkappa\lambda}$) is strongly normal in the sense of 3.5.* ■

3.7. Remark. Notice that by 3.4 (2) and 3.5 we have that if $\lambda^{<\varkappa} = \lambda$ and $\varkappa$ has the λ-Shelah property, then

(1) for any bijection $\varphi\colon P_\varkappa\lambda \to \lambda$, $(\exists A \in NSh^*_{\varkappa\lambda})(\forall x, y \in A)(x < y \to \varphi(x) \in y)$, and

(2) $NSh_{\varkappa\lambda}$ is strongly normal in the sense that for any $X \in NSh^+_{\varkappa\lambda}$ and any $f\colon X \to P_\varkappa\lambda$ such that $(\forall x \in X)(f(x) < x)$, $(\exists y \in P_\varkappa\lambda)(f^{-1}(\{y\}) \in NSh^+_{\varkappa\lambda})$.

If we could also assert that under the above assumptions on $\varkappa$ and λ,

$$(\exists B \in NSh^+_{\varkappa\lambda})(\forall x, y \in B)(x \subset y \to x < y)\,,$$

we could conclude that there is an $S \in NSh^+_{\varkappa\lambda}$ (namely $S = A \cap B$) and a one-one function $c\colon S \to \lambda$ such that $(\forall x, y \in S)(x \subset y \to c(x) \in y)$ and hence that there is a stationary coding set as defined by Zwicker in [18].

If we could assert that there is a $B \in NSh^*_{\varkappa\lambda}$ such that $(\forall x, y \in B)(x \subset y \to x < y)$, then we could conclude that for any $X \in NSh^+_{\varkappa\lambda}$ and any $f\colon X \to A \cap B$ such that $(\forall x \in X)(f(x) \subset x)$, $(\exists y \in A \cap B)(f^{-1}(\{y\}) \in NSh^+_{\varkappa\lambda})$, and hence that $NSh_{\varkappa\lambda}$ is "set normal with witness index $A \cap B$" as defined by Zwicker in [19].

As yet we do not know if there is a set $B \in NSh^+_{\varkappa\lambda}$ such that

$$(\forall x, y \in B)(x \subset y \to x < y)\,.$$

However, we know that *there is no such B in any of $NSh^*_{\varkappa\lambda}$, $NAI^*_{\varkappa\lambda}$, $NIn^*_{\varkappa\lambda}$*. To see this, recall that if $\varkappa$ is supercompact and $\lambda > \varkappa$ is measurable, then there is a λ-supercompact ultrafilter on $P_\varkappa\lambda$ which does *not* have the partition property (Di Prisco [9]). Thus there is a λ-supercompact ultrafilter U on $P_\varkappa\lambda$ such that there is no $B \in U$ with the above property (Menas [16]). Since every λ-supercompact ultrafilter on $P_\varkappa\lambda$ extends $NIn^*_{\varkappa\lambda} \supseteq NAIn^*_{\varkappa\lambda} \supseteq NSh^*_{\varkappa\lambda}$, it is clear that there is no such B in any of these filters.

4. Partition-theoretic consequences of almost λ-ineffability nad λ-ineffability. As we remarked in 1.5, an easy modification of Magidor's proof in [14] that if $\mathrm{Part}^*(\varkappa, \lambda)$ holds then $\varkappa$ is λ-ineffable shows that for any $X \subseteq P_\varkappa\lambda$, if $X \to (NS^+_{\varkappa\lambda})^2$ holds, then X is λ-ineffable. We shall use Theorem 3.6 above to prove that the converse of this is true too if $\lambda^{<\varkappa} = \lambda$ (Theorem 4.2). An easy modification of this argument will also yield a partition-theoretic consequence of almost λ-ineffability. Our proof of 4.2 requires the following easy preliminary.

4.1. Proposition. *If $\lambda^{<\varkappa} = \lambda$, then for any $X \subseteq P_\varkappa\lambda$,*

(1) *if X is λ-ineffable, then for any $(B_x\colon x \in X)$ such that $(\forall x \in X)\,(B_x \subseteq P_{\varkappa_x}x)$, $(\exists B \subseteq P_\varkappa\lambda)(\{x \in X\colon B_x = B \cap P_{\varkappa_x}x\} \in NS^+_{\varkappa\lambda})$, and*

(2) *if X is almost λ-ineffable, then for any $(B_x\colon x \in X)$ such that*

$$(\forall x \in X)(B_x \subseteq P_{\varkappa_x}x),\ (\exists B \subseteq P_\varkappa\lambda)(\{x \in X\colon B_x = B \cap P_{\varkappa_x}x\} \in I^+_{\varkappa\lambda})\,.$$

Proof. We shall just prove (1) here; (2) is proved similarly. Pick $X \in NIn^+_{\varkappa\lambda}$ and $(B_x\colon x \in X)$ such that $(\forall x \in X)(B_x \subseteq P_{\varkappa_x}x)$. Further, let $\varphi\colon P_\varkappa\lambda \to \lambda$ be a bijection, and recall that by 3.6 $\{x \in P_\varkappa\lambda\colon \varphi''(P_{\varkappa_x}x) = x\} \in NIn^*_{\varkappa\lambda}$. Thus we may assume w.l.o.g. that $(\forall x \in X)(\varphi''(P_{\varkappa_x}x) = x)$. For each $x \in X$, define $A_x \subseteq x$ by $A_x = \varphi''(B_x)$, and then let $A \subseteq \lambda$ be such that $H = \{x \in X\colon A_x = A \cap x\} \in NS^+_{\varkappa\lambda}$. Now set $B = \varphi^{-1}(A)$. It is easy to see that $(\forall x \in H)(B_x = B \cap P_{\varkappa_x}x)$. ∎

4.2. Theorem. (1) *If $\lambda^{<\varkappa} = \lambda$ and $\varkappa$ is λ-ineffable, then $X \to (NS^+_{\varkappa\lambda})^2$ holds for every $X \in NIn^+_{\varkappa\lambda}$.*

(2) *If $\lambda^{<\varkappa} = \lambda$ and $\varkappa$ is almost λ-ineffable, then $X \to (I^+_{\varkappa\lambda})^2$ holds for every $X \in NAI^+_{\varkappa\lambda}$.*

Proof. Again, we shall just prove (1); (2) is proved similarly. Pick $X \in NIn^+_{\varkappa\lambda}$ and $f\colon (X)^2 \to 2$. Further, let $\varphi\colon P_\varkappa\lambda \to \lambda$ be a bijection. As in the proof of 4.1, we may assume w.l.o.g. that $(\forall x \in X)(\varphi''(P_{\varkappa_x}x) = x)$. For each $x \in X$, define $B_x \subseteq P_{\varkappa_x}x$ by $B_x = \{z \in P_{\varkappa_x}x\colon f(x, z) = 1\}$, and then let $B \subseteq P_\varkappa\lambda$ be such that $H = \{x \in X\colon B_x = B \cap x\} \in NS^+_{\varkappa\lambda}$. Now, either $H \cap B \in NS^+_{\varkappa\lambda}$ or else $H - B \in NS^+_{\varkappa\lambda}$. It is easy to see that $f''(H \cap B)^2 = \{1\}$ and $f''(H-B)^2 = \{0\}$. ∎

As a consequence of Theorem 4.2 and the remark preceding 4.1, we have

4.3. Corollary. *If $\lambda^{<\varkappa} = \lambda$, then $\varkappa$ is λ-ineffable iff $P_\varkappa\lambda \to (NS^+_{\varkappa\lambda})^2$.* ∎

4.4. Remark. It is clear that an easy modification of the proofs of 4.1, 4.2 yield the fact that *if* $\lambda^{<\varkappa} = \lambda$ and $P_\varkappa\lambda$ has the property that for every $(A_x\colon x \in P_\varkappa\lambda)$ such that $(\forall x \in P_\varkappa\lambda)(A_a \subseteq x)$, $(\exists A \subseteq \lambda)(\{x \in P_\varkappa\lambda\colon A_x = A \cap x\} \in SNS^+_{\varkappa\lambda})$ *then* $P_\varkappa\lambda \to (SNS^+_{\varkappa\lambda})^2$. Combining this with Remark 1.8, we obtain the fact that *if $\lambda^{<\varkappa} = \lambda$ then $P_\varkappa\lambda \to (SNS^+_{\varkappa\lambda})^2$ holds iff for every $(A_x\colon x \in P_\varkappa\lambda)$ such that* $(\forall x \in P_\varkappa\lambda)(A_x \subseteq x)$. $(\exists A \subseteq \lambda)(\{x \in P_\varkappa\lambda\colon A_x = A \cap x\} \in SNS^+_{\varkappa\lambda})$.

5. A partition-theoretic consequence of the λ-Shelah property. A consequence of Baumgartner's work in [2] is that if $\lambda^{<\varkappa} = \lambda$ and $\varkappa$ has the λ-Shelah property, then $P_\varkappa\lambda \to (uhf, NS^+_{\varkappa\lambda})^2$ holds. We will sharpen that result here by proving that under the above assumptions on $\varkappa$ and λ, $NSh^+_{\varkappa\lambda} \to (uhf, NSh^+_{\varkappa\lambda})^2$ holds, i.e. $X \to (uhf, NSh^+_{\varkappa\lambda})^2$ for every $X \in NSh^+_{\varkappa\lambda}$ (Theorem 5.4 below).

The proof of Theorem 5.4 requires 3.4 and 3.5 together with the Π^1_1-indescribability characterization of the λ-Shelah property given in [7]. We recall the particulars of the characterization here.

5.1. Definition. For any uncountable regular cardinal $\varkappa$ and any cardinal $\lambda \geqslant \varkappa$, the sequence $(V_\alpha(\varkappa, \lambda)\colon \alpha < \varkappa)$ is defined inductively as follows:

Set $V_0(\varkappa, \lambda) = \lambda$, $V_{\alpha+1}(\varkappa, \lambda) = P_\varkappa(V_\alpha(\varkappa, \lambda)) \cup V_\alpha(\varkappa, \lambda)$, and

$$V_\gamma = \bigcup \{V_\alpha(\varkappa, \lambda)\colon \alpha < \lambda\} \quad \text{if } \lim(\gamma)\,.$$

Finally, the set $V_\varkappa(\varkappa, \lambda) = \bigcup \{V_\alpha(\varkappa, \lambda)\colon \alpha < \varkappa\}$.

Properties of the structure $(V_\varkappa(\varkappa, \lambda), \in)$ are studied in [7].

5.2. Definition. For any finite m, n, $X \subseteq P_\varkappa\lambda$ is said to be *Π^n_m-indescribable* iff for any finite sequence $R_1, \ldots, R_k$ of subsets of $V_\varkappa(\varkappa, \lambda)$ and any Π^n_m-sentence φ, if $(V_\varkappa(\varkappa, \lambda), \in, R_1, \ldots, R_k) \models \varphi$, then

$(\exists x \in X)[x \cap \varkappa = \varkappa_x$ and

$$(V_{\varkappa_x}(\varkappa_x, x), \in, R_1 \cap V_{\varkappa_x}(\varkappa_x, x), \ldots, R_k \cap V_{\varkappa_x}(\varkappa_x, x)) \models \varphi]\,.$$

In [7, Theorem 4.7] we established the following fact.

5.3. Theorem. *If $\varkappa$ is inaccessible and $\lambda^{<\varkappa} = \lambda$, then $X \subseteq P_\varkappa\lambda$ has the λ-Shelah property iff X is Π^1_1-indescribable.* ∎

5.4. Theorem. *If $\lambda^{<\varkappa} = \lambda$ and $\varkappa$ has the λ-Shelah property, then $X \to (uhf, NSh^+_{\varkappa\lambda})^2$ holds for every $X \in NSh^+_{\varkappa\lambda}$.*

Proof. Pick $X \in NSh^+_{\varkappa\lambda}$. Since $\{x \in P_\varkappa\lambda\colon X \cap \varkappa$ is an inaccessible cardinal$\}$ $\in NSh^*_{\varkappa\lambda}$ by 3.4 (1) above, we may assume w.l.o.g. that $(\forall x \in X)$ ($x \cap \varkappa$ is an inaccessible cardinal).

Let $f\colon (X)^2 \to 2$. For each $x \in X$ we define (if possible) a *uhf* $h_x\colon P_{\varkappa_x}x \to X \cap P_{\varkappa_x}x$ of color 0 inductively as follows. Pick $x \in X$ and $\beta < \varkappa_x$, set $O_\beta x = \{z < x\colon ot(z) < \beta\}$ and suppose that we've been able to define $h_x\colon O_\beta x \to X \cap P_{\varkappa_x}x$ so that

(i) $(\forall z \in O_\beta x)(z < h_x(z) \wedge f(h_x(z), x) = 0)$, and

(ii) $(\forall z_0, z_1 \in O_\beta x)(z_0 < z_1 \to h_x(z_0) < h_x(z_1) \wedge f(h_x(z_0), h_x(z_1)) = 0)$.

Suppose that for *every* $y \in P_{\varkappa_x}x$ of order type β there is a $z \in X \cap P_{\varkappa_x}x$ such that

1) $\bigcup \{h_x(u)\colon u < y\} < z$ and $y < z$,

2) $(\forall u < y)(f\ h_x(u), z) = 0)$ and

3) $f(z, x) = 0$.

Then for each $y \in P_{\varkappa_x}x$ of order type β, let $h_x(y)$ be a set z in $X \cap P_{\varkappa_x}x$ witnessing (1), (2), (3) above for y.

If any of (1), (2), (3) fail to hold for some $y \in P_{\varkappa_x}x$ of order type β, then stop; the construction of h_x cannot be completed.

Set $X_1 = \{x \in X$: the construction of h_x cannot be completed$\}$. If $X_1 \in NSh_{\varkappa\lambda}$, then $Y = \{x \in X$: there is a $uhf\, h_x$: $P_{\varkappa_x}x \to X \cap P_{\varkappa_x}x$ of color $0\} \in NSh^*_{\varkappa\lambda}$. By 5.3 above, Y is Π^1_1-indescribable, so there is a $uhf\, h$: $P_\varkappa\lambda \to X$ of color 0.

Now suppose that $X_1 \in NSh^+_{\varkappa\lambda}$. For each $x \in X_1$, pick $v_x \in P_{\varkappa_x}x$ of minimal order type $< \varkappa_x$ such that $h_x(v_x)$ cannot be defined, and then let $v \in P_\varkappa\lambda$ be such that $X_2 = \{x \in X_1: v_x = v\} \in NSh^+_{\varkappa\lambda}$; such a v exists by 3.5 above. Since

$$(\forall x \in X_2)(|v| < \varkappa_x \wedge \varkappa_x \text{ is inaccessible}),$$

we have that $(\forall x \in X_2)(\bigcup \{h_x(u): u < v\} < x)$. Now let $w \in P_\varkappa\lambda$ be such that $X_3 = \{x \in X_2: \bigcup \{h_x(u): u < v\} = w\} \in NSh^+_{\varkappa\lambda}$; such a w exists by 3.5 again. Notice that $(\forall x \in X_3)(\{(u, h_x(u)): u < v\} \subseteq P(v) \times P(w))$. Since $\varkappa$ is inaccessible, $(\exists h \subseteq P(v) \times P(w))(H = \{x \in X_3: \{(u, h_x(u)): u < v\} = h\} \in NSh^+_{\varkappa\lambda})$. We will show that H is 1-homogeneous for f.

Pick $x, y \in H$ such that $x < y$. Since $v_y = v = v_x$ and

$$\{(u, h_y(u)): u < v_y\} = h = \{(u, h_x(u)): u < v\},$$
$$\{h_y(u): u < v_y\} = \{h_x(u): u < v\} < x,$$

and

$$(\forall u < v_y)(f(h_y(u), x) = f(h_x(u), x) = 0).$$

Thus if $f(x, y) = 0$, we could define $h_y(v_y)$ to be x; thus $f(x, y) = 1$. ∎

5.5. **Remark.** An immediate consequence of 5.4 is that if $\lambda^{<\varkappa} = \lambda$ and $\varkappa$ has the λ-Shelah property, then $P_\varkappa\lambda \to (uhf, NS^+_{\varkappa\lambda})^2$ holds. It would be nice if we could establish the converse of this too.

Although we can prove that if $P_\varkappa\lambda \to (uhf, SNS^+_{\varkappa\lambda})^3$ holds, then $\varkappa$ has the λ-Shelah property, repeated efforts to obtain the λ-Shelah property from $P_\varkappa\lambda \to (uhf, NS^+_{\varkappa\lambda})^2$ or even from $P_\varkappa\lambda \to (I^+_{\varkappa\lambda}, NS^+_{\varkappa\lambda})^2$ have failed.

References

[1] J. E. Baumgartner, *Ineffability properties of cardinals*, II, in R. Butts and J. Hintikka (eds.), *Logic, Foundations of Mathematics and Computability Theory*, D. Reidel Publishing Co., Dordrecht–Holland 1977, 87–106.

[2] — *Generalizing weak compactness to the* $(\varkappa, \lambda)$ *context*, handwritten notes.

[3] D. M. Carr, *The minimal normal filter on* $P_\varkappa\lambda$, Proc. Amer. Math. Soc. 86 (1982), 316–320.

[4] — *Ineffability properties of* $P_\varkappa\lambda$, Amer. Math. Soc. Abstracts, 2 (1981), p. 486, abstract no. 789-04-75.

[5] — *Ineffability properties of* $P_\varkappa\lambda$, Ph. D. disertation, Mc Master University, 1981.

[6] — *The Structure of ineffability properties of* $P_\varkappa\lambda$, Acta Math. Hung. 47 (3–4) (1986), 325–332.

[7] — $P_\varkappa\lambda$ *generalizations of weak compactness*, Z. Math. Logic Grundlag. Math. 31 (1985), 393–401.

[8] — $P_\varkappa\lambda$ *partition relations*, J. Symbolic Logic, 49 (1984), 675 (abstract only).

[9] C. A. Di Prisco, *Supercompact cardinals and a partition property*, Adv. Math. 25 (1977), 46–55.

[10] — handwritten note.

[11] C. A. Di Prisco and W. S. Zwicker, *Flipping properties and supercompact cardinals*, Fund. Math. 109 (1980), 31–36.

[12] T. J. Jech, *Some combinatorial problems concerning uncountable cardinals*, Ann. Math. Logic 5 (1973), 165–198.

[13] — *Set Theory*, Academic Press, New York 1978.

[14] M. Magidor, *Combinatorial characterization of supercompact cardinals*, Proc. Amer. Math. Soc. 49 (1974), 279–285.

[15] T. K. Menas, *On strong compactness and supercompactness*, Ann. Math. Logic 7 (1974), 327–359.

[16] — *A combinatorial property of* $P_\varkappa\lambda$, J. Symbolic Logic 42 (1976), 225–233.

[17] S. Shelah, *Weakly compact cardinals: a combinatorial proof*, J. Symbolic Logic 44 (1979), 559–562.

[18] W. S. Zwicker, $P_\varkappa\lambda$ *combinatorics*, I, in J. E. Baumgartner, D. A. Martin, S. Shelah (eds.), *Axiomatic Set Theory*, American Mathematical Society, Providence, R. I., 243–259.

[19] — *Structural properties of ideals on* $P_\varkappa\lambda$ notes for talk presented at the 1984 A. S. L. European Summer Meeting, Manchester U. K., 16–24 July 1984.

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