STUDIA MATHEMATICA, T. LXXXV. (1987)

# On the functors $\mathrm{Ext}^1(E, F)$ for Fréchet spaces

by

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**Abstract.** Necessary as well as sufficient conditions are given for those pairs $(E, F)$ of Fréchet spaces for which every sequence $0 \to F \to G \to E \to 0$ splits or, equivalently, for which $\mathrm{Ext}^1(E, F) = 0$. This is based on a systematic study of the derived functors $\mathrm{Ext}^1(E, \cdot)$. It is used to determine for certain $F_0$ (resp. $E_0$), which are important for applications, the classes of all Fréchet spaces $E$ (resp. $F$) such that $\mathrm{Ext}^1(E, F_0) = 0$ (resp. $\mathrm{Ext}^1(E_0, F) = 0$).

In the present article we study the derived functors $\mathrm{Ext}^1(E, \cdot)$ of the functors $L(E, \cdot)$, $E$ fixed, acting from the category of Fréchet spaces to the category of linear spaces. In particular, we give necessary as well as sufficient conditions for $\mathrm{Ext}^1(E, F) = 0$, which means that every exact sequence $0 \to F \to G \to E \to 0$ splits.

The fact that this is true in various nontrivial cases (e.g. $E = F = (s)$, $C^\infty[0, 1]$, $H(C)$ etc. or, more generally, $E \in (\mathrm{DN})$, $F \in (\Omega)$, one of them nuclear) is one of the basic tools in the structure theory of nuclear Fréchet spaces as developed for power series spaces of infinite type in [31], [42], [43] (cf. [32]) and for $L_f(\alpha, +\infty)$-spaces in [1]. In particular, the (DN)-, $(\Omega)$-case or, equivalently, the case where $E$ is a subspace of $(s)$ and $F$ a quotient of $(s)$, led also to various applications to analytical problems (cf. [4], [15], [16], [28], [37], [38]). Other related work can be found in [12], [22], [30].

In § 1 we give a systematic analysis of the functor $\mathrm{Ext}^1(E, \cdot)$ following to some extent Palamodov [21], § 9. We use it in § 2 to prove a sufficient condition $(\mathrm{S}_1)$ and a necessary condition $(\mathrm{S}_2)$ for $\mathrm{Ext}^1(E, F) = 0$. They are related to the conditions of Palamodov [20], [21] and Retakh [24] for the vanishing of the first derived functor of the projective limit functor. If one of the spaces $E$ or $F$ is either nuclear or a Köthe space (of $\lambda$- or $\lambda^\infty$-type respectively), then conditions $(\mathrm{S}_1)$ and $(\mathrm{S}_2)$ which are set inclusions between balls in spaces of linear operators can be turned into the (dual) conditions $(\mathrm{S}_1^*)$ and $(\mathrm{S}_2^*)$ (see § 3). These are inequalities and therefore accessible for calculations. Condition $(\mathrm{S}_1^*)$ is essentially the same as the condition (S) defined by Apiola ([1], Def. 1.1), who proved sufficiency if one of the spaces is a Köthe space and certain assumptions concerning nuclearity and existence of continuous norms are satisfied. If $E$ and $F$ are both Köthe spaces, then $(\mathrm{S}_2^*)$ is necessary and sufficient for $\mathrm{Ext}^1(E, F) = 0$ (see [14]).

Conditions $(S_1^*)$ and $(S_2^*)$ are used to derive complete characterizations of certain acyclicity classes. More precisely: We determine for fixed $F_0$ (resp. $E_0$) the class of all Fréchet spaces $E$ (resp. $F$) such that $\operatorname{Ext}^1(E, F_0) = 0$ (resp. $\operatorname{Ext}^1(E_0, F) = 0$). In § 4 this is done for (shift-) stable power series spaces $E_0$ and $F_0$ and in § 5 for Fréchet spaces $E_0, F_0$ which satisfy certain assumptions which are chosen so as to be fulfilled for many of the Fréchet spaces of real or complex analytic functions occurring in analysis. For instance, they are fulfilled for the spaces of zero solutions of elliptic partial differential operators with constant coefficients, for the space of all holomorphic functions on a Stein manifold and for the ideals associated to analytic varieties (always in the role of $F_0$, which is the interesting case). These examples are treated in § 7 (cf. [37], [38]). In all these cases the acyclicity classes can be described in terms of the invariants (DN) and $(\Omega)$ from the structure theory of power series spaces and the invariant $(\bar{\bar{\Omega}})$, which occurs also in [35].

In § 6 it is shown that for an exact sequence $0 \to F \to G \xrightarrow{q} H \to 0$ of Fréchet spaces, without further assumptions, condition $(\Omega)$ for $F$ implies solvability of the equation $q(g) = h$ with polynomial bounds, $(\Omega)$ for $F$ and (DN) for $H$ imply solvability with linear bounds (by a linear operator if $F$ is locally injective or $H$ is locally projective).

**Notation.** We use the common terminology on locally convex spaces (see e.g. [10], [13], [26]). For nuclear spaces and power series spaces see [23] and [6], for concepts from homological algebra see [21] and [19].

For an infinite matrix $A = (a_{j,k})_{j,k\in N}$ with $0 \leqslant a_{j,k} \leqslant a_{j,k+1}$, $\sup_k a_{j,k} > 0$ for all $j$ and $k$, we define the Köthe sequence spaces

$$\lambda(A) = \{x = (x_1, x_2, \ldots)\colon \|x\|_k = \sum_j |x_j|\, a_{j,k} < +\infty \text{ for all } k\}$$

$$\lambda^\infty(A) = \{x = (x_1, x_2, \ldots)\colon \|x\|_k = \sup_j |x_j|\, a_{j,k} < +\infty \text{ for all } k\}.$$

Equipped with their respective seminorms $\|\ \|_k$, $k \in N$, they are Fréchet spaces. They coincide iff one of them is (both are) nuclear.

If $A$ has the form $a_{j,k} = \varrho_k^{\alpha_j}$, where $\alpha$ is a sequence with $0 \leqslant \alpha_j \leqslant \alpha_{j+1} \to +\infty$ and $0 < \varrho_k < \varrho_{k+1} \to r$, $r \in \{1, +\infty\}$, then $\lambda(A)$ (resp. $\lambda^\infty(A)$) is called *power series space of finite type* for $r = 1$, of *infinite type* for $r = +\infty$ and denoted by $\Lambda_r(\alpha)$ (resp. $\Lambda_r^\infty(\alpha)$). In the nuclear case, i.e. $\limsup_n (\log n)/\alpha_n = 0$ for $r = 1$, $\limsup_n (\log n)/\alpha_n < +\infty$ for $r = +\infty$, $\Lambda_r(\alpha)$ and $\Lambda_r^\infty(\alpha)$ coincide. We put $(s) = \Lambda_\infty(\alpha)$ with $\alpha_j = \log j$.

A locally convex space $E$ is called *quasinormable* if for every equicontinuous set $A \subset E'$ there exists a neighbourhood $V$ of zero in $E$ such that on $A$ the strong topology $\beta(E', E)$ and the (norm-) topology of uniform



convergence on $V$ coincide, or equivalently (see [10], p. 176), if for every neighbourhood $U$ of zero in $E$ there exists a neighbourhood $V$ of zero such that for every $\varepsilon > 0$ there is a bounded set $M \subset E$ such that $V \subset M + \varepsilon U$.

Every Banach space and every Schwartz space is quasinormable (see [10]; they even generate in a certain sense the class of these spaces, cf. [17]), hence every nuclear space and every power series space is quasinormable. Since in $\lambda^\infty(A)$ all bounded sets are of the form $M = \{x\colon |x| \leqslant \lambda\}$, where $\lambda \in \lambda^\infty(A)$, $\lambda_j \geqslant 0$ for all $j$, it can easily be proved that $\lambda^\infty(A)$ is quasinormable iff for any $k \in N$ there is $l \in N$ such that for every $\varepsilon > 0$ there is $\lambda \in \lambda^\infty(A)$, $\lambda_j \geqslant 0$ for all $j$, such that

$$\frac{1}{a_{j,l}} \leqslant \max\left(\lambda_j, \varepsilon \frac{1}{a_{j,k}}\right)$$

for all $j$ (see [3], Prop. 3.5 and 3.2).

A Fréchet space $E$ is called *countably normed* if there is a fundamental system $\|\ \|_1 \leqslant \|\ \|_2 \leqslant \ldots$ of seminorms such that the natural maps $\varrho_{l,k}\colon E_l \to E_k$ are injective, where $E_k = (E/\ker\|\ \|_k, \|\ \|_k)^\wedge$. There are non-countably normed (even nuclear) Fréchet spaces with a fundamental system $(\|\ \|_k)_k$ consisting of norms (see [6], VI, 3; cf. [36]).

Following [2] we call a Fréchet space $E$ a *quojection* if it is isomorphic to a projective limit of a sequence of surjective operators on Banach spaces or equivalently (see [2], Prop. 3) if for every continuous seminorm $\|\ \|$ on $E$ the space $E/\ker\|\ \|$ is a Banach space ($= E$ relatively complete, see [27]). Equivalent descriptions which we shall use can be found in [5] and [27], see also [9], II, § 4.

An important notational convention which we will use throughout this paper is the following: In all relations between two Fréchet spaces as $(S_1)$, $(S_2)$, $(S_1^*)$, $(S_2^*)$ the spaces are called $E$ and $F$ and these variables are always used in the same way, namely such that the statement "$E$, $F$ satisfy $(S_1)$" etc. corresponds to $\operatorname{Ext}^1(E, F) = 0$. Moreover, the indices, i.e. the letters $n_0, m, n, \mu, k, K$, are, in connection with these conditions and if not stated otherwise, always used in the same way.

We use the common rules for calculations with nonnegative extended real numbers, in particular $0\cdot(+\infty) = 0$, $\frac{0}{0} = 0$, $\frac{\alpha}{0} = +\infty$ for $\alpha > 0$, etc.

**1.** Let $\mathscr{F}$ be the category of $(F)$-spaces with continuous linear maps as morphisms, $\mathscr{L}$ the category of linear spaces with linear maps as morphisms. $E, F, G, H, \ldots$ always denote $(F)$-spaces, $L(E, F)$ the linear space of continuous linear maps from $E$ to $F$. Let $E$ be fixed.

A space $I$ in $\mathscr{F}$ is called *injective* iff for each space $E_1$ in $\mathscr{F}$, each closed subspace $E_0 \subset E_1$ and each $\varphi \in L(E_0, I)$ there exists an extension $\Phi \in L(E_1, I)$. Examples of injective spaces are the Banach spaces $l^\infty(M)$, where $M$ is an index set. Products of injective spaces are obviously injective.

An *injective resolution* of $F$ is an exact sequence

$$0 \to F \to I_0 \xrightarrow{i_0} I_1 \xrightarrow{i_1} I_2 \to \dots . \tag{1}$$

where $I_k$ is injective for all $k$. Every $F \in \mathcal{F}$ has an injective resolution. The $I_k$ can be chosen as products of spaces $l^\infty(M)$, and if $F$ is a Banach space, as spaces $l^\infty(M)$ ([21], p. 23).

We denote by $\mathrm{Ext}^k(E, \cdot)$ the right derived functors of the functor $L(E, \cdot)$ acting from $\mathcal{F}$ to $\mathcal{L}$ (see [21], § 8 and p. 13).

Hence we have by definition for any injective resolution (1) of $F$

$$\mathrm{Ext}^k(E, F) \cong \ker j_k / \operatorname{im} j_{k-1}, \quad k = 1, 2, \dots \tag{2}$$

where $j_k$: $L(E, I_k) \to L(E, I_{k+1})$ is defined by $j_k(A) = i_k \circ A$ for $A \in L(E, I_k)$; $\mathrm{Ext}^0(E, F) = L(E, F)$.

The functors $\mathrm{Ext}^k(E, \cdot)$ have the following properties:

(I) For every short exact sequence $0 \to F \xrightarrow{\iota} G \xrightarrow{q} H \to 0$ we have linear maps $\delta^k$: $\mathrm{Ext}^k(E, H) \to \mathrm{Ext}^{k+1}(E, F)$, $k = 0, 1, \dots$ such that

$$0 \to L(E, F) \xrightarrow{\iota^*} L(E, G) \xrightarrow{q^*} L(E, H) \xrightarrow{\delta^0} \mathrm{Ext}^1(E, F) \to$$
$$\to \mathrm{Ext}^1(E, G) \to \mathrm{Ext}^1(E, H) \xrightarrow{\delta^1} \mathrm{Ext}^2(E, F) \to \dots$$

is exact and depends functorially on the short exact sequence.

(II) For every injective space $I$ in $\mathcal{F}$ we have $\mathrm{Ext}^k(E, I) = 0$ for $k = 1, 2, \dots$

In (I) the arrows not given by $\delta^k$ are derived from $\iota$ and $q$ by the respective functor, in particular we have $\iota^* A = \iota \circ A$, $q^* B = q \circ B$. The functors $\mathrm{Ext}^k(E, \cdot)$ are (up to natural equivalence) uniquely determined by (I) and (II). Hence one could use (I) and (II) also as a definition.

A Banach space $P$ is called *projective* if for every Banach space $E_1$, quotient space $E_2$ and $\varphi \in L(P, E_2)$ there exists a lifting $\Phi \in L(P, E_1)$. Examples of projective spaces are the Banach spaces $l^1(M)$ where $M$ is an index set.

Projective spaces in $\mathcal{F}$ would be defined in an analogous way. But the only ones are the finite-dimensional spaces (see [8]). Hence we do not have projective resolutions in $\mathcal{F}$ and cannot define right derived functors of the contravariant functor $L(\cdot, F)$, $F$ fixed. Nevertheless, we have (see [21], p. 49):

(III) For every short exact sequence $0 \to E_0 \xrightarrow{\iota} E_1 \xrightarrow{q} E_2 \to 0$ in $\mathcal{F}$ we have an exact sequence

$$0 \to L(E_2, F) \xrightarrow{q_*} L(E_1, F) \xrightarrow{\iota_*} L(E_0, F) \to \mathrm{Ext}^1(E_2, F) \to \mathrm{Ext}^1(E_1, F) \to$$
$$\to \mathrm{Ext}^1(E_0, F) \to \mathrm{Ext}^2(E_2, F) \to \dots$$

which depends functorially on the short exact sequence.



In particular, (III) implies that $\mathrm{Ext}^k(\cdot, F)$ can be considered as a contravariant functor in the first variable. We used the notation $\iota_* A = A \circ \iota$, $q_* B = B \circ q$.

A projective spectrum $\varrho_{l,k}$: $F_l \to F_k$ $(l > k)$ of Banach spaces is called a *fundamental system of Banach spaces* for the (F)-space $F$ if

(i) $F = \lim\operatorname{proj} F_k$,

(ii) $\forall k\ \exists l > k$: $\varrho_k F$ is dense in $\varrho_{l,k} F_l$,

where $\varrho_k$: $F \to F_k$ denotes the canonical map.

A fundamental system of Banach spaces for $F$ is given for instance by the Banach spaces $F_k$ obtained from a fundamental system of seminorms. There are other important cases. If $F$ is nuclear and $1 \leqslant p \leqslant +\infty$ then $F$ has a fundamental system of Banach spaces isomorphic to $l^p$, since every nuclear map between Banach spaces factors through $l^p$ (cf. [9], II, § 1, Remarque, or [23], III, 7.3). For $p = +\infty$ these cannot be Banach spaces obtained by completion with respect to seminorms since $F$ is separable whereas $l^\infty$ is not. We state the case $p = 1, +\infty$.

Remark. If $F$ is nuclear then it has a fundamental system of injective Banach spaces and a fundamental system of projective Banach spaces.

Another important case is contained in the following remark, the first part of which is easy to prove.

Remark. (1) Every Köthe space $\lambda(A)$ has a fundamental system of projective Banach spaces.

(2) Every quasinormable Köthe space $\lambda^\infty(A)$ has a fundamental system of injective Banach spaces.

The second part of the above remark is contained in the following.

Remark. $\lambda^\infty(A)$ is quasinormable if and only if the spaces

$$\lambda_k^\infty = \{x = (x_1, x_2, \dots): \|x\|_k = \sup_j |x_j|\, a_{j,k} < +\infty\}$$

with their identical imbeddings are a fundamental system of Banach spaces.

Proof. We use the characterization of quasinormability described in the notational section at the beginning of this paper.

Let $\lambda^\infty(A)$ be quasinormable. For given $k$ we choose $l$ according to this characterization. Let $x \in \lambda_l^\infty$, $\|x\|_l = b > 0$, and $\varepsilon > 0$ be given. Again using the characterization we choose $\lambda$ for $\varepsilon/b$ (instead of $\varepsilon$). We put $J_0 = \{j: |x_j| \leqslant \lambda_j\}$, $J_1 = N \setminus J_0$. So we have $|x_j| \leqslant b/a_{j,l} \leqslant \max(\lambda_j, \varepsilon/a_{j,k})$ for all $j$, hence

$$|x_j| \leqslant \varepsilon/a_{j,k}$$

for $j \in J_1$. We put

$$y_j = \begin{cases} x_j & \text{for } j \in J_0, \\ 0 & \text{for } j \in J_1. \end{cases}$$

Then $y = (y_j)_j \in \lambda^\infty(A)$ and

$$\|x-y\|_k = \sup_{j\in J_1} |x_j|\, a_{j,k} \leqslant \varepsilon.$$

If on the other hand the $\lambda_k^\infty$ are a fundamental system of Banach spaces, then we choose for given $k$ an $l \geqslant k$ according to the definition. Let $\varepsilon > 0$ be given. Since $x = (x_j)_j$ with $x_j = 1/a_{j,l}$ for $a_{j,l} > 0$, $x_j = 0$ otherwise, is in $\lambda_l^\infty$, we can find $\lambda \in \lambda^\infty(A)$ such that $\|x-\lambda\|_k \leqslant \varepsilon$. Hence

$$1/a_{j,l} \leqslant |\lambda_j| + \varepsilon/a_{j,k}$$

for all $j$ with $a_{j,l} > 0$. The inequality is trivial otherwise.

The exact sequence defined in the following lemma we call the *canonical resolution* of $F$ with respect to the given fundamental system of Banach spaces. The following lemma is contained in [21], Th. 5.2 (or Cor. 5.1). We give a direct proof.

1.1. Lemma. *If $\varrho_{l,k}\colon F_l \to F_k$ is a fundamental system of Banach spaces for the space $F$ in $\mathscr{F}$, then the sequence*

$$0 \to F \xrightarrow{\iota} \prod_k F_k \xrightarrow{q} \prod_k F_k \to 0$$

*is exact, where $\iota\colon x \mapsto (\varrho_k x)_k$ and $q\colon (x_k)_k \mapsto (\varrho_{k+1,k} x_{k+1} - x_k)_k$.*

Proof. The only part to show is the surjectivity of $q$. We proceed in three steps.

*Case* 1: *$\varrho_k F$ is dense in $F_k$ for all $k$.* Let $\|\ \|_k$ be the norm in $F_k$. We can assume that $\|\varrho_{j,k} x\|_k \leqslant \|x\|_j$ for all $j > k$.

For $y_k \in F_k$, $k = 1, 2, \ldots$ we choose inductively a sequence $v_k \in F_k$, $k = 1, 2, \ldots$ Put $v_1 = 0$. If $v_k$ is chosen then there exists $v_{k+1} \in F_{k+1}$ such that

$$\|y_k + v_k - \varrho_{k+1,k} v_{k+1}\|_k \leqslant 2^{-k}.$$

We put $u_k = y_k + v_k - \varrho_{k+1,k} v_{k+1}$; then $\|u_k\|_k \leqslant 2^{-k}$. So we can define

$$x_k := v_k - \sum_{j=k}^{\infty} \varrho_{j,k} u_k.$$

The series converges in $F_k$. An easy calculation gives

$$\varrho_{k+1,k} x_{k+1} - x_k = \varrho_{k+1,k} v_{k+1} - v_k + u_k = y_k$$

for all $k$.

*Case* 2: *$\varrho_k F$ is dense in $\varrho_{k+1,k} F_{k+1}$ for all $k$.* We apply the previous case to the fundamental system $G_k := \overline{\varrho_k F} = \overline{\varrho_{k+1,k} F_{k+1}}$ and deduce that $q|\prod_k G_k$ is surjective onto $\prod_k G_k$, hence $\prod_k G_k \subset \operatorname{im} q$. For $y = (y_k)_k \in \prod_k F_k$ we have

$$y_k = \varrho_{k+1,k} y_{k+1} - (\varrho_{k+1,k} y_{k+1} - y_k)$$

and therefore $y \in \prod_k G_k + \operatorname{im} q = \operatorname{im} q$.



*General case.* We choose inductively $k(\nu)$, $\nu = 1, 2, \ldots$ such that $k(1) = 1$ and $\varrho_{k(\nu)} F$ is dense in $\varrho_{k(\nu+1),k(\nu)} F_{k(\nu+1)}$. We apply case 2 to the fundamental system $F_{k(\nu)}$ (with $\varrho_{k(\mu),k(\nu)}$) and conclude that for each $(\eta_\nu)_\nu \in \prod_\nu F_{k(\nu)}$ we can find $(\xi_\nu)_\nu \in \prod_\nu F_{k(\nu)}$ such that $\eta_\nu = \varrho_{k(\nu+1),k(\nu)} \xi_{\nu+1} - \xi_\nu$ for all $\nu$.

If $(y_k)_k \in \prod_k F_k$ is given we find $(\xi_\nu)_\nu$ for

$$\eta_\nu = y_{k(\nu)} + \sum_{j=k(\nu)+1}^{k(\nu+1)-1} \varrho_{j,k(\nu)} y_j$$

and put

$$x_k = \begin{cases} \xi_\nu & \text{for } k = k(\nu),\\ \varrho_{k(\nu+1),k}\, \xi_{\nu+1} - \sum\limits_{j=k}^{k(\nu+1)-1} \varrho_{j,k} y_j & \text{for } k(\nu) < k < k(\nu+1) \end{cases}$$

with $\varrho_{j,j} = \mathrm{id}$ for all $j$. Straightforward calculation shows that $\varrho_{k+1,k} x_{k+1} - x_k = y_k$ for all $k$.

We call a space $F$ in $\mathscr{F}$ *$E$-acyclic* if $\mathrm{Ext}^k(E, F) = 0$ for $k = 1, 2, \ldots$

The following theorem gives a description of the linear spaces $\mathrm{Ext}^k(E, F)$ which we shall use in the sequel. (1) is closely related to [20], 7.1 (cf. [21], p. 51), (2) can be considered as an analogue to the Leray lemma in sheaf cohomology.

1.2. Theorem. *Let $\tilde{F} = (F_n)_n$ be a fundamental system of $E$-acyclic Banach spaces for $F$. Then we have*

(1) $\mathrm{Ext}^k(E, F) = 0$ *for* $k \geqslant 2$.

(2) $\mathrm{Ext}^1(E, F) \cong \prod_k L(E, F_k)/B(E, \tilde{F})$ *where*

$$B(E, \tilde{F}) = \{(A_k)_k \in \prod_k L(E, F_k)\colon \text{there exists } (B_k)_k \in \prod_k L(E, F_k) \text{ such that } A_k = \varrho_{k+1,k} \circ B_{k+1} - B_k \text{ for all } k\}.$$

Proof. We apply the long cohomology sequence (I) to the canonical resolution, i.e. the short exact sequence from 1.1, given by the $(F_k)_k$.

By taking as injective resolution for $\prod_k F_k$ the product of injective resolutions of the $F_k$ we see from (2) at the beginning of this section that

$$\mathrm{Ext}^n(E, \prod_k F_k) = \prod_k \mathrm{Ext}^n(E, F_k) = 0$$

for $n = 1, 2, \ldots$ Hence we obtain

$$0 \to L(E, F) \to \prod_k L(E, F_k) \xrightarrow{q^*} \prod_k L(E, F_k) \to \mathrm{Ext}^1(E, F) \to 0 \to$$
$$\to 0 \to \mathrm{Ext}^2(E, F) \to 0 \to \ldots \to 0 \to \mathrm{Ext}^k(E, F) \to 0 \to \ldots$$

(1) follows immediately. For $A = (A_k)_k$ we have $q^*(A) = q \circ A$ with $q$ as defined in 1.1. So $q^*(A) = (\varrho_{k+1,k} \circ A_{k+1} - A_k)_k$ which proves (2).

Remark. Without any assumption on $E$ or $(F_k)_k$ we can conclude from $\operatorname{Ext}^1(E, F) = 0$ that $\prod_k L(E, F_k) = B(E, \tilde{F})$.

Since injective Banach spaces are $E$-acyclic for every $E$ we have

1.3. Corollary. *If $(F_k)_k$ is a fundamental system of injective Banach spaces for $F$ then the assertion of* 1.2 *holds.*

The preceding corollary is important especially if $F = \lambda^\infty(A)$ or also if $F$ is nuclear. Another important situation is the following.

1.4. Lemma. *If $E$ has a fundamental system of projective Banach spaces then every Banach space $F$ is $E$-acyclic.*

Proof. $F$ has an injective resolution

$$0 \to F \to I_0 \xrightarrow{\iota_0} I_1 \xrightarrow{\iota_1} I_2 \to \ldots$$

of Banach spaces. Let $(E_n)_n$ be a fundamental system of projective Banach spaces for $E$. Then every continuous linear map $\varphi$ from $E$ into the Banach space $\ker \iota_k$ factors through one of the $E_n$ and can therefore be lifted to $\Phi \in L(E, I_{k-1})$. Hence $\operatorname{Ext}^k(E, F) = 0$.

We call a space *locally projective (injective)* if it has a fundamental system of projective (injective) Banach spaces.

1.5. Corollary. *If $E$ is locally projective (e.g. if $E$ is nuclear or a Köthe space $\lambda(A)$) then the assertion of* 1.2 *holds for every fundamental system $(F_k)_k$.*

A consequence are the following permanence properties of acyclicity classes.

1.6. Proposition. *If $E$ is locally projective (e.g. if $E$ is nuclear or a Köthe space $\lambda(A)$) then the class of all spaces $F$ with $\operatorname{Ext}^1(E, F) = 0$ is closed under quotients.*

Proof. Assume $\operatorname{Ext}^1(E, F) = 0$. Let $q\colon F \to F_0$ be a quotient map. Then we apply the long cohomology sequence (I) to the short exact sequence $0 \to K = \ker q \to F \to F_0 \to 0$ and obtain

$$\ldots \to 0 = \operatorname{Ext}^1(E, F) \to \operatorname{Ext}^1(E, F_0) \to \operatorname{Ext}^2(E, K) = 0 \to \ldots$$

which implies the result.

The same proof gives:

1.6.′ Proposition. *If $E$ is arbitrary but fixed then the class of all nuclear spaces $F$ with $\operatorname{Ext}^1(E, F) = 0$ is closed under quotients.*

In a completely analogous way by use of the long cohomology sequence (III) one proves:

1.7. Proposition. *If $F$ is locally injective (e.g. if $F$ is nuclear or a quasinormable Köthe space $\lambda^\infty(A)$) then the class of all spaces $E$ with $\operatorname{Ext}^1(E, F) = 0$ is closed under subspaces.*



1.7.′ Proposition. *If $F$ is arbitrary but fixed then the class of all nuclear spaces $E$ with $\operatorname{Ext}^1(E, F) = 0$ is closed under subspaces.*

We finish this section with some equivalent formulations of the relation $\operatorname{Ext}^1(E, F) = 0$ which show the motivation of its investigation and its relevance for applications.

1.8. Theorem. *The following are equivalent*:

(1) $\operatorname{Ext}^1(E, F) = 0$.

(2) *For each exact sequence $0 \to F \to G \xrightarrow{q} H \to 0$ and $\varphi \in L(E, H)$, there exists a lifting $\psi \in L(E, G)$, i.e. a map $\psi$ with $\varphi = q \circ \psi$.*

(3) *Each exact sequence $0 \to F \to G \to E \to 0$ splits.*

(4) *For each exact sequence $0 \to H \xrightarrow{\iota} G \to E \to 0$ and $\varphi \in L(H, F)$, there exists an extension $\Phi \in L(G, F)$, i.e. a map $\Phi$ with $\varphi = \Phi \circ \iota$.*

Proof. (1)$\Rightarrow$(2) follows from the long cohomology sequence (I).

(2)$\Rightarrow$(1). Let $0 \to F \to I_0 \xrightarrow{\iota_0} I_1 \xrightarrow{\iota_1} I_2 \to \ldots$ be an injective resolution of $F$. We apply the assertion of (2) to $0 \to F \to I_0 \to \operatorname{im} \iota_0 = \ker \iota_1 \to 0$. This gives (see (2) at the beginning of this section) $\operatorname{Ext}^1(E, F) = 0$.

(2)$\Rightarrow$(3). Put $H = E$, $\varphi = \mathrm{id}_E$.

(3)$\Rightarrow$(2). Put $\tilde{G} = \{(x, y) \in G \times E\colon qx = \varphi y\}$. The map $\pi(x, y) = y$ gives an exact sequence

$$0 \to F \to \tilde{G} \xrightarrow{\pi} E \to 0$$

which splits iff $\varphi$ has a lifting.

(3)$\Rightarrow$(4). Put $\tilde{G} = F \times G/\{(\varphi x, \iota x)\colon x \in H\}$. The map $jx = [(x, 0)]$, $[\cdot]$ equivalence class, $x \in F$, gives an exact sequence

$$0 \to F \xrightarrow{j} \tilde{G} \to E \to 0$$

which splits iff $\varphi$ has an extension.

(4)$\Rightarrow$(3). Put $H = F$, $\varphi = \mathrm{id}_F$.

**2.** We will now establish the necessary and sufficient conditions for $\operatorname{Ext}^1(E, F) = 0$ following from 1.2. Let $(F_k)_k$ be a fundamental system of Banach spaces for $F$ and $(E_\nu)_\nu$ a fundamental system of Banach spaces for $E$. $B(E_\nu, F_k)$ denotes the unit ball in $L(E_\nu, F_k)$.

For the sake of simplicity we omit the spectral maps $\varrho_{l,k}$. If $A \in L(E_\nu, F_k)$, $B \in L(E_\mu, F_l)$ then $A + B \in L(E_{\max(\nu,\mu)}, F_{\min(k,l)})$ is defined in the natural manner. Also the equation $A = B$ has a natural meaning (in $L(E_{\max(\nu,\mu)}, F_{\min(k,l)})$). Hence the inclusion relation in the following conditions makes sense.

We use the following conditions:

$(S_1)$ $\quad \exists n_0\ \forall \mu\ \exists k\ \forall K, m, R > 0\ \exists n, S$:

$$B(E_m, F_k) \subset S\,B(E_n, F_K) + \frac{1}{R} B(E_{n_0}, F_\mu).$$

(S$_2$) $\quad \forall \mu\ \exists n_0, k\ \forall K, m\ \exists n, S$:

$$B(E_m, F_k) \subset S\big(B(E_n, F_K) + B(E_{n_0}, F_\mu)\big).$$

For the next proposition (2.1) we assume that the $F_k$ are $E$-acyclic and we obtain

2.1. Proposition. (S$_1$) *implies* $\operatorname{Ext}^1(E, F) = 0$.

Proof. Without restriction of generality we can assume $n_0 = 1$, $k = k(\mu) = \mu + 1$. Otherwise we would change the fundamental systems of Banach spaces by omitting some spaces and changing the numeration which does not touch the assumptions. So we can assume that for all $k \geq 2$ and $R > 0$ we have

$$(*) \qquad L(E, F_k) \subset L(E, F_{k+1}) + \frac{1}{R} B(E_1, F_{k-1}).$$

We apply 1.2. We consider a sequence $A_k \in L(E, F_k)$, $k = 1, 2, \ldots$, and have to produce a sequence $B_k \in L(E, F_k)$ such that $A_k = B_{k+1} - B_k$.

We determine inductively maps $U_k$, $k = 3, 4, \ldots$, and $V_k$, $k = 2, 3, \ldots$, with the following properties:

(1) $V_2 = 0$,

(2) $U_{k+1} \in 2^{-k} B(E_1, F_{k-1})$,

(3) $V_{k+1} \in L(E, F_{k+1})$,

(4) $U_{k+1} + V_{k+1} = A_k + V_k$.

Obviously $(*)$ guarantees the induction.

We define

$$B_k = -A_k + V_{k+1} - \sum_{l=k+2}^{\infty} U_l \quad \text{for } k = 1, 2, \ldots$$

The convergence of the series in $L(E_1, F_k)$ follows from (2). Hence we have $B_k \in L(E, F_k)$. We get from (4)

$$B_{k+1} - B_k = -A_{k+1} + A_k + V_{k+2} - V_{k+1} + U_{k+2} = A_k$$

for $k = 1, 2, \ldots$. Notice that by our convention equality takes place in $L(E, F_k)$.

Remark. (a) In the proof of 2.1 in fact we used the possibly somewhat weaker condition

($\tilde{\mathrm{S}}_1$) $\quad \exists n_0\ \forall \mu\ \exists k\ \forall K, R > 0$:

$$L(E, F_k) \subset L(E, F_K) + \frac{1}{R} B(E_{n_0}, F_\mu).$$



Under reasonable assumptions on $E$ or $F$ condition ($\tilde{\mathrm{S}}_1$) or even the condition

$$\exists n_0\ \forall \mu\ \exists k\ \forall K: \quad L(E, F_k) \subset L(E, F_K) + L(E_{n_0}, F_\mu)$$

will be equivalent to (S$_1$). See § 3, in particular 3.9, and the proof of 2.2.

(b) Condition ($\tilde{\mathrm{S}}_1$) is obviously satisfied if $E$ is locally projective, $F$ a quojection and $(F_k)_k$ a fundamental system of Banach spaces such that the $\varrho_{l,k}$ are surjective. Hence for any locally projective space $E$ and quojection $F$ we have $\operatorname{Ext}^1(E, F) = 0$.

To establish (S$_2$) as a necessary condition for $\operatorname{Ext}^1(E, F) = 0$ we need an equivalent reformulation of (S$_2$).

2.2. Lemma. (S$_2$) *is equivalent to the following condition*:

($\tilde{\mathrm{S}}_2$) $\quad \forall \mu\ \exists n_0, k\ \forall K: \quad L(E, F_k) \subset L(E, F_K) + L(E_{n_0}, F_\mu)$.

Proof. One implication is obvious. For the other we apply Grothendieck's factorization theorem ([9], I, p. 16) to the locally convex space $L_b(E, F_\mu)$, and to the Banach spaces $L(E_m, F_k)$ and $L(E_n, F_K) \oplus L(E_{n_0}, F_\mu)$, $n = 1, 2, \ldots$ We can assume $k, K \geq \mu$. By assumption the canonical image of the first Banach space in $L_b(E, F_\mu)$ is contained in the union of the canonical images of the others. We obtain $n$ and $S$ such that

$$B(E_m, F_k) \subset S\big(B(E_n, F_K) + B(E_{n_0}, F_\mu)\big).$$

2.3. Proposition. $\operatorname{Ext}^1(E, F) = 0$ *implies* (S$_2$).

Proof. If ($\tilde{\mathrm{S}}_2$) is not true we have the following:

$$\exists \mu\ \forall n_0, k\ \exists K: \quad L(E, F_k) \not\subset L(E, F_K) + L(E_{n_0}, F_\mu).$$

We can assume $\mu = 1$. We determine inductively sequences $L(k)$, $k = 0, 1, \ldots$, $K(k)$, $k = 1, 2, \ldots$, in $N$ and $A_k \in L(E, F_{K(k)})$, $k = 1, 2, \ldots$ We put $L(0) = 1$, $K(1) = 1$. If $L(k-1)$, $K(k)$ is chosen then by the assumption applied to $n_0 = L(k-1)$ and $K(k)$ instead of $k$ we can find $A_k \in L(E, F_{K(k)})$ such that there exists $K(k+1)$ with

$$A_k \notin L(E, F_{K(k+1)}) + L(E_{L(k-1)}, F_1).$$

There exists $L(k)$ such that $A_k \in L(E_{L(k)}, F_{K(k)})$. We can assume $L(k) < L(k+1)$, $K(k) < K(k+1)$ and by omitting some spaces and renumeration we can assume $L(k) = k$, $k = 0, 1, \ldots, K(k) = k$, $k = 1, 2, \ldots$ Hence we have obtained a sequence of maps with

(i) $A_k \in L(E_k, F_k)$,

(ii) $A_k \notin L(E, F_{k+1}) + L(E_{k-1}, F_1)$.

From $\mathrm{Ext}^1(E, F) = 0$ we get a sequence $B_k \in L(E, F_k)$ with $B_{k+1} - B_k = A_k$ (cf. remark after Th. 1.2). Addition gives

$$A_k = B_{k+1} - B_1 - \sum_{l=1}^{k-1} A_l$$

hence for $k$ so large that $B_1 \in L(E_{k-1}, F_1)$

$$A_k \in L(E, F_{k+1}) + L(E_{k-1}, F_1)$$

which yields a contradiction.

The proof of the following lemma is straightforward. We omit it.

2.4. Lemma. *$(S_1)$ and $(S_2)$ do not depend on the fundamental systems of Banach spaces $(E_\nu)_\nu$ and $(F_k)_k$.*

We call a space $F$ in $\mathscr{F}$ *locally $E$-acyclic* if it has a fundamental system of $E$-acyclic Banach spaces. From 2.1, 2.3 and 2.4 we obtain immediately the following theorem. $(E_\nu)_\nu$, $(F_n)_n$ are arbitrary fundamental systems of Banach spaces.

2.5. Theorem. *If $F$ is locally $E$-acyclic then the following implications hold*:

$$(S_1) \Rightarrow \mathrm{Ext}^1(E, F) = 0 \Rightarrow (S_2).$$

*No assumption is needed for the second implication.*

The assumption of 2.5 is satisfied e.g. if $E$ is locally projective or $F$ is locally injective. Hence it is satisfied in each one of the following four cases (i) $E = \lambda(A)$, (ii) $F = \lambda^\infty(A)$ and quasinormable, (iii) $E$ nuclear, (iv) $F$ nuclear.

**3.** Our next step will be to bring, at least in the four standard cases mentioned at the end of the last section, conditions $(S_1)$ and $(S_2)$ into a form which makes calculation easier. Therefore we choose fundamental systems of seminorms $\| \ \|_1 \leqslant \| \ \|_2 \leqslant \ldots$ in $E$ and $F$. We put $U_k = \{x \in F\colon \|x\|_k \leqslant 1\}$, $V_k = \{x \in E\colon \|x\|_k \leqslant 1\}$ and $\|y\|_k^* = \sup_{\|x\|_k \leqslant 1} |y(x)|$ for $y \in E'$ or $F'$.

Unless stated otherwise, in this section $E_k$ and $F_k$ are assumed to be the natural Banach spaces generated by the $\| \ \|_k$, in particular $F'_k = \{y \in F'\colon \|y\|_k^* < +\infty\} \subset F'$.

We define two conditions $(S_1^*)$ and $(S_2^*)$(1). Condition $(S_1^*)$, or more precisely: condition $(S_1^*)_0$ defined in Lemma 3.3, is Apiola's splitting condition (S) ([1], Def. 1.1), up to a minor change discussed in the remark at the end of this section. He used it to show splitting theorems by a direct proof, which generalizes proofs in [31] and [41]. His results say (in our terminology) that $\mathrm{Ext}^1(E, F) = 0$ if $E$ or $F$ is a Köthe space and certain

(1) In [40], p. 359, a slightly different definition is used; see Prop. 3.10.



additional assumptions about nuclearity and existence of continuous norms are satisfied. Condition $(S_2^*)$ is shown in [14] to be necessary and sufficient for $\mathrm{Ext}^1(E, F) = 0$ if $E$ and $F$ are Köthe spaces.

We will show that $(S_1^*)$ is sufficient for $\mathrm{Ext}^1(E, F) = 0$ if $E = \lambda(A)$ or $F = \lambda^\infty(B)$ or if one of the spaces is nuclear, and that $(S_2^*)$ is always necessary.

For the sake of simplicity, in the following conditions we will always assume that $n_0 \leqslant m \leqslant n$, and $\mu \leqslant k \leqslant K$. See also the convention at the end of the introduction.

$(S_1^*)$ $\quad \exists n_0 \ \forall \mu \ \exists k \ \forall K, m \ \exists n, S \ \forall x \in E_n, y \in F'_k$:

$$\|x\|_m \|y\|_k^* \leqslant S(\|x\|_n \|y\|_K^* + \|x\|_{n_0} \|y\|_\mu^*).$$

$(S_2^*)$ $\quad \forall \mu \ \exists n_0, k \ \forall K, m \ \exists n, S \ \forall x \in E_n, y \in F'_k$:

$$\|x\|_m \|y\|_k^* \leqslant S(\|x\|_n \|y\|_K^* + \|x\|_{n_0} \|y\|_\mu^*).$$

Remark. Conditions $(S_1^*)$ and $(S_2^*)$ do not depend on the choice of the fundamental system of seminorms.

The easier part of what we want to prove is the following. We need no special assumptions.

3.1. Proposition. *$(S_2)$ implies $(S_2^*)$.*

Proof. This follows immediately from

$$\|x\|_m \|y\|_k^* = \sup\{|y(Ax)|\colon A \in B(E_m, F_k)\}$$

for all $m, k, x \in E_m, y \in F'_k$.

For the converse direction we treat the four cases separately. But first we need some preparation.

3.2. Lemma. *$(S_2^*)$ (and therefore also $(S_1^*)$) implies that either $E$ is countably normed or $F$ is a quojection.*

Proof. If $E$ is not countably normed, then for any $n_0$ there exists an $m$ and $x \in E_m$ such that $\|x\|_m \neq 0$ but $\|x\|_{n_0} = 0$. $(S_2^*)$ yields for any $\mu$ a $k \geqslant \mu$ such that with appropriate $x$

$$\forall K \ \exists n, S \ \forall y \in F'_k\colon \quad \|x\|_m \|y\|_k^* \leqslant S \|x\|_n \|y\|_K^*.$$

Hence all norms $\| \ \|_K^*$, $K \geqslant k$, are equivalent on $F'_k$. This means that for a subsequence $(j(k))_k$ the $F'_{j(k)}$, $k = 1, 2, \ldots$, are a strict inductive spectrum, which implies that $F$ is a quojection (see [5]).

3.3. Lemma. *If $E$, $F$ satisfy $(S_1^*)$ and $E$ is a proper (not normable) Fréchet space, then $E$, $F$ also satisfy the following condition*:

$(S_1^*)_0$ $\quad \exists n_0 \ \forall \mu \ \exists k \ \forall K, m, R > 0 \ \exists n, S \ \forall x \in E_n, y \in F'_k$:

$$\|x\|_m \|y\|_k^* \leqslant S \|x\|_n \|y\|_K^* + \frac{1}{R} \|x\|_{n_0} \|y\|_\mu^*.$$

Proof. We fix $n_0$ according to $(S_1^*)$ and $m > n_0$ such that $\|\ \|_{n_0}$ and $\|\ \|_m$ are inequivalent. Then $(S_1^*)$ yields:

$$\forall \mu\ \exists k\ \forall K\ \exists n, S\ \forall x\in E,\ \|x\|_m \neq 0,\ y\in F_k':$$

$$\|y\|_k^* \leqslant S\frac{\|x\|_n}{\|x\|_m}\|y\|_K^* + S\frac{\|x\|_{n_0}}{\|x\|_m}\|y\|_\mu^*.$$

Provided $\mu$, $k$, $K$, $n$, $S$ are chosen, let $\varepsilon > 0$ be given. There is an $x\in E$ such that $S\|x\|_{n_0} \leqslant \varepsilon\|x\|_m \neq 0$. We put $M = S\|x\|_n/\|x\|_m$; we have proved:

(Q) $\forall\mu\ \exists\mu_0\ \forall K, \varepsilon>0\ \exists M\ \forall y\in F_k'$:

$$\|y\|_{\mu_0}^* \leqslant M\|y\|_K^* + \varepsilon\|y\|_\mu^*.$$

We have to prove that $(S_1^*)$ implies $(S_1^*)_0$. $n_0$ is chosen according to $(S_1^*)$. Let $\mu$ be given. We choose $\mu_0$ according to (Q) and apply $(S_1^*)$ to $\mu_0$ instead of $\mu$. We obtain $k$. Let $K$, $m$, $R > 0$ be given. We get $n$, $S$ from $(S_1^*)$. We put $\varepsilon = 1/(SR)$ and obtain $M$ from (Q). For $x\in E_n$, $y\in F_k'$ we have:

$$\|x\|_m\|y\|_k^* \leqslant S\|x\|_n\|y\|_K^* + S\|x\|_{n_0}\|y\|_{\mu_0}^*$$
$$\leqslant (S+M)\|x\|_n\|y\|_K^* + \frac{1}{R}\|x\|_{n_0}\|y\|_\mu^*.$$

This completes the proof.

Remark. Condition (Q) is equivalent to $F$ being quasinormable (see [17]; (Q) means existence of a property $(\Omega_\varphi)$, cf. [40], 5.9). For a Banach space $E$ condition $(S_1^*)$ is always satisfied, while condition $(S_1^*)_0$ is satisfied iff $F$ satisfies (Q), i.e. iff $F$ is quasinormable. For a proper Fréchet space $E$ the lemma says that $(S_1^*)$ and $(S_1^*)_0$ are equivalent.

3.4. Proposition. *If $E = \lambda(A)$ admits a continuous norm, then* $(S_1^*)_0$ *implies* $(S_1)$.

Proof. In $(S_1^*)_0$ we put $x = e_j$, the $j$th basis vector of $E = \lambda(A)$. We obtain

$$a_{j,m}\|y\|_k^* \leqslant S a_{j,n}\|y\|_K^* + \frac{1}{R} a_{j,n_0}\|y\|_\mu^*$$

and by polarization as in [41] we obtain (cf. [1], 1.2)

$$(*)\qquad a_{j,m}U_k \subset S a_{j,n}U_K + \frac{1}{R} a_{j,n_0}U_\mu$$

with the same quantifiers (but perhaps changed $n$ and $S$). Put

$$E_m = \{x=(x_1, x_2, \ldots):\ \|x\|_m = \sum_j |x_j|\, a_{j,m} < +\infty\}$$



and let $F_k$ be the Banach space generated by $\|\ \|_k$, $\hat{U}_k$ the unit ball in $F_k$. A map $A\in B(E_m, F_k)$ can be written as

$$(**)\qquad Ax = \sum_j x_j A_j$$

with $A_j \in a_{j,m}\hat{U}_k$ for all $j$ and for every such $(A_j)_j$ (**) defines a map in $B(E_m, F_k)$. Because of density we can find $A_j^0\in a_{j,m}U_k$, $R_j\in(2R)^{-1}a_{j,n_0}U_\mu$ with $A_j = \varrho_k A_j^0 + R_j$, $\varrho_k$: $F\to F_k$ the canonical map. According to (*) with $S$, $n$ chosen for $2R$ we can write $A_j^0 = B_j^0 + C_j^0$ with $B_j^0 \in S a_{j,n}U_k$, $C_j^0 \in (2R)^{-1} a_{j,n_0}U_\mu$. Then $B_j = \varrho_K B_j^0$, $C_j = \varrho_\mu C_j^0 + R_j$ define maps $B$ and $C$ which prove the inclusion in $(S_1)$.

3.5. Proposition. *If $F = \lambda^\infty(B)$, $E \neq \{0\}$ then* $(S_1^*)_0$ *implies* $(S_1)$ *and that $F$ is quasinormable.*

Proof. In $(S_1^*)_0$ we put $y = f_j$, the $j$th dual "basis" vector in $F' = \lambda^\infty(B)'$. We obtain

$$(*)\qquad \frac{\|x\|_m}{b_{j,k}} \leqslant S\frac{\|x\|_n}{b_{j,K}} + \frac{1}{R}\frac{\|x\|_{n_0}}{b_{j,\mu}}$$

and by polarization as in [31] we obtain (cf. [1], 1.2)

$$(**)\qquad \frac{1}{b_{j,k}}V_m^0 \subset S\frac{1}{b_{j,K}}V_n^0 + \frac{1}{R}\frac{1}{b_{j,\mu}}V_{n_0}^0$$

with the same quantifiers (but perhaps changed $n$ and $S$).

We first prove the second assertion. We fix $a \neq 0$, $a\in E$. For given $\mu$ we find $k$ according to $(S_1^*)_0$. We choose $m$ such that $\|a\|_m > 0$. If $\varepsilon > 0$ is given we put $R = 1+2\|a\|_{n_0}/(\varepsilon\|a\|_m)$; then for every $K$ we have $n = n(K)$, $S = S(K)$ such that (*) holds. We put

$$\lambda_j = \inf_K \frac{2S(K)\|a\|_{n(K)}}{\|a\|_m b_{j,K}}$$

and obtain $\lambda = (\lambda_1, \lambda_2, \ldots)\in\lambda^\infty(B)$ such that

$$1/b_{j,k} \leqslant \max(\lambda_j, \varepsilon/b_{j,\mu})$$

for all $j$.

To prove $(S_1)$ we now may assume (see remark before 1.1) that

$$F_k = \{x = (x_1, x_2, \ldots):\ \|x\|_k = \sup_j |x_j|\, b_{j,k} < +\infty\}.$$

Let $E_m$ be the Banach space generated by $\|\ \|_m$. A map $B\in B(E_m, F_k)$ can be written as

$$(***)\qquad Bx = (B_1 x, B_2 x, \ldots)$$

with $B_j \in (1/b_{j,k}) V_m^0$ for all $j$ and for every such $(B_j)_j$ (***) defines a map in $B(E_m, F_k)$. Straightforward application of (**) gives the first assertion.

3.6. PROPOSITION. *If $E$ is nuclear and countably normed, then $(\mathrm{S}_1^*)_0$ implies* $(\mathrm{S}_1)$.

Proof. Let $E_m$, $F_m$ be the Banach spaces belonging to the seminorms $\| \ \|_m$ in $E$, $F$ respectively. The norms $\| \ \|_m$ in $E_m$ can be assumed to be generated by scalar products $\langle \ , \ \rangle_m$, i.e. the $E_m$ are Hilbert spaces. The $\varrho_{n+1,n}$ can be assumed to be Hilbert–Schmidt and injective.

We will show that for $n > m > n_0$

$$(*) \qquad \|x\|_m \|y\|_k^* \leqslant S\|x\|_n \|y\|_K^* + \frac{1}{R}\|x\|_{n_0}\|y\|_\mu^* \quad \text{for all } x \in E_n,\ y \in F_k'$$

implies

$$B(E_m, F_k) \subset S' B(E_{n+1}, F_K) + \frac{C(n_0)}{R} B(E_{n_0+1}, F_\mu)$$

with $S' = 3S\sigma(\varrho_{n+1,n})$, $C(n_0) = 3\sigma(\varrho_{n_0+1,n_0})$ where $\sigma(\cdot)$ is the Hilbert–Schmidt norm of an operator.

Condition (*) means, explicitly written (cf. remark at the end of this section):

$$\|\varrho_{n,m} x\|_m \|y\|_k^* \leqslant S\|x\|_n \|y\|_K^* + \frac{1}{R}\|\varrho_{n,n_0} x\|_{n_0}\|y\|_\mu^*$$

for all $x \in E_n$, $y \in F_\mu'$. As in 3.4 this leads to

$$(**) \qquad \|\varrho_{n,m} x\|_m U_k \subset 3S\|x\|_n U_K + \frac{2}{R}\|\varrho_{n,n_0} x\|_{n_0} U_\mu$$

for all $x \in E_n$.

Since $\varrho_{n+1,n_0+1}$ is Hilbert–Schmidt and injective, the spectral theorem gives us complete orthonormal systems $(e_j)_j$ in $E_{n+1}$, $(f_j)_j$ in $E_{n_0+1}$ and a sequence $a_j > 0$ such that

$$\varrho_{n+1,n_0+1} x = \sum_j a_j \langle x, e_j\rangle_{n+1} f_j$$

for all $x \in E_{n+1}$. In particular, $\varrho_{n+1,n_0+1} e_j = a_j f_j$. For given $A \in B(E_m, F_k)$ we put $A_j = A\varrho_{n+1,m} e_j \in F_k$. We proceed as in the proof of 3.2 and obtain $B_j \in 3S\|\varrho_{n+1,n} e_j\|_n U_K$, $C_j \in (3/R)\|\varrho_{n+1,n_0} e_j\|_{n_0} U_\mu$ such that $A_j = B_j + C_j$ (canonical maps for $F$ are omitted). We set

$$Bx = \sum_j \langle x, e_j\rangle_{n+1} B_j \qquad \text{for } x \in E_{n+1},$$

$$Cx = \sum_j \frac{1}{a_j}\langle x, f_j\rangle_{n_0+1} C_j \qquad \text{for } x \in E_{n_0+1}.$$



The series converge and $B$, $C$ define continuous linear maps because of

$$\|Bx\|_K \leqslant 3S\sum_j |\langle x, e_j\rangle_{n+1}|\,\|\varrho_{n+1,n} e_j\|_n \leqslant 3S\sigma(\varrho_{n+1,n})\|x\|_{n+1},$$

$$\begin{aligned}\|Cx\|_\mu &\leqslant \frac{3}{R}\sum_j \frac{1}{a_j}|\langle x, f_j\rangle_{n_0+1}|\,\|\varrho_{n+1,n_0} e_j\|_{n_0}\\ &= \frac{3}{R}\sum_j |\langle x, f_j\rangle_{n_0+1}|\,\|\varrho_{n_0+1,n_0} f_j\|_{n_0}\\ &\leqslant \frac{3}{R}\sigma(\varrho_{n_0+1,n_0})\|x\|_{n_0+1}.\end{aligned}$$

The constants in the estimates prove the assertion.

3.7. PROPOSITION. *If $F$ is nuclear then $(\mathrm{S}_1^*)_0$ implies* $(\mathrm{S}_1)$.

Proof. Let again $E_m$, $F_m$ be the Banach spaces generated by the seminorms $\| \ \|_m$ in $E$, $F$ respectively. We assume that the norms $\| \ \|_m$ in $F_m$ come from scalar products $\langle \ , \ \rangle_m$, i.e. that the $F_m$ are Hilbert spaces, and that $\varrho_{n+1,n}$ is nuclear for all $n$.

We will show that for $K > k > \mu + 2$

$$(*) \qquad \|x\|_m \|y\|_k^* \leqslant S\|x\|_n \|y\|_{K+2}^* + \frac{1}{R}\|x\|_{n_0}\|y\|_{\mu+2}^* \quad \text{for all } x \in E,\ y \in F_{\mu+2}'$$

implies

$$B(E_m, F_k) \subset S' B(E_n, F_K) + \frac{C(\mu)}{R} B(E_{n_0}, F_\mu)$$

with $S' = 2\sigma(\varrho_{K+1,K})\nu(\varrho_{K+2,K+1})$, $C(\mu) = 2\sigma(\varrho_{\mu+1,\mu})\nu(\varrho_{\mu+2,\mu+1})$ where $\sigma(\cdot)$ is the Hilbert–Schmidt norm and $\nu(\cdot)$ the nuclear norm of an operator.

By polarization as in 3.5 we obtain from (*)

$$(**) \qquad \|y\|_k^* V_m^0 \subset S\|y\|_{K+2}^* V_n^0 + \frac{1}{R}\|y\|_{\mu+2}^* V_{n_0}^0$$

for every $y \in F_{\mu+2}'$.

Since $\varrho_{K+1,\mu+1}$ is nuclear the spectral theorem gives us orthonormal systems $(e_j)_j$ in $F_{K+1}$, $(f_j)_j$ in $F_{\mu+1}$ and a sequence $a_j > 0$ such that $(f_j)_j$ is complete and

$$\varrho_{K+1,\mu+1} x = \sum_j a_j \langle x, e_j\rangle_{K+1} f_j$$

for all $x \in F_{K+1}$. In particular, $\langle \varrho_{K+1,\mu+1} x, f_j\rangle_{\mu+1} = a_j \langle x, e_j\rangle_{K+1}$.

For given $\varphi \in B(E_m, F_k)$ we put $\varphi_j = y_j \circ \varrho_{k,\mu+1} \circ \varphi$ where $y_j(x) = \langle x, f_j\rangle_{\mu+1}$; $y_j \in F_{\mu+1}' \subset F_{\mu+2}' \subset F'$. Since $\|\varphi_j\|_m^* \leqslant \|y_j\|_k^*$, i.e. $\varphi_j \in \|y_j\|_k^* V_m^0$, we obtain from (**) $\psi_j \in S\|y_j\|^*_{K+2} V^0_n$ and $\chi_j \in (1/R)\|y\|^*_{\mu+2} V^0_{n_0}$ such that $\varphi_j = \psi_j + \chi_j$ for all $j$. We set

$$\psi(x) = \sum_j \frac{1}{a_j} \psi_j(x) \varrho_{K+1,K} e_j \quad \text{for } x \in E_n,$$
$$\chi(x) = \sum_j \chi_j(x) \varrho_{\mu+1,\mu} f_j \quad \text{for } x \in E_{n_0},$$

and we have to show that the series converge and define continuous maps with the desired estimates.

We need some additional estimates involving the maps $\varrho_{\nu+1,\nu}$. Since they are nuclear, they are in particular Hilbert–Schmidt. We have for every $\nu$ and every orthonormal system $(g_j)_j$ in $F_{\nu+1}$

$$\sum_j |\langle x, g_j\rangle_{\nu+1}| \, \|\varrho_{\nu+1,\nu} g_j\|_\nu \leqslant \sigma(\varrho_{\nu+1,\nu}) \|x\|_{\nu+1},$$

which means that $\sigma_{\nu+1}: x \mapsto (\langle x, g_j\rangle_{\nu+1} \|\varrho_{\nu+1,\nu} g_j\|_\nu)_j$ is a continuous linear map from $F_{\nu+1}$ into $l^1$, $\sigma_{\nu+1} \circ \varrho_{\nu+2,\nu+1}$ is nuclear. Hence there exists $\lambda^{(\nu)}_j \in l^1$, $\lambda^{(\nu)}_j > 0$, such that

$$\sum_j \lambda^{(\nu)}_j \leqslant 2\nu(\sigma_{\nu+1} \circ \varrho_{\nu+2,\nu+1}) \leqslant 2\|\sigma_{\nu+1}\| \, \nu(\varrho_{\nu+2,\nu+1})$$
$$\leqslant 2\sigma(\varrho_{\nu+1,\nu}) \nu(\varrho_{\nu+2,\nu+1}) =: C(\nu)$$

and

$$|\langle x, g_j\rangle_{\nu+1}| \, \|\varrho_{\nu+1,\nu} g_j\|_\nu \leqslant \lambda^{(\nu)}_j \quad \text{for all } \|x\|_{\nu+2} \leqslant 1.$$

With $\gamma^{(\nu)}_j = (1/\lambda^{\nu}_j)\|\varrho_{\nu+1,\nu} g_j\|_\nu$ we obtain

$$(\alpha) \qquad \|\varrho_{\nu+1,\nu} x\|_\nu \leqslant \sum_j |\langle x, g_j\rangle_{\nu+1}| \, \|\varrho_{\nu+1,\nu} g_j\|_\nu \leqslant C(\nu) \sup_j \gamma^{(\nu)}_j |\langle x, g_j\rangle_{\nu+1}|$$

for all $x \in F_{\nu+1}$ and

$$(\beta) \qquad \sup_j \gamma^{(\nu)}_j |\langle \varrho_{\nu+2,\nu+1} x, g_j\rangle_{\nu+1}| \leqslant \|x\|_{\nu+2}.$$

For $x \in F_{K+2}$ we have

$$y_j(\varrho_{K+2,\mu+1} x) = \langle \varrho_{K+1,\mu+1} \varrho_{K+2,K+1} x, f_j\rangle_{\mu+1} = a_j \langle \varrho_{K+2,K+1} x, e_j\rangle_{K+1}.$$

Hence $(\beta)$ for $\nu = K$, $g_j = e_j$ implies

$$(\beta') \qquad \sup_j \gamma^{(\nu)}_j \frac{1}{a_j} \|y_j\|^*_{K+2} \leqslant 1.$$

For $\nu = \mu$, $g_j = f_j$, $(\beta)$ says

$$(\beta'') \qquad \sup_j \gamma^{(\nu)}_j \|y_j\|^*_{\mu+2} \leqslant 1.$$



Returning to the maps $\psi$ and $\chi$ we obtain as in $(\alpha)$ and from $(\beta')$

$$\|\psi x\|_K \leqslant \sum_j \frac{1}{a_j} |\psi_j(x)| \, \|\varrho_{K+1,K} e_j\|_K \leqslant C(K) \sup_j \gamma^{(K)}_j \frac{1}{a_j} |\psi_j(x)|$$
$$\leqslant C(K) S \sup_j \gamma^{(K)}_j \frac{1}{a_j} \|y_j\|^*_{K+2} \|x\|_n \leqslant C(K) S \|x\|_n$$

and from $(\beta'')$

$$\|\chi x\|_\mu \leqslant \sum_j |\chi_j(x)| \, \|\varrho_{\mu+1,\mu} f_j\|_\mu \leqslant C(\mu) \sup_j \gamma^{(\mu)}_j |\chi_j(x)|$$
$$\leqslant C(\mu) \frac{1}{R} \sup_j \gamma^{(\mu)}_j \|y_j\|^*_{\mu+2} \|x\|_{n_0} \leqslant C(\mu) \frac{1}{R} \|x\|_{n_0}.$$

Since obviously $\varphi = \psi + \chi$ (in the sense of our convention) this proves the assertion.

We are now ready to prove the main theorem of this section.

3.8. THEOREM. *Under each of the following assumptions*:

(i) $E = \lambda(A)$, (ii) $F = \lambda^\infty(B)$, (iii) $E$ *nuclear*, (iv) $F$ *nuclear*,

*we have the implications*:

$$(S^*_1)_0 \Rightarrow \mathrm{Ext}^1(E, F) = 0 \Rightarrow (S^*_2).$$

*If $E$ is a proper (non-Banach) Fréchet space, then $(S^*_1)_0$ may be replaced by $(S^*_1)$.*

Proof. The first implication follows from 3.4, ..., 3.7 together with 2.5, in cases (i) and (iii) provided $E$ is countably normed (see 3.4 and 3.6). However, if $E$ is not countably normed, then $(S^*_1)$ implies that $F$ is a quojection (Lemma 3.2) and Remark (b) after 2.1 gives the result, since under assumptions (i) and (iii), $E$ is locally injective. In case (ii) we have used the fact that because of 3.5, $\lambda^\infty(B)$ is quasinormable, which according to Remark (2) preceding Lemma 1.1 implies that $\lambda^\infty(B)$ is locally injective. Lemma 3.3 gives the last assertion of the theorem.

The second implication is a consequence of 3.1 and the second part of 2.5. Since the assumptions are not needed there, we state the result separately:

3.9. THEOREM. *For any two spaces $E$ and $F$ in $\mathscr{F}$*, $\mathrm{Ext}^1(E, F) = 0$ *implies* $(S^*_2)$.

Finally, it should be remarked that we have proved only inclusions relevant for 3.8 and 3.9. Nevertheless, $(S_1) \Rightarrow (S^*_1)$ is always true and $(S^*_2) \Rightarrow (S_2)$ holds under the four standard assumptions. We close this section by showing that in certain cases "$x \in E_n$, $y \in F'_k$" in $(S^*_1)$ and $(S^*_2)$ can be replaced by "$x \in E$, $y \in F'_\mu$", i.e. we do not need the completions and $\|y\|^*_j$, $j = \mu, k, K$, may always be assumed finite.

3.10. PROPOSITION. *If $E$ is countably normed or $F$ reflexive then in* $(\mathrm{S}_1^*)$, $(\mathrm{S}_1^*)_0$, $(\mathrm{S}_2^*)$, *"$x \in E_n$, $y \in F_k'$" can be replaced by "$x \in E$, $y \in F_\mu'$". If $E$ is countably normed, also by "$x \in E$, $y \in F'$".*

Proof. Let $E$ be countably normed. Since $\|x\|_j \neq 0$, $j = n_0, m, n$, for $x \neq 0$, the inequality becomes trivial for $y \notin F_\mu'$. But for $y \in F_\mu'$ we need to require the inequality only for $x \in E$. For $x \in E_n$ it then follows by continuity.

If $E$ is not countably normed and $E$, $F$ satisfy one of the conditions with "$x \in E$, $y \in F_\mu'$", then the analogue [40], 5.3, of our Lemma 3.2 says that $F$ fails property $(*)$ of Bellenot and Dubinsky [2]. If $F$ is reflexive this implies that $F$ is a quojection (see [2], Cor. 3). Hence even $(\mathrm{S}_1^*)_0$ is satisfied.

Remark. In view of the above result Apiola's splitting relation (S) (see [1]) is equivalent to $(\mathrm{S}_1^*)_0$ if $E$ is countably normed or $F$ reflexive, which is the case in his theorems 1.6 and 1.8. For the equivalence to $(\mathrm{S}_1^*)_0$ notice that he assumes that the $\{x: \|x\|_k \leqslant 1\}$ are a basis of neighbourhoods of zero.

**4.** In this section we apply Th. 3.8 to the case of $E$ or $F$ being a power series space. We determine the exact acyclicity classes in this case.

$\| \ \|_1 \leqslant \| \ \|_2 \leqslant \ldots$ always denotes a fundamental system of seminorms in $E$ or $F$. If $E = \Lambda_r(\alpha)$ we assume the norms to be of the form

$$\|x\|_k = \sum_j |x_j| \varrho_k^{\alpha_j} = \sum_j |x_j| e^{\sigma_k \alpha_j}$$

for some fixed sequence $0 < \varrho_k \nearrow r$. We always set $\sigma_k = \log \varrho_k$. If $F = \Lambda_r^\infty(\alpha)$ then we assume the norms to be of the form

$$\|x\|_k = \sup_j |x_j| \varrho_k^{\alpha_j} = \sup_j |x_j| e^{\sigma_k \alpha_j}$$

with $\varrho_k$ and $\sigma_k$ as above.

For the exponent sequence $(\alpha_j)_j$ we make the general assumption

$$\sup_j \frac{\alpha_{j+1}}{\alpha_j} = d < +\infty .$$

The part of the results where we do not need this assumption is stated in Th. 4.5. In Th. 4.2 we need something more, namely that

$$\lim_j \frac{\alpha_{j+1}}{\alpha_j} = 1 .$$

These assumptions are fulfilled in most of the interesting examples.

Notice that in the sufficiency parts of 4.1, 4.2, 4.3 we always know that $E$ is countably normed. In particular, this is implied by (DN). Hence we may show $(\mathrm{S}_1^*)$ in one of the modified forms of 3.10. It would also be easy to show directly $(\mathrm{S}_1^*)_0$.

For our first result we use the following property $(\Omega)$ (see [41], Def. 1.1 and Cor. 2.2), which is under the additional assumption of nuclearity



characteristic for the quotient spaces of $(s)$ ([41], 1.8). For the nonnuclear case see also [39].

$(\Omega)$ $\quad \forall p \ \exists q \ \forall k \ \exists \nu, C \ \forall y \in F'$:

$$\|y\|_q^{*1+\nu} \leqslant C \|y\|_k^* \|y\|_p^{*\nu}.$$

It is equivalent to the existence of a logarithmic convex fundamental system of dual norms (see [39]).

4.1. THEOREM. $\mathrm{Ext}^1(\Lambda_\infty(\alpha), F) = 0$ *iff* $F$ *has property* $(\Omega)$.

Proof. First we prove necessity: For given $\mu$ we choose $n_0$, $k$ according to $(\mathrm{S}_2^*)$ and then a fixed $m > n_0$. We put the basis vector $e_j$ into the inequality and obtain

$\forall K \ \exists n > m, S \ \forall j \in N, y \in F'$:

$$\|y\|_k^* \leqslant S(e^{(\sigma_n - \sigma_m)\alpha_j} \|y\|_K^* + e^{(\sigma_{n_0} - \sigma_m)\alpha_j} \|y\|_\mu^*).$$

For given $r \geqslant e^{(\sigma_m - \sigma_{n_0})\alpha_2}$ we choose $j$ such that $(\sigma_m - \sigma_{n_0})\alpha_{j-1} \leqslant \log r \leqslant (\sigma_m - \sigma_{n_0})\alpha_j$ which gives because of $a_j \leqslant d a_{j-1}$

$$(*) \qquad \|y\|_k^* \leqslant S\left(r^\nu \|y\|_K^* + \frac{1}{r} \|y\|_\mu^*\right)$$

with

$$\nu = d \frac{\sigma_n - \sigma_m}{\sigma_m - \sigma_{n_0}}.$$

We can increase $S$ so that the inequality holds for all $r > 0$. Calculation of the minimum of the function of $r$ on the right side gives the result.

To prove sufficiency we use $(\mathrm{S}_1^*)$ and put $n_0 = 1$. For given $\mu$ we find a $k$ such that for all $K$ there exists $\nu$ with

$$\|y\|_k^{*1+\nu} \leqslant C \|y\|_K^* \|y\|_\mu^{*\nu}.$$

Then we have either

$$(**) \qquad e^{\sigma_m \alpha_j} \|y\|_k^* \leqslant e^{\sigma_{n_0} \alpha_j} \|y\|_\mu^*$$

or

$$(***) \qquad \begin{aligned} e^{\sigma_m \alpha_j} \|y\|_k^{*1+\nu} &\leqslant C e^{\sigma_m \alpha_j} \|y\|_K^* \|y\|_\mu^{*\nu} \\ &\leqslant C e^{[\sigma_m + \nu(\sigma_m - \sigma_{n_0})]\alpha_j} \|y\|_K^* \|y\|_k^{*\nu}. \end{aligned}$$

We put $S = C + 1$ and choose $n$ such that $\sigma_n \geqslant \sigma_m + \nu(\sigma_m - \sigma_{n_0})$. We obtain from $(**)$ and $(***)$

$$e^{\sigma_m \alpha_j} \|y\|_k^* \leqslant S(e^{\sigma_n \alpha_j} \|y\|_K^* + e^{\sigma_{n_0} \alpha_j} \|y\|_\mu^*).$$

For given $x=(x_1, x_2, \ldots)\in \Lambda_\infty(\alpha)$ we multiply the above inequality by $|x_j|$ and add up. This proves the inequality in $(S_1^*)$.

The analogous class corresponding to a finite type power series space is described by an invariant of similar structure:

$(\bar{\Omega})$ $\quad \forall p\ \exists q\ \forall k, \varepsilon>0\ \exists C\ \forall y\in F'$:

$$\|y\|_q^{*1+\varepsilon} \leqslant C\|y\|_k^*\|y\|_p^{*\varepsilon}.$$

The same invariant $(\bar{\Omega})$ describes the class of all $(F)$-spaces $F$ such that every continuous linear map from $F$ into a finite type power series space $\Lambda_1^\infty(\alpha)$ is bounded ([35], Satz 4.2). Since $(\bar{\Omega}) \Rightarrow (\tilde{\Omega}) \Rightarrow (\mathrm{LB}^\infty)$ (see [35], Prop. 5.3) it describes on account of [35], Satz 5.2, also the class of all $(F)$-spaces $F$ such that every continuous linear map from $F$ into any power series space is bounded, i.e. maps some neighbourhood of zero into a bounded set. An infinite-dimensional space of this type can never be a subspace of any power series space.

For Theorem 4.2 we assume that

$$\lim_j \frac{\alpha_{j+1}}{\alpha_j}=1.$$

4.2. Theorem. $\mathrm{Ext}^1(\Lambda_1(\alpha), F)=0$ *iff* $F$ *has property* $(\bar{\Omega})$.

Proof. We can essentially use the proof of 4.1. For the proof of necessity the only change is that we choose $m$ in dependence of $\varepsilon>0$ and that for given $r\geqslant \exp[(\sigma_m-\sigma_{n_0})\alpha_{j_0+1}]$ we choose $j\geqslant j_0$. Then we get the inequality $(*)$ with

$$\nu=d_{j_0}\frac{\sigma_n-\sigma_m}{\sigma_m-\sigma_{n_0}}$$

where $d_{j_0}=\sup_{j\geqslant j_0}(\alpha_{j+1}/\alpha_j)$. If $m$ and $j_0$ are large enough then $\nu\leqslant\varepsilon$.

The sufficiency part is all the same except that we have to choose $\nu>0$ so small that $\sigma_m+\nu(\sigma_m-\sigma_{n_0})<0$, i.e. that there exists $n$ with $\sigma_m+\nu(\sigma_m-\sigma_{n_0})\leqslant\sigma_n$.

Now we assume $F=\Lambda_r^\infty(\alpha)$ to be fixed. In this case we need again only the weaker assumption on $(\alpha_j)_j$.

We use the following condition (DN) on $(F)$-spaces $E$ (see [31], Def. 1.1 and Satz 2.1). It is, under the additional assumption of nuclearity, characteristic for the subspaces of $(s)$ ([31], Satz 1.3). For the nonnuclear case see also [39].

(DN) $\qquad \exists n_0\ \forall m\ \exists n, C\ \forall x\in E$: $\|x\|_m^2\leqslant C\|x\|_{n_0}\|x\|_n$.



It can easily be seen that the following versions are equivalent:

$$\left.\begin{array}{l}\exists n_0, \delta>0\ \forall m\ \exists n, C\ \forall x\in E\\ \exists n_0\ \forall m, \delta\ \exists n, C\ \forall x\in E\end{array}\right\}: \quad \|x\|_m^{1+\delta}\leqslant C\|x\|_{n_0}^\delta\|x\|_n.$$

(DN) is further equivalent to the existence of a logarithmic convex fundamental system of seminorms.

4.3. Theorem. $\mathrm{Ext}^1(E, \Lambda_r^\infty(\alpha))=0$ *iff* $E$ *has property* (DN).

Proof. First we prove necessity. We apply $(S_2^*)$ with $\mu=1$ and obtain $k$ such that we obtain by putting the dual basis vectors in $F'=\Lambda_r^\infty(\alpha)'$ into the inequality:

$$\exists n_0\ \forall m\ \exists n, S:\quad \|x\|_m\leqslant S(e^{(\sigma_k-\sigma_{k+1})\alpha_j}\|x\|_n+e^{(\sigma_k-\sigma_1)\alpha_j}\|x\|_{n_0}).$$

For given $r\geqslant e^{(\sigma_k-\sigma_1)\alpha_j}$ we choose $j$ such that

$$(\sigma_k-\sigma_1)\alpha_j\leqslant\log r\leqslant(\sigma_k-\sigma_1)\alpha_{j+1}\leqslant d(\sigma_k-\sigma_1)\alpha_j$$

which gives

$$\|x\|_m\leqslant S\left(r\|x\|_{n_0}+\frac{1}{r^\delta}\|x\|_n\right)$$

with $\delta=\dfrac{1}{d}\dfrac{\sigma_{k+1}-\sigma_k}{\sigma_k-\sigma_1}$. We can increase $S$ so that the inequality holds for all $r>0$. Calculation of the minimum of the function of $r$ on the right side gives (DN) in the second form mentioned above.

For the proof of sufficiency we use the third (sharpest) form of (DN), which by the above argument can be given the form

$$\exists n_0\ \forall m, \delta\ \exists n, C\ \forall x\in E, r>0:\quad \|x\|_m\leqslant r\|x\|_{n_0}+\frac{C}{r^\delta}\|x\|_n.$$

We prove $(S_1^*)$. We take $n_0$ from (DN). For given $\mu$ we choose $k=\mu+1$. If then $K>k$, $m$ is given we set $\delta=(\sigma_K-\sigma_k)/(\sigma_k-\sigma_\mu)$ and obtain $n$, $C$ from (DN). We put $r=\exp[(\sigma_k-\sigma_\mu)\alpha_j]$ into the inequality. This gives

$$\begin{aligned}\|x\|_m&\leqslant e^{(\sigma_k-\sigma_\mu)\alpha_j}\|x\|_{n_0}+Ce^{\delta(\sigma_\mu-\sigma_k)\alpha_j}\|x\|_n\\ &=e^{(\sigma_k-\sigma_\mu)\alpha_j}\|x\|_{n_0}+Ce^{(\sigma_k-\sigma_K)\alpha_j}\|x\|_n.\end{aligned}$$

We multiply the inequality by $\exp(-\sigma_k\alpha_j)$ and obtain with $S=C+1$

$$e^{-\sigma_k\alpha_j}\|x\|_m\leqslant S(e^{-\sigma_K\alpha_j}\|x\|_n+e^{-\sigma_\mu\alpha_j}\|x\|_{n_0}).$$

For given $y=(y_1, y_2, \ldots)\in\Lambda_r^\infty(\alpha)'$ we multiply the above inequality by $|y_j|$ and add up. This proves the inequality in $(S_1^*)$.

It should be remarked in this connection that since we assume $\alpha_j\nearrow+\infty$ we have $\lim_j|x_j|\varrho^{\alpha_j}=0$ for each $0<\varrho<r$. Hence the Banach spaces attached

to our standard norms are weighted $c_0$-spaces and their duals weighted (by $\varrho^{-\alpha_j}$, $j = 1, 2, \ldots$) $l^1$-spaces. More precisely: $\|y\|_k^* = \sum_j |y_j| e^{-\sigma_k \alpha_j}$ etc.

An interesting special case we obtain by considering two power series spaces. We assume that either $\lim_j (\alpha_{j+1}/\alpha_j) = 1$ or $\sup_j (\beta_{j+1}/\beta_j) < +\infty$.

4.4. Corollary. $\operatorname{Ext}^1(\Lambda_r(\alpha), \Lambda_\varrho(\beta)) = 0$ *iff* $r = +\infty$.

In all of the preceding proofs the condition on the sequence $(\alpha_j)_j$ was only needed for the necessity part.

4.5. Theorem. *In* 4.1, ..., 4.4 *the respective conditions are sufficient for* $\operatorname{Ext}^1(E, F) = 0$ *or* $\operatorname{Ext}^1(\Lambda_r(\alpha), \Lambda_\varrho(\beta)) = 0$ *without the assumptions on* $(\alpha_j)_j$ *or* $(\beta_j)_j$.

Remark. The sufficiency part of Th. 4.1 is for nuclear $\Lambda_\infty(\alpha)$ in one of the equivalent formulations of Th. 1.8 contained in [41], Satz 1.4, cf. also [32], Th. 2.3, which can easily be generalized to the nonnuclear case. The sufficiency part of 4.3 is again in one of the equivalent formulations for $\Lambda_\infty(\alpha) = (s)$ contained in [31], Satz 1.5. This proof can also easily be generalized to arbitrary $\Lambda_\infty(\alpha)$. For analogous sufficiency results for $L_f(\alpha, \infty)$-spaces see [1].

**5.** In this section we want to discuss the (DN), ($\Omega$)-situation more closely. We recall the following theorem (cf. the introduction at the beginning of this paper): if $E$ and $F$ are nuclear, $E$ has property (DN) and $F$ has property ($\Omega$) then $\operatorname{Ext}^1(E, F) = 0$. This is a consequence of [31], 1.3, and [41], 1.4, and stated e.g. in [32], 7.2. It has been generalized by Petzsche in [22] with a proof which does not use sequence spaces.

We assume that one of our four standard assumptions is satisfied: (i) $E = \lambda(A)$, (ii) $F = \lambda^\infty(B)$, (iii) $E$ nuclear, (iv) $F$ nuclear. We obtain from Theorem 3.8

5.1. Theorem. *If $E$ has property* (DN) *and $F$ has property* ($\Omega$) *then* $\operatorname{Ext}^1(E, F) = 0$.

Proof. We use Theorem 3.8 and (DN) and ($\Omega$) in the form:

$$\exists n_0\ \forall m, \delta\ \exists n, C\ \forall x \in E:\quad \|x\|_m^{1+\delta} \leqslant C \|x\|_{n_0}^\delta \|x\|_n,$$

$$\forall \mu\ \exists k\ \forall K\ \exists \nu, D\ \forall y \in F':\quad \|y\|_k^{*1+\nu} \leqslant D \|y\|_K^* \|y\|_\mu^{*\nu}.$$

From (DN) we get $n_0$. For given $\mu$ we choose $k$ according to ($\Omega$). If then $K$ and $m$ are given we choose first $\nu$ and $D$ according to ($\Omega$) and then for $m$ and $\nu = \delta$ numbers $n$ and $C$ according to (DN).

For $x \in E$, $y \in F'$ we have either

$$(*)\qquad \|x\|_m \|y\|_k^* \leqslant \|x\|_{n_0} \|y\|_\mu^*$$

or

$$(**)\qquad \|x\|_m \|y\|_k^* > \|x\|_{n_0} \|y\|_\mu^* \geqslant 0.$$



By multiplication of the (DN)- and ($\Omega$)-inequalities, and by use of $(**)$ we obtain

$$(\|x\|_m \|y\|_k^*)^{1+\nu} \leqslant CD \|x\|_n \|y\|_K^* (\|x\|_{n_0} \|y\|_\mu^*)^\nu \leqslant CD \|x\|_n \|y\|_K^* (\|x\|_m \|y\|_k^*)^\nu$$

which yields

$$(***)\qquad \|x\|_m \|y\|_k^* \leqslant CD \|x\|_n \|y\|_K^*.$$

From $(*)$ and $(***)$ we obtain the desired inequality with $S = CD + 1$. Notice that (DN) implies countably normedness. Therefore 3.10 can be applied.

We will now derive necessary conditions for $\operatorname{Ext}^1(E, F) = 0$ which generalize the results of Section 4 to a basis free setting and turn out to be useful in concrete situations (see [37], and Section 7).

We need the following property ($\underline{\text{DN}}$), a weakened version of (DN) which is under additional nuclearity assumptions characteristic for the subspaces of power series spaces of finite type (see [33], [34]). It is for instance under very general assumptions satisfied for spaces of (real) analytic functions (see [33], § 5).

$$(\underline{\text{DN}})\qquad \exists n_0\ \forall m\ \exists n, d > 0, C:\quad \|\ \|_m^{1+d} \leqslant C \|\ \|_{n_0}^d \|\ \|_n.$$

For two absolutely convex subsets $U \subset V$ of a linear space $H$ the $\nu$th *Kolmogorov diameter* $\delta_\nu(U, V)$, $\nu = 0, 1, 2, \ldots$, is defined as the infimum of all $\delta > 0$ such that there exists a linear subspace $F \subset H$ with dimension at most $\nu$ and $U \subset \delta V + F$. A Fréchet space is a *Schwartz space* iff for every $k$ there exists a $K > k$ such that $\lim_\nu \delta_\nu(U_K, U_k) = 0$ where $U_k = \{x: \|x\|_k \leqslant 1\}$.

5.2. Theorem. *Let $F$ be infinite-dimensional and nuclear or $F = \lambda^\infty(B)$ and a Schwartz space, let moreover $F$ have property* ($\underline{\text{DN}}$), *and satisfy the following condition*:

$$\exists \mu_0\ \forall \mu \geqslant \mu_0\ \exists K_0\ \forall K \geqslant K_0:\quad \limsup_{\nu \to \infty} \frac{\log \delta_{\nu+1}(U_K, U_\mu)}{\log \delta_\nu(U_K, U_\mu)} < +\infty.$$

*Then* $\operatorname{Ext}^1(E, F) = 0$ *implies that $E$ has property* (DN).

Proof. We carry out the proof only for the case of $F$ being nuclear. The other case is similar but easier or can be considered as a direct generalization of one part of 4.3. We can assume that all $\|\ \|_k$ on $F$ are norms (because of ($\underline{\text{DN}}$)) and are generated by scalar products $\langle\ ,\ \rangle_k$, i.e. $\|\ \|_k^2 = \langle\ ,\ \rangle_k$. Hence the canonical Banach spaces $F_k$ are Hilbert spaces.

We choose $\mu_0$ as in the assumption and $\mu \geqslant \mu_0$ such that for every $k$ there exist $K, d > 0, C$ with

$$(1)\qquad \|x\|_k^{1+d} \leqslant C \|x\|_\mu^d \|x\|_K$$

for all $x \in F$. Given this $\mu$ we obtain by use of Theorem 3.9 from $(S_2^*)$ numbers $n_0$ and $k$. We choose $K$ so large that

$$\lim_{\nu} \delta_\nu(U_K, U_\mu) = 0, \tag{2}$$

$$\limsup_{\nu} \frac{\log \delta_{\nu+1}(U_K, U_\mu)}{\log \delta_\nu(U_K, U_\mu)} = D < +\infty \tag{3}$$

and that (1) holds for all $x \in F$ with appropriate $d > 0$, $C > 0$.

With these choices we have for every $m$ numbers $n$ and $S$ such that for all $x \in E$ and $y \in F'$

$$\|x\|_m \|y\|_k^* \leqslant S(\|x\|_n \|y\|_K^* + \|x\|_{n_0} \|y\|_\mu^*). \tag{4}$$

The canonical map $\varrho_{K,\mu}\colon F_K \to F_\mu$ is compact by (2). Hence we obtain by the spectral theorem

$$\varrho_{K,\mu} x = \sum_{j=0}^{\infty} a_j \langle x, e_j \rangle_K f_j$$

with orthonormal systems $(e_j)_j$, $(f_j)_j$ in $F_K$, $F_\mu$ respectively and $a_0 \geqslant a_1 \geqslant \ldots \ldots \geqslant 0$. Since $a_j = \delta_j(\mathring{U}_K, U_\mu)$ (see [23], 8.3.2) we have with $\alpha_j = -\log a_j$

$$\lim_{j} \alpha_j = +\infty, \tag{2'}$$

$$\limsup_{j} \frac{\alpha_{j+1}}{\alpha_j} = D < +\infty. \tag{3'}$$

For $x \in F$ we put

$$y_j(x) = \frac{1}{a_j} \langle \varrho_\mu x, f_j \rangle_\mu = \langle \varrho_K x, e_j \rangle_K$$

and obtain

$$\|y_j\|_K^* = 1, \qquad \|y_j\|_\mu^* = 1/a_j.$$

We put $x_j = a_j^{-d/(1+d)} \varrho_{K,k} e_j$. We obtain from (1)

$$\|x_j\|_k \leqslant C \|a_j^{1/(1+d)} f_j\|_\mu^{d/(1+d)} \|a_j^{-d/(1+d)} e_j\|_K^{1/(1+d)} = C$$

and hence

$$\|\tilde{y}_j\|_k^* \geqslant \frac{1}{C} |\tilde{y}_j(x_j)| = \frac{1}{C} a_j^{-d/(1+d)}$$

where $\tilde{y}_j$ denotes the canonical extension of $y_j$ to $F_k$.

Applying (4) to $y_j$ we get for every $j$

$$\|x\|_m \frac{1}{C} a_j^{-d/(1+d)} \leqslant S\left(\|x\|_n + \|x\|_{n_0} \frac{1}{a_j}\right)$$



and by use of (3′) for large $j$

$$\begin{aligned} \|x\|_m &\leqslant SC(e^{-d\alpha_j/(1+d)} \|x\|_n + e^{\alpha_j/(1+d)} \|x\|_{n_0}) \\ &\leqslant SC(e^{-d\alpha_{j+1}/[2D(1+d)]} \|x\|_n + e^{\alpha_j/(1+d)} \|x\|_{n_0}). \end{aligned}$$

For $r > 0$ large enough we choose $j$ such that

$$e^{\alpha_j/(1+d)} \leqslant r \leqslant e^{\alpha_{j+1}/(1+d)}$$

and obtain

$$\|x\|_m \leqslant M\left(r \|x\|_{n_0} + \frac{1}{r^\delta} \|x\|_n\right)$$

with $\delta = d/(2D)$. If $M \geqslant SC$ is large enough this inequality holds for all $r > 0$ and $x \in E$. Taking the minimum of the function of $r$ on the right-hand side we have with modified $M$

$$\|x\|_m^{1+\delta} \leqslant M \|x\|_{n_0}^\delta \|x\|_n$$

which proves the assertion.

The following remarks are useful for the application of the preceding theorem. We state them without proof.

5.3. **Remarks.** (a) The assumption on the Kolmogorov diameters in 5.2 is satisfied if there exists an increasing sequence $(\alpha_\nu)_\nu$ with $\lim_\nu \alpha_\nu = +\infty$ and $\sup_\nu (\alpha_{\nu+1}/\alpha_\nu) < +\infty$ such that the following is true:

(P) $\quad \exists \mu_0 \ \forall \mu \geqslant \mu_0 \ \exists K_0 \ \forall K \geqslant K_0 \ \exists D > 0, r > 1, R > 1 \ \forall \nu$:

$$\frac{1}{D} R^{-\alpha_\nu} \leqslant \delta_\nu(U_K, U_\mu) \leqslant D r^{-\alpha_\nu}.$$

(b) Condition (P) in (a) is satisfied if $F$ is $\Lambda_1(\alpha)$-nuclear (see [25], [33]; we assume $\lim_n (\log n)/\alpha_n = 0$) and

$$\exists \mu, K_0 \ \forall K \geqslant K_0 \ \exists R > 1\colon \ \inf_\nu R^{\alpha_\nu} \delta_\nu(U_K, U_\mu) > 0.$$

(c) If $F$ has $(\underline{DN})$ and $(\Omega)$ then the argument of [42], 4.3 (cf. [33], 3.5) and a modification for the $(\Omega)$-case (cf. [34], 7.1) says that the conditions in (b) are satisfied if there exists $n_0$ in the sense of $(\underline{DN})$, and $r > 1$ such that $\sup_\nu r^{\alpha_\nu} \delta_\nu(U_m, U_{n_0}) < +\infty$ and if moreover there exists for some $p$ a $q$ in the sense of $(\Omega)$ and $R > 1$ such that $\inf_\nu R^{\alpha_\nu} \delta_\nu(U_q, U_p) > 0$.

(d) Another useful set of conditions which implies (b) is: $\sup_\nu (\alpha_{2\nu}/\alpha_\nu) <$

$+\infty$, $F$ $\Lambda_1(\alpha)$-nuclear and there exists an isomorphic imbedding $\Lambda_\infty(\alpha) \hookrightarrow F^N$. For the proof we refer to [33], 1.4.

Just as in 5.2 we can obtain necessary conditions on $F$ for $\operatorname{Ext}^1(E, F) = 0$ given some information on $E$.

5.4. Theorem. *Let $E$ be infinite-dimensional and nuclear or $E = \lambda(A)$ and a Schwartz space. Let moreover $E$ have property $(\Omega)$ and satisfy the condition on the Kolmogorov diameters given in 5.2. Then $\operatorname{Ext}^1(E, F) = 0$ implies that $F$ has property $(\Omega)$.*

Proof. Again we carry out the proof only in the nuclear case. We can assume that all $\|\ \|_k$ on $E$ are norms, because otherwise $F$ is a quojection (see 3.6) and hence trivially satisfies $(\Omega)$, and are generated by scalar products $\langle\ ,\ \rangle_k$. Hence the canonical Banach spaces $E_k$ are Hilbert spaces. We proceed as in the proof of 5.2 using Theorem 3.8.

Given $\mu$ we obtain from $(S_2^*)$ numbers $k$ and $n_0$. We can assume $n_0 \geqslant \mu_0$ where $\mu_0$ is as in the assumption on the Kolmogorov diameters. For $p = n_0$ we choose $q$ according to $(\Omega)$. Then given $K$ we apply condition $(S_2^*)$ to $K$ and $m = 0$, and obtain $n$ and $S$ such that

$$\|x\|_m \|y\|_k^* \leqslant S(\|x\|_n \|y\|_K^* + \|x\|_{n_0} \|y\|_\mu^*) \tag{1}$$

for all $x \in E$, $y \in F'$. Finally, putting $n$ (instead of $k$) into $(\Omega)$ we obtain $\nu$ and $C$ such that

$$\|\eta\|_m^{*1+\nu} \leqslant C \|\eta\|_n^* \|\eta\|_{n_0}^{*\nu} \tag{2}$$

for all $\eta \in E'$. We can assume $n_0 < m < n$ and $n \geqslant K_0$ where $K_0$ is as in the assumption on the Kolmogorov diameters, and so large that $\lim_\nu \delta_\nu(U_n, U_{n_0}) = 0$.

Hence the canonical map $\varrho_{n,n_0}\colon E_n \to E_{n_0}$ is compact and we obtain by the spectral theorem

$$\varrho_{n,n_0} x = \sum_{j=0}^{\infty} a_j \langle x, e_j\rangle_n f_j$$

with orthonormal systems $(e_j)_j$, $(f_j)_j$ in $E_n$, $E_{n_0}$ respectively and $a_0 \geqslant a_1 \geqslant \ldots \geqslant 0$. Since again $a_j = \delta_j(U_n, U_{n_0})$ (see [23], 8.3.2) we have with $\alpha_j = -\log a_j$

$$\lim_j \alpha_j = +\infty, \qquad \limsup_j \frac{\alpha_{j+1}}{\alpha_j} = D < +\infty.$$

By definition we have

$$\|e_j\|_n = 1, \qquad \|\varrho_{n,n_0} e_j\|_{n_0} = a_j.$$

To estimate the norm of $\varrho_{n,m} e_j$ we define for $x \in E$

$$\eta_j(x) = \langle \varrho_n x, e_j\rangle_n = \frac{1}{a_j} \langle \varrho_{n_0} x, f_j\rangle_{n_0}$$



and obtain

$$\|\eta_j\|_n^* = 1, \qquad \|\eta_j\|_{n_0}^* = \frac{1}{a_j},$$

hence by (2)

$$\|\eta_j\|_m^{*1+\nu} \leqslant C \|\eta_j\|_n^* \|\eta_j\|_{n_0}^{*\nu} = C \frac{1}{a_j^\nu}.$$

Therefore we have

$$\|e_j\|_m \geqslant C^{-1/(1+\nu)} a_j^{-\nu/(1+\nu)} |\eta_j(e_j)| = C^{-1/(1+\nu)} a_j^{\nu/(1+\nu)}.$$

Putting $e_j$ into (1) we obtain for all $y \in F'$ and $j$

$$C^{-1/(1+\nu)} a_j^{\nu/(1+\nu)} \|y\|_k^* \leqslant S(\|y\|_K^* + a_j \|y\|_\mu^*)$$

or also

$$\|y\|_k^* \leqslant C^{1/(1+\nu)} S(e^{\nu\alpha_j/(1+\nu)} \|y\|_K^* + e^{-\alpha_j/(1+\nu)} \|y\|_\mu^*).$$

Proceeding as in the proof of 5.2 we finally obtain

$$\|y\|_k^{*1+\delta} \leqslant M \|y\|_K^* \|y\|_\mu^{*\delta}$$

for all $y \in F'$ with appropriate $\delta > 0$ and $M$. This proves the assertion.

**6.** We continue the discussion of the $(\Omega)$, (DN)-situation and assume in this section a fixed exact sequence $0 \to F \to G \xrightarrow{q} H \to 0$ to be given.

In this section $\Lambda$ always denotes a space of the form

$$\Lambda = \Lambda(M, (a_k)_{k\in N}) = \{f \in K^M\colon \|f\|_k = \sum_{t\in M} |f(t)|\, a_k(t) < +\infty \text{ for all } k\}$$

where $M$ is a set, $K = \mathbf{R}$ or $\mathbf{C}$ the scalar field and $a_1 \leqslant a_2 \leqslant \ldots$, $\sup_k a_k(t) > 0$ for all $t$, is a sequence of nonnegative functions on $M$. These are the analogues to Köthe spaces defined on a not necessarily countable index set.

6.1. Proposition. *The following are equivalent:*

(1) *There exist sequences $H_n$, $m(n)$, $M(n)$ such that for every $h \in H$ we can find $g \in G$ with $h = qg$ and*

$$\|g\|_n \leqslant H_n \max(1, \|h\|_{m(n)}^{M(n)}).$$

(2) *For every $\Lambda$ with (DN) and $\varphi \in L(\Lambda, H)$ there exists $\psi \in L(\Lambda, G)$ such that $\varphi = q \circ \psi$.*

Proof. Given (1) and $\Lambda$ with (DN) we may assume that $n_0 = 1$. Then $a_1(t) > 0$ for all $t$. We apply (1) to the vectors $h_t = (1/a_1(t))\varphi(e_t)$, $t \in M$, where $e_t(\tau) = \delta_{t,\tau}$. We obtain vectors $g_t$ and put $\psi(e_t) = a_1(t) g_t$. This defines a map $\psi \in L(\Lambda, G)$ such that

$$\sum_t |f(t)|\,\|\psi(e_t)\|_n \leqslant H_n \sum_t |f(t)|\, a_1(t) \max(1, \|h_t\|_{m(n)}^{M(n)})$$
$$\leqslant \tilde{H}_n \sum_t |f(t)| \frac{a_{l(n)}(t)^{M(n)}}{a_1(t)^{M(n)-1}}$$
$$\leqslant C_n \sum_t |f(t)|\, a_{K(n)}(t).$$

Here $l(n)$ comes from the continuity estimates for $\varphi$ and $K(n)$ is chosen according to (DN) for $m = l(n)$, $\delta = M(n)-1$; one should notice that $\|e_t\|_n = a_n(t)$. $\psi$ obviously satisfies (2).

To prove the other implication we put $M = H$, $a_k(h) = (1+\|h\|_k)^k$ and obtain $\varphi \in L(\Lambda, H)$ by $\varphi(e_h) = h$. We get $\psi \in L(\Lambda, G)$ with $\varphi = q \circ \psi$ by (2). Then $g = \psi(e_h)$ for $h \in H$ fulfills the requirements in (1).

A modification of 5.1 for $E = \Lambda$ instead of $E = \lambda(A)$ together with 6.1, (2) $\Rightarrow$ (1), gives immediately

6.2. Proposition. *If $F$ has $(\Omega)$ then* 6.1(1) *is satisfied.*

This proposition can also be proved directly by a modification of the proof of [32], 2.3 (cf. [41], 1.4). One has to replace there the basis vector $e_j \in (s)$ by $h \in H$.

The preceding propositions explain a lifting property for certain maps in $L(\Lambda, H)$ (resp. the property $(\Omega)$ for $F$) by the possibility of finding solutions for the equation $\varphi g = h$ with certain estimates. If $G$ has property $(\Omega)$ then 6.2 can be improved to an equivalence.

6.3. Theorem. *If $G$ has property $(\Omega)$ then the following are equivalent:*

(1) *$F$ has property $(\Omega)$.*

(2) *The condition in* 6.1(1).

Proof. (2) follows from (1) by 6.2. From (2) we conclude by 6.1, (1) $\Rightarrow$ (2), applied to $\Lambda = (s)$ that the map $\mathrm{Ext}^1(s, F) \to \mathrm{Ext}^1(s, G)$ induced by $F \to G$ is injective (see § 1, I). Since $\mathrm{Ext}^1(s, G) = 0$ by 4.1 we have $\mathrm{Ext}^1(s, F) = 0$ and hence again by 4.1 that $F$ has property $(\Omega)$.

One cannot expect a theorem like 6.3 without assumption on $G$ as the canonical sequence $0 \to F \to F \oplus H \to H \to 0$ shows. On the other hand, in concrete situations often $q$ is an operator defined on a well-known space $G$ with $(\Omega)$ whereas one is interested in properties of $H = \ker q$ resp. in estimates for solutions of $qg = h$.

It is interesting to discuss the analogue to 6.1(1) where we have linear estimates. It helps us to get information for the (DN), $(\Omega)$-situation in the case of a nonsplitting sequence, i.e. if neither $F$ is locally injective nor $H$ is locally projective.

6.4. Proposition. *The following are equivalent:*

(1) *There exist sequences $H_n$, $m(n)$ such that for every $h \in H$ we can find $g \in G$ with $h = qg$ and $\|g\|_n \leqslant H_n \|h\|_{m(n)}$.*



(2) *For every $\Lambda$ and $\varphi \in L(\Lambda, H)$ there exists $\psi \in L(\Lambda, G)$* such that *$\varphi = q \circ \psi$.*

Proof. Given (1), $\Lambda$ and $\varphi \in L(\Lambda, H)$ we apply (1) to $h_t = \varphi(e_t)$, $t \in M$. We obtain $g_t \in G$ and define $\psi \in L(\Lambda, G)$ by $\psi(e_t) = g_t$.

For the other implication we use the space $\Lambda$ with $M = H$ and $a_k(h) = \|h\|_k$ and proceed as in the proof of 6.1.

If $H$ has (DN) the space $\Lambda$ just constructed obviously also has (DN). Hence by a modification of 5.1 for $E = \Lambda$ instead of $E = \lambda(A)$ we conclude that every $\varphi \in L(\Lambda, H)$ can be lifted to $\psi \in L(\Lambda, G)$ which, as before, implies 6.4(1). So we have proved:

6.5. Proposition. *If $F$ has $(\Omega)$ and $H$ has* (DN) *then* 6.4(1) *is satisfied.*

This implies by 6.4 that every $\varphi \in L(\Lambda, H)$ for any $\Lambda$, in particular for any Köthe sequence space $\lambda(A)$, can be lifted to $\psi \in L(\Lambda, G)$.

6.5 is a weakened form of 5.1 for the case when $F$ is not locally injective and $H$ is not locally projective. Clearly 6.4(1) follows from the existence of a right inverse of $q$, hence from $\mathrm{Ext}^1(H, F) = 0$. Any nonsplitting sequence of Banach spaces shows that nonsplitting sequences exist with $F$ having property $(\Omega)$ and $H$ having property (DN).

7. We finish the discussion of the (DN), $(\Omega)$-case by applying the results of § 5 to determine the exact acyclicity classes for spaces of solutions of elliptic differential operators (cf. [37]) and for spaces of holomorphic functions (cf. [38]).

First we recall a result of [33]. Let $X$ be a connected $N$-dimensional $\sigma$-compact real-analytic manifold, $\mathscr{A}$ the sheaf of complex-valued real-analytic functions on $X$, $\mathscr{G} \subset \mathscr{A}$ a subsheaf such that for every open set $U \subset X$ the space $G(U)$ is complete in the compact-open topology. Then [33], Satz 5.1 ff., say that the $(F)$-space $G(X)$ has property $(\underline{\mathrm{DN}})$.

A. Let $P(D)$ be an elliptic linear partial differential operator with constant coefficients on $\boldsymbol{R}^N$, $N \geqslant 2$. In [37], 2.4, it is shown that for any open $\Omega \subset \boldsymbol{R}^N$ the space $\mathscr{N}(\Omega) = \{f \in C^\infty(\Omega)\colon P(D)f = 0\}$ has property $(\Omega)$ in the induced topology of $C^\infty(\Omega)$ which is equivalent to the compact-open topology. From [33], 4.5, we know that $\mathscr{N}(\Omega)$ is $\Lambda_1(\alpha)$-nuclear with $\alpha_n = n^{1/(N-1)}$ and from [44], Kap. 2, § 2, Satz 9, that $\mathscr{N}(\boldsymbol{R}^N) \cong \Lambda_\infty(\alpha)$ with the same $\alpha$.

For the following theorem cf. [37], 2.6.

7.1. Theorem. $\mathrm{Ext}^1(E, \mathscr{N}(\Omega)) = 0$ *if and only if $E$ has property* (DN).

Proof. One implication follows from 5.1 since $\mathscr{N}(\Omega)$ is nuclear and has property $(\Omega)$. If on the other hand $\mathrm{Ext}^1(E, \mathscr{N}(\Omega)) = 0$ then the same is true for every connected component $\omega \subset \Omega$. $\mathscr{N}(\omega)$ has $(\underline{\mathrm{DN}})$ and is $\Lambda_1(\alpha)$-nuclear with $\alpha$ as above. There exist $x_1, x_2, \ldots \in \boldsymbol{R}^N$ such that $\boldsymbol{R}^N = \bigcup_k (x_k + \omega)$. Since $\mathscr{N}(x_k+\omega) \cong \mathscr{N}(\omega)$ we have an isomorphic imbedding

$$\Lambda_\infty(\alpha) \cong \mathscr{N}(\boldsymbol{R}^N) \to \prod_k \mathscr{N}(x_k+\omega) \cong \mathscr{N}(\omega)^{\boldsymbol{N}}.$$

Therefore the other implication follows from 5.2 and 5.3 (d).

We remark that in [37], 2.1, it is shown that the canonical map $\operatorname{Ext}^1(E, \mathcal{N}(\Omega)) \to \operatorname{Ext}^1(E, C^\infty(\Omega))$ induced by $\iota\colon \mathcal{N}(\Omega) \hookrightarrow C^\infty(\Omega)$ is zero. Therefore we have an exact sequence

$$0 \to L(E, \mathcal{N}(\Omega)) \xrightarrow{\iota^*} L(E, C^\infty(\Omega)) \xrightarrow{P(D)^*} L(E, C^\infty(\Omega)) \to \operatorname{Ext}^1(E, \mathcal{N}(\Omega)) \to 0.$$

Hence $\operatorname{Ext}^1(E, \mathcal{N}(\Omega)) = 0$ is necessary and sufficient for the surjectivity of $P(D)^*$.

Consequences of 7.1 are lifting theorems for $P(D)$, the existence of right inverses for $P(D)$ on continuously imbedded (DN)-subspaces of $C^\infty(\Omega)$ or $C(\Omega)$ and necessary and sufficient conditions for the solvability of $P(D)g_\lambda = f_\lambda$ where $f_\lambda$ and $g_\lambda$ are parametrized families of $C^\infty$-functions and the parameter $\lambda$ runs through the dual of an $(F)$-space, e.g. germs of holomorphic functions or a distribution space or a Gevrey class (see [37]).

The question for which $F$ we have $\operatorname{Ext}^1(\mathcal{N}(\Omega), F) = 0$ is not so smoothly solvable. It also seems to be less important. From 5.4 we deduce that property $(\Omega)$ is a necessary condition. It is not sufficient in general. If e.g. $\Omega$ is bounded and convex we know from [34], 7.7 that $\mathcal{N}(\Omega) \cong \Lambda_1(\alpha)$, $\alpha$ as above. Hence in this case $\operatorname{Ext}^1(\mathcal{N}(\Omega), F) = 0$ if and only if $F$ has property $(\bar{\Omega})$ (see 4.2), whereas $\operatorname{Ext}^1(\mathcal{N}(\boldsymbol{R}^N), F) = 0$ if and only if $F$ has property $(\Omega)$, because $\mathcal{N}(\boldsymbol{R}^N) \cong \Lambda_\infty(\alpha)$ (see [44]).

**B.** Let $X$ be a connected $N$-dimensional Stein manifold. Then the nuclear $(F)$-space $\mathcal{H}(X)$ of holomorphic functions on $X$ has property $(\underline{DN})$. This follows from the remarks at the beginning of this section. We can imbed $X$ as a closed subvariety into $\boldsymbol{C}^M$, $M$ large enough. Then the restriction map $\mathcal{H}(\boldsymbol{C}^M) \to \mathcal{H}(X)$ is surjective. Since $\mathcal{H}(\boldsymbol{C}^M) \cong \Lambda_\infty(\beta)$ with $\beta_n = n^{1/M}$ the space $\mathcal{H}(X)$ has property $(\Omega)$ (cf. [38] and § 7, C below). Using local coordinates we can imbed $\mathcal{H}(X) \to \mathcal{H}(D^N)^N \cong \Lambda_1(\alpha)^N$, where $D^N$ is the $N$-dimensional polydisc and $\alpha = n^{1/N}$. Hence $\mathcal{H}(X)$ is $\Lambda_1(\alpha)$-nuclear. On the other hand, let $\varphi = (\varphi_1, \ldots, \varphi_N)$ be local coordinates at some point $x_0 \in X$ given by global holomorphic functions on $X$ (see [11], p. 105). Let $U \subset \varphi X$ be open such that $\varphi^{-1}$ exists on $U$ and $z_1, z_2, \ldots \in \boldsymbol{C}^N$ such that $\bigcup_j (z_j + U) = \boldsymbol{C}^N$, then $f \mapsto (f \circ (z_j + \varphi))_{j=1,2,\ldots}$ imbeds $\Lambda_\infty(\alpha) \cong \mathcal{H}(\boldsymbol{C}^N)$ into $\mathcal{H}(X)^N$. From 5.1, 5.2 and 5.3 (d) we conclude:

7.2. THEOREM. $\operatorname{Ext}^1(E, \mathcal{H}(X)) = 0$ *if and only if $E$ has property* (DN).

This result, in particular for $E = \mathcal{H}(X)$, is interesting in connection with investigations of Grothendieck on the topological properties of the space $L_b(E, F)$ (cf. [9], II, § 7 and [40], § 4).

For $\operatorname{Ext}^1(\mathcal{H}(X), F) = 0$ the situation is similar to that in part A of this section.

**C.** Let now $X$ be an analytic subvariety of $\boldsymbol{C}^N$, $\mathcal{G}_X$ the sheaf of ideals of $X$, $\mathcal{J}_X = \Gamma(\boldsymbol{C}^N, \mathcal{G}_X)$ the space of all functions in $\mathcal{H}(\boldsymbol{C}^N)$ which vanish on $X$.



According to the remarks at the beginning of this section $\mathcal{J}_X$ has property $(\underline{DN})$. In [38] it is shown that for any coherent analytic sheaf $\mathcal{G}$ on $\boldsymbol{C}^N$ the nuclear space $\Gamma(\boldsymbol{C}^N, \mathcal{G})$ has property $(\Omega)$. Clearly $\mathcal{J}_X \subset \mathcal{H}(\boldsymbol{C}^N)$ is $\Lambda_1(\alpha)$-nuclear with $\alpha_n = n^{1/N}$. On the other hand, we choose $g_0 \in \mathcal{J}_X$, $g_0 \not\equiv 0$, open $U \subset \boldsymbol{C}^N$ such that $\inf_{z \in U} |g_0(z)| > 0$ and $z_1, z_2, \ldots \in \boldsymbol{C}^N$ such that $\bigcup_j (z_j + U) = \boldsymbol{C}^N$. The map $f \mapsto (g_j)_{j \in \boldsymbol{N}}$ with $g_j(z) = g_0(z) f(z - z_j)$ imbeds $\Lambda_\infty(\alpha) \cong \mathcal{H}(\boldsymbol{C}^N)$ into $\mathcal{J}_X^{\boldsymbol{N}}$. From 5.1, 5.2 and 5.3 (d) we obtain

7.3. THEOREM. $\operatorname{Ext}^1(E, \mathcal{J}_X) = 0$ *if and only if $E$ has property* (DN).

Most interesting is the case $E = \mathcal{H}(X)$. It shows that there exists a continuous linear extension map $\mathcal{H}(X) \mapsto \mathcal{H}(\boldsymbol{C}^N)$ if and (since $\mathcal{H}(\boldsymbol{C}^N)$ has (DN)) only if $\mathcal{H}(X)$ has property (DN). A close investigation of this case can be found in [38] (cf. [45]). In particular, the manifolds $X$ such that $\mathcal{H}(X)$ has (DN) can be completely characterized.

## References


[1] H. Apiola, *Characterization of subspaces and quotients of nuclear $L_f(\alpha, \infty)$-spaces*, Compositio Math. 50 (1983), 65–81.

[2] S. Bellenot and E. Dubinsky, *Fréchet spaces with nuclear Köthe quotients*, Trans. Amer. Math. Soc. 273 (1982), 579–594.

[3] K. D. Bierstedt, R. Meise and W. H. Summers, *Köthe sets and Köthe sequence spaces*, in: Functional Analysis, Holomorphy and Approximation Theory, J. A. Barroso (ed.), North-Holland Math. Stud. 71, 1982, 27–90.

[4] E. Bierstone and G. W. Schwarz, *Continuous linear division and extension of $C^\infty$-functions*, Duke Math. J. 50 (1983), 233–271.

[5] S. Dierolf and D. N. Zarnadze, *A note on strictly regular Fréchet spaces*, Arch. Math. 42 (1984), 549–556.

[6] E. Dubinsky, *The Structure of Nuclear Fréchet Spaces*, Lecture Notes in Math. 720, Springer, 1979.

[7] E. Dubinsky and M. S. Ramanujan, *On λ-nuclearity*, Mem. Amer. Math. Soc. 128 (1972).

[8] W. Gejler, *On extending and lifting continuous linear mappings in topological vector spaces*, Studia Math. 62 (1978), 295–303.

[9] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1953).

[10] —, *Topological Vector Spaces*, Gordon and Breach, 1973.

[11] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, Van Nostrand, 1966.

[12] T. Ketonen and K. Nyberg, *Twisted sums of nuclear Fréchet spaces*, Ann. Acad. Sci. Fenn. Ser. AI Math. 7 (1982), 323–335.

[13] G. Köthe, *Topologische lineare Räume*, Springer, 1960.

[14] J. Krone and D. Vogt, *The splitting relation for Köthe spaces*, Math. Z. 190 (1985), 387–400.

[15] R. Meise, *Structure of closed linear translation invariant subspaces of $A(C)$ and kernels of convolution operators*, in: Functional Analysis: Surveys and Recent Results III, K. D. Bierstedt and B. Fuchssteiner (eds.), North-Holland Math. Stud. 90, 1984, 331–347.

[16] R. Meise and D. Vogt, *Counterexamples in holomorphic functions on nuclear Fréchet spaces*, Math. Z. 182 (1983), 167–177.

[17] R. Meise and D. Vogt, *A characterization of quasi-normable Fréchet spaces*, Math. Nachr. 122 (1985), 141–150.

[18] B. S. Mityagin and G. M. Henkin, *Linear problems of complex analysis* (in Russian), Uspekhi Mat. Nauk 26 (4) (1971), 93–152, English transl.: Russian Math. Surveys 26 (4) (1971), 99–164.

[19] D. G. Northcott, *An Introduction to Homological Algebra*, Cambridge University Press, 1972.

[20] V. P. Palamodov, *The projective limit functor in the category of linear topological spaces* (in Russian), Mat. Sb. 75 (1968), 567–603, English transl.: Math. USSR-Sb. 4 (1968), 529–558.

[21] —, *Homological methods in the theory of locally convex spaces* (in Russian), Uspekhi Mat. Nauk 26 (1) (1971), 3–66, English transl.: Russian Math. Surveys 26 (1) (1971), 1–64.

[22] H. J. Petzsche, *Some results of Mittag-Leffler type for vector valued functions and spaces of class* $\mathscr{A}$, in: Functional Analysis: Surveys and Recent Results II, K. D. Bierstedt and B. Fuchssteiner (eds.), North-Holland Math. Stud. 38, 1980, 183–204.

[23] A. Pietsch, *Nuclear Locally Convex Spaces*, Ergeb. Math. Grenzgeb. 66, Springer, 1972.

[24] V. S. Retakh, *On the dual of a subspace of a countable inductive limit* (in Russian), Dokl. Akad. Nauk SSSR 184 (1) (1969), 44–46, English transl.: Soviet Math. Dokl. 10 (1) (1969), 39–41.

[25] W. Robinson, *On* $\Lambda_1(\alpha)$*-nuclearity*, Duke Math. J. 40 (1973), 541–546.

[26] H. H. Schaefer, *Topological Vector Spaces*, Springer, 1966.

[27] W. Słowikowski and W. Zawadowski, *Note on relatively complete* $B_0$*-spaces*, Studia Math. 15 (1956), 267–272.

[28] M. Tidten, *Fortsetzungen von* $C^\infty$*-Funktionen, welche auf einer abgeschlossenen Menge in* $\boldsymbol{R}^n$ *definiert sind*, Manuscripta Math. 27 (1979), 291–312.

[29] D. Vogt, *Vektorwertige Distributionen als Randverteilungen holomorpher Funktionen*, ibid. 17 (1975), 267–290.

[30] —, *Tensorprodukte von (F)- mit (DF)-Räumen und ein Fortsetzungssatz*, preprint.

[31] —, *Charakterisierung der Unterräume von s*, Math. Z. 155 (1977), 109–117.

[32] —, *Subspaces and quotient spaces of (s)*, in: Functional Analysis: Surveys and Recent Results, K.D. Bierstedt and B. Fuchssteiner (eds.), North-Holland Math. Stud. 27, 1977, 167–187.

[33] —, *Charakterisierung der Unterräume eines nuklearen stabilen Potenzreihenraumes von endlichem Typ*, Studia Math. 71 (1982), 251–270.

[34] —, *Eine Charakterisierung der Potenzreihenräume von endlichem Typ und ihre Folgerungen*, Manuscripta Math. 37 (1982), 269–301.

[35] —, *Fréchеträume, zwischen denen jede stetige lineare Abbildung beschränkt ist*, J. Reine Angew. Math. 345 (1983), 182–200.

[36] —, *An example of a nuclear Fréchet space without the bounded approximation property*, Math. Z. 182 (1983), 265–267.

[37] —, *On the solvability of* $P(D)f = g$ *for vector valued functions*, RIMS Kokyoroku 508 (1983), 168–182.

[38] —, *Über die Existenz von Ausdehnungsoperatoren für holomorphe Funktionen auf analytischen Mengen in* $C^N$, preprint.

[39] —, *On two classes of (F)-spaces*, Arch. Math. 45 (1985), 255–266.

[40] —, *Some results on continuous linear maps between Fréchet spaces*, in: Functional Analysis: Surveys and Recent Results III, K. D. Bierstedt and B. Fuchssteiner (eds.), North-Holland Math. Studies 90, 1984, 349–381.

[41] D. Vogt and M. J. Wagner, *Charakterisierung der Quotientenräume von s und eine Vermutung von Martineau*, Studia Math. 67 (1980), 225–240.

[42] —, —, *Charakterisierung der Unterräume und Quotientenräume der nuklearen stabilen Potenzreihenräume von unendlichem Typ*, ibid. 70 (1981), 65–80.

[43] M. J. Wagner, *Quotientenräume von stabilen Potenzreihenräumen endlichen Typs*, Manuscripta Math. 31 (1980), 97–109.

[44] G. Wiechert, *Dualitäts- und Strukturtheorie der Kerne von linearen Differentialoperatoren*, Dissertation, Wuppertal 1982.

[45] V. P. Zakharyuta, *Isomorphism of spaces of analytic functions* (in Russian), Dokl. Akad. Nauk SSSR 255 (1980), 11–14, English transl.: Soviet Math. Dokl. 22 (1980), 631–634.


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*Received October 28, 1985* (2105)