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The Bergman projection on harmonic functions

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Abstract. The behavior of the Bergman projection on various spaces of harmonic functions is studied. It is proved that if D is a strictly pseudoconvex bounded domain in C^n with boundary of class C^4 then its Bergman projection extends continuously to a projection from L^1 Harm(D) onto $L^1 H(D)$ and maps continuously $L^\infty(D)$ onto the space of Bloch holomorphic functions, the space of Bloch functions onto the space of Bloch holomorphic functions, and the space of bounded harmonic functions onto the dual of $H^1(\partial D)$. Analogous results are proved for orthogonal projections onto the spaces of pluriharmonic functions.

1. Introduction and the statement of results. Let D be a bounded domain in C^n. We shall assume that $D = \{z \in C^n\colon \varrho(z) < 0\}$, $\varrho \in C^\infty(C^n)$ (or $\varrho \in C^k(C^n)$) and $\operatorname{grad}\varrho \neq 0$ on ∂D. Each such ϱ will be called a *defining function* for D. The *Bergman projection* B is the orthogonal projection from $L^2(D)$ onto the space $L^2 H(D)$ of square-integrable holomorphic functions. We shall denote by L^2 Harm(D) the space of square-integrable harmonic functions.

It was proved in [18] that if B is bounded in the sth Sobolev norm then its restriction to L^2 Harm(D) is bounded in the negative Sobolev norm $\| \ \|_{-s}$, where s is a positive integer. It was also shown that the norm $\| \ \|_{-s}$ is equivalent on harmonic functions to the $L^2(D, \varrho^{2s})$ norm, where ϱ is a defining function for D.

In [20] it was proved that if ∂D is of class $\Lambda_{4+\alpha_0}$ and B is bounded in the αth Hölder norm, $\alpha < \alpha_0$, $0 < \alpha - [\alpha] < 1$, then the restriction of B to L^2 Harm(D) is bounded in the $L^1(D, |\varrho|^\alpha)$ norm where ϱ is a defining function for D. The expression "B is bounded in some norm $\| \ \|$" means as usual that there exists a constant c such that for every $u \in L^2$ Harm(D)

$$\|Bu\| \leqslant c\,\|u\|.$$

The above results yield in particular that if D is a strictly pseudoconvex domain with C^∞-smooth boundary then B extends to a continuous projection from L^2 Harm(D, ϱ^{2s}) onto $L^2 H(D, \varrho^{2s})$ and to a continuous projection from $\mathring{L}^1$ Harm$(D, |\varrho|^\alpha)$ onto $\mathring{L}^1 H(D, |\varrho|^\alpha)$, where $\mathring{L}^1$ Harm$(D, |\varrho|^\alpha)$ denotes the closure of L^2 Harm(D) in $L^1(D, |\varrho|^\alpha)$ and $\mathring{L}^1 H(D, |\varrho|^\alpha)$ denotes the closure of $L^2 H(D)$ in $L^1(D, |\varrho|^\alpha)$.

This means that the Bergman projection "behaves better" on the spaces of harmonic functions than on arbitrary functions.

The aim of the present paper is to give some further examples of such behavior of the Bergman projection. We are going to prove the following

THEOREM 1. *Let D be a strictly pseudoconvex domain with boundary of class C^4. Then the Bergman projection B extends to a continuous projection from* $L^1\operatorname{Harm}(D)$ *onto* $L^1 H(D)$.

Note that B cannot be extended to a continuous projection from $L^1(D)$ onto $L^1 H(D)$ even if D is the unit ball in C^n (see [22]).

In the proof of Theorem 1 we shall use spaces of Bloch functions. A differentiable function f on D is called a *Bloch function* iff

$$\|f\|_{\mathrm{Bl}(D)} = \sup_{z\in D}\big(|\varrho(z)|\,|f(z)|+|\varrho(z)|\,|\operatorname{grad} f(z)|\big) < \infty .$$

We shall denote by $\mathrm{Bl}(D)$ the space of Bloch functions on D, and by $\mathrm{Bl}H(D)$ the space of holomorphic Bloch functions on D.

The proof of Theorem 1 is based on the following two facts:

PROPOSITION 1. *Let D be a strictly pseudoconvex domain with boundary of class C^4. Then B maps continuously $L^\infty(D)$ onto* $\mathrm{Bl}H(D)$.

PROPOSITION 2. *If D is a strictly pseudoconvex domain with boundary of class C^4 then* $\mathrm{Bl}H(D)$ *represents the dual space $(L^1 H(D))^*$ via the pairing*

$$\langle u, v\rangle_1 = \langle u, L^1 v\rangle_0 = \int_D u\overline{L^1 v}, \quad u\in L^1 H(D),\ v\in \mathrm{Bl}H(D).$$

We shall also prove the following

PROPOSITION 3. *If D is as above, then the Bergman projection B maps continuously* $\mathrm{Bl}(D)$ *onto* $\mathrm{Bl}H(D)$.

The statement of Proposition 2 needs some explanation. In [2] S. Bell constructed, for a C^∞-smooth domain D, a family of operators L^s: $C^\infty(\bar D)\to C^\infty(\bar D)$ such that $L^s f$ vanishes on ∂D up to order $s-1$ and $f-L^s f$ is orthogonal to $L^2\operatorname{Harm}(D)$. In this paper we need only the operator L^1 which is defined as follows:

$$L^1 f = f-\Delta\left(\frac{f\varphi\varrho^2}{2|\nabla\varrho|^2}\right),$$

where φ is an arbitrarily chosen C^∞ function equal to 1 in a neighborhood of ∂D and equal to 0 in a neighborhood of the set $\{\nabla\varrho = 0\}$.

It is easy to see that the construction of $L^1 f$ can be done if the boundary of D is of class C^2.

The operators L^s were used to study various duality relations between spaces of holomorphic and harmonic functions (see [1], [4], [10], [18], [19], [20] and for the earlier versions of Bell's duality theorem, [3]).

It follows from the results of [18] that L^1 extends to a continuous mapping from $L^2\operatorname{Harm}(D)$ into $L^2(D)$ and that for all $u, v\in L^2\operatorname{Harm}(D)$, $\langle u, v\rangle_1 = \langle u, v\rangle_0$ where $\langle\ ,\ \rangle_0$ is the $L^2(D)$ scalar product.

The very definition of L^1 implies that if $u\in \mathrm{Bl}\operatorname{Harm}(D)$ then $L^1 u\in L^\infty(D)$ and $\|L^1 u\|_\infty \lesssim \|u\|_{\mathrm{Bl}}$.

As the image of $L^\infty(D)$ in the projection B is equal to $\mathrm{Bl}H(D)$, the question arises: what is the image of the space $\operatorname{Harm}^\infty(D)$ of bounded harmonic functions? The answer is the following:

THEOREM 2. *Let D be a strictly pseudoconvex domain with boundary of class C^4. Then the Bergman projection B maps continuously* $\operatorname{Harm}^\infty(D)$ *onto the space $(H^1(\partial D))^*$, the dual to the Hardy space $H^1(\partial D)$.*

Note that $(H^1(\partial D))^*$ consists of holomorphic functions which satisfy a BMO-condition on ∂D if D is the unit ball (see Krantz [12] and also [1] for various estimates concerning Sobolev spaces of holomorphic functions and BMO). The proof of Theorem 2 is based on the fact that the difference between the Bergman and Szegö projections restricted to the space of harmonic functions is a smoothing singular integral operator.

This fact can be easily checked in the case when D is the unit ball in C^n. If ∂D is of class C^∞ then this fact is an easy consequence of the deep results of Boutet de Monvel and Sjöstrand [4a]. In our case we shall use the Kerzman–Stein [9] results for the Szegö operator and analogous results for the Bergman projection proved in [17]. As a by-product we get the following

THEOREM 3. *Let D be a strictly pseudoconvex domain with C^4 boundary. Then B maps continuously the Hardy space* $\operatorname{Harm}^p(\partial D)$ *of harmonic functions onto the Hardy space $H^p(\partial D)$ of holomorphic functions for $2\leqslant p<\infty$.*

Recall that the Hardy space $\operatorname{Harm}^p(\partial D)$ is the space of harmonic functions on D whose boundary values exist and belong to $L^p(\partial D)$. In other words, $\operatorname{Harm}^p(\partial D)$, $p>1$, consists of harmonic extensions by the Poisson formula of functions from $L^p(\partial D)$. The norm on $\operatorname{Harm}^p(\partial D)$ is the $L^p(\partial D)$ norm. The Hardy space $H^p(\partial D)$ is the subspace of $\operatorname{Harm}^p(\partial D)$ consisting of holomorphic functions. The *Szegö projection* is the orthogonal projection from $L^2(\partial D)$ onto $H^2(\partial D)$. $H^1(\partial D)$ is the space of holomorphic extensions to D of functions φ from $L^1(\partial D)$ such that $\int_D \varphi\omega = 0$ if $\omega\in C^\infty_{(n,n-1)}(\bar D)$ and $\bar\partial\omega = 0$.

We shall apply the above-given results to the study of the behavior of orthogonal projections onto other spaces of pluriharmonic functions (see [19]). A function f is called *pluriharmonic* if $\bar\partial\partial f = 0$. We shall consider the orthogonal projection Q from $L^2(D)$ onto the space $L^2 PH(D)$ of square-integrable pluriharmonic functions, the orthogonal projection S_r from the space of real functions $L^2_r(D)$ onto the space $\operatorname{Re} L^2 H(D)$ of the real parts of functions from $L^2 H(D)$, and its complexification S, which is the orthogonal projection from $L^2(D)$ onto $\operatorname{Re} L^2 H(D)\otimes C$.

Let $\mathrm{Bl}\,PH(D)$ denote the space of Bloch pluriharmonic functions on D. $L^1 PH(D)$ denotes the space of pluriharmonic functions from $L^1(D)$ and

$PH^p(\partial D)$ the Hardy space of pluriharmonic functions. We shall prove the following

THEOREM 4. *Let D be a strictly pseudoconvex domain with boundary of class C^4. Then*

(a) *The projection Q maps continuously*

1) $\mathrm{Bl}(D)$ *onto* $\mathrm{Bl}PH(D)$,

2) $L^\infty(D)$ *onto* $\mathrm{Bl}PH(D)$,

3) $L^1\,\mathrm{Harm}(D)$ *onto* $L^1PH(D)$,

4) $\mathrm{Harm}^\infty(D)$ *onto the dual space* $(PH^1(D))^*$,

5) $\mathrm{Harm}^p(\partial D)$ *onto* $PH^p(\partial D)$ *for* $2 \leqslant p < \infty$.

(b) *The space* $\mathrm{Bl}PH(D)$ *represents the dual space to* $L^1\,PH(D)$ *via the pairing* $\langle\ ,\ \rangle_1$.

Now, let $\mathrm{Bl}\,\mathrm{Harm}_r(D)$ denote the space of real harmonic Bloch functions, and let $\mathrm{Re}\,\mathrm{Bl}H(D)$, $\mathrm{Re}L^1\,H(D)$ and $\mathrm{Re}H^p(\partial D)$ denote the real Banach spaces of the real parts of functions from the corresponding classes. We shall also denote by $L_r^\infty(D)$ the real Banach space of real functions from $L^\infty(D)$ and by $\mathrm{Harm}_r^\infty(D)$ the space of real harmonic bounded functions. We shall prove the following

THEOREM 5. *Let D be a strictly pseudoconvex domain with boundary of class C^4. Then*

(a) *The projection S_r maps continuously*

1) $\mathrm{Bl}_r(D)$ *onto* $\mathrm{Re}\,\mathrm{Bl}\,H(D)$,

2) $L_r^\infty(D)$ *onto* $\mathrm{Re}\,\mathrm{Bl}H(D)$,

3) $L^1\mathrm{Harm}_r(D)$ *onto* $\mathrm{Re}\,L^1\,H(D)$,

4) $\mathrm{Harm}_r^\infty(D)$ *onto* $(\mathrm{Re}H^1(\partial D))^*$, *the dual of* $\mathrm{Re}H^1(\partial D)$,

5) $\mathrm{Harm}_r^p(\partial D)$ *onto* $\mathrm{Re}H^p(\partial D)$ *for* $2 \leqslant p < \infty$.

(b) *The projection S maps continuously*

1) $\mathrm{Bl}(D)$ *onto* $\mathrm{Re}\,\mathrm{Bl}H(D)\otimes C$,

2) $L^\infty(D)$ *onto* $\mathrm{Re}\,\mathrm{Bl}H(D)\otimes C$,

3) $L^1\mathrm{Harm}(D)$ *onto* $\mathrm{Re}L^1\,H(D)\otimes C$,

4) $\mathrm{Harm}^\infty(D)$ *onto* $(\mathrm{Re}H^1(\partial D)\otimes C)^*$,

5) $\mathrm{Harm}^p(\partial D)$ *onto* $\mathrm{Re}H^p(\partial D)\otimes C$, $2 \leqslant p < \infty$.

(c) *The space* $\mathrm{Re}\,\mathrm{Bl}H(D)$ *represents the dual of* $\mathrm{Re}L^1\,H(D)$ *via the pairing* $\langle\ ,\ \rangle_1$. *The space* $\mathrm{Re}\,\mathrm{Bl}H(D)\otimes C$ *represents the dual of* $\mathrm{Re}L^1\,H(D)\otimes C$ *via the pairing* $\langle\ ,\ \rangle_1$.

We have also: $S_r(\mathrm{Harm}_r^\infty(D)) = (\mathrm{Re}H^1(\partial D))^* = \mathrm{Re}(H^1(\partial D))^*$.

2. Proofs.

Proof of Proposition 1. In [17] an explicit projection G: $L^2(D) \to L^2H(D)$ was constructed. It was proved that $B = G(I-(G^*-G))^{-1}$. In the course of proof of Theorem 1 in [17] it was proved that G^*-G maps continuously $L^\infty(D)$ into the Hölder space $\Lambda_{1/2}(D)$ and that $\mathrm{Ker}[I-(G^*-G)] = \{0\}$. Thus $I-(G^*-G)$ is a Fredholm isomorphism of $L^\infty(D)$ and therefore it suffices to prove that the operator G maps $L^\infty(D)$ into $\mathrm{Bl}H(D)$.

We shall proceed as in the proof of Theorem 1 from [17]. Let us recall briefly the definition of G. Let ϱ denote a defining function of D which is strictly plurisubharmonic in a neighborhood of $\bar{D}$ and of class C^4 on C^n. Denote by $L\varrho(z)$ the Levi form of the function ϱ. There exist ε_0 and δ_0 such that $L\varrho(z, z-w) \geqslant c|z-w|^2$ if $\varrho(z) < \delta_0$ and $|z-w| < \varepsilon_0$. Let

$$F_1(w, z) = \sum_i \frac{\partial\varrho}{\partial z_i}(z)(z_i-w_i) + \tfrac{1}{2}\sum_{i,j}\frac{\partial^2\varrho}{\partial z_i\,\partial z_j}(w_i-z_i)(z_j-w_j).$$

We have

$$\mathrm{Re}\,F_1(w, z) - \varrho(z) \geqslant -\frac{\varrho(z)+\varrho(w)}{2} + \frac{c}{2}|z-w|^2 \text{ if } \varrho(z) < \delta_0 \text{ and } |z-w| < \varepsilon_0.$$

Let $\psi(t)$ be a cut-off function such that $\psi(t) = 1$ if $t < \varepsilon_0/4$ and $\psi(t) = 0$ if $t > \varepsilon_0/2$.

We put $t = |z-w|$ and define

$$F(w-z) = \psi(t)\,F_1(w, z) + (1-\psi(t))|w-z|^2,$$

$$g_i(w, z) = \psi(t)\left(\frac{\partial\varrho}{\partial z_i}(z) + \tfrac{1}{2}\sum_j \frac{\partial^2\varrho}{\partial z_i\,\partial z_j}(w_j-z_j)\right) + (1-\psi(t))(\bar{z}_i-\bar{w}_i).$$

We have

$$\mathrm{Re}F(w, z) - \varrho(z) \geqslant c\left(-\varrho(z)-\varrho(w)+|z-w|^2\right)$$

for $w \in D_{\delta_0} = \{s \in C^n\colon \varrho(s) < \delta_0\}$ and $z \in \bar{D}$. The functions $F(w, z)$ and $g_i(w, z)$ are of class C^∞ in w and C^2 in z. Let

$$N(w, z) = c\sum_i (-1)^{i-1}\frac{g_i(w, z)}{(F(w, z)-\varrho(z))^n}\,\bar\partial_z g_1 \wedge \ldots \wedge \widehat{\bar\partial_z g_i} \wedge \ldots \wedge \bar\partial_z g_n \wedge dz.$$

$N(w, z)$ is a Cauchy–Fantappié form for $z \in \partial D$ and thus for every holomorphic function $f \in C^1(\bar{D})$

$$f(w) = \int_{\partial D} N(w, z) f(z)\,d\sigma_z = \int_D \bar\partial_z N(w, z) f(z)\,dV_z.$$

$N(w, z)$ and $\bar\partial_z N(w, z)$ are of class C^∞ in w and C^1 in z. Now we define

$$G(w, z) = \bar\partial_z N(w, z) - P_w(\bar\partial_w \bar\partial_z N(w, z)) = \bar\partial_z N(w, z) + Q(w, z)$$

where P_w is a Hörmander operator solving the $\bar\partial$-problem on D_δ, for $\delta < \delta_0$ such that $\bar\partial_w\bar\partial_z N(w, z)$ is of class $C^\infty \times C^1$ on $\bar{D}_\delta \times \bar{D}$. Thus $G(w, z)$ is holomorphic in w and since $Q(w,z)$ is of class $C^\infty\times C^1$ on $\bar D\times\bar D$ we have

$$G(w,z)=nc\frac{\det\begin{bmatrix}-\varrho(z) & \dfrac{\partial\varrho}{\partial z_i}(z)\\ -\dfrac{\partial\varrho}{\partial\bar z_j}(z) & \dfrac{\partial^2\varrho}{\partial z_i\,\partial\bar z_j}(z)\end{bmatrix}+O(|z-w|)}{(F(w,z)-\varrho(z))^{n+1}}$$
$$+\text{nonsingular terms.}$$

The above construction was given by Kerzman and Stein in [9] and adapted to the case of area integrals over D in [17].

In order to prove that G maps $L^\infty(D)$ into $\mathrm{Bl}H(D)$ we shall proceed as in [17] and show that

$$|\operatorname{grad} Gf(w)|\leqslant\frac{C\|f\|_\infty}{|\varrho(w)|}.$$

We have

$$\operatorname{grad} Gf(w)=\int_D f(z)\operatorname{grad}_w G(w,z)\,dV_z.$$

If w is near z then the kernel on the right-hand side is dominated by

$$\frac{c_1}{\left(\dfrac{-\varrho(z)-\varrho(w)}{2}+c|z-w|^2\right)^{n+2}}.$$

Now, we can simply repeat the estimates given by S. Krantz in [11]. Proceeding in exactly the same manner as S. Krantz we can find a suitable σ, $0<\sigma<\varepsilon_0/6$, and a $C^\infty(C^n\times C^n)$ cut-off function $h(w,z)$, $h(w,z)=1$ if $-\varrho(z)-\varrho(w)+|w-z|<\sigma/2$, $h(w,z)=0$ if $-\varrho(z)-\varrho(z)+|w-z|>\sigma$.

Let G_1 be the operator with kernel $(1-h(w,z))G(w,z)$ and G_2 the operator with kernel $h(w,z)G(w,z)$.

The operator G_1 is nonsingular and therefore C^1-smoothing. We have

$$|\operatorname{grad}_w G_2f(w)|\lesssim\|f\|_\infty\int_D\frac{1}{[(-\varrho(w)-\varrho(z))/2+c|z-w|^2]^{n+2}}\,dV_z.$$

After the same change of coordinates as in Krantz's [11] paper the integral on the right can be estimated by $C+I_{\delta,\beta,\lambda}$ where

$$I_{\delta,\beta,\gamma}=\int_{-R}^{|\varrho(w)|}dt_1\int_0^{\sqrt{R^2-t_1^2}}dr\int_{-1}^{1}\frac{r^\gamma\,ds}{(t_1^2+r^2)^\beta([2|\varrho(w)|-t_1+t_1^2+r^2]^2+r^2s^2)^\delta}$$

with $\delta=(n+2)/2$, $\beta=0$, $\gamma=2n-2$. Krantz proved by elementary calculations that

$$I_{\delta,\beta,\gamma}\leqslant c|\varrho(w)|^m+c,\qquad m=\tfrac12\gamma-\beta-2\delta+2,$$

if $\gamma-2\beta>0$, $\delta\geqslant0$ and $m\neq0$. In our case we have $m=n-1-n-2+2=-1$ and this implies the needed estimate for $\operatorname{grad} Gf$. Thus Proposition 1 is proved, since the fact that B is onto $\mathrm{Bl}H(D)$ follows from the fact mentioned in the introduction that if $u\in\mathrm{Bl}H(D)$ then $L^1u\in L^\infty(D)$ and $u=B(L^1u)$.

Proof of Proposition 2. We shall need the following fact: the space $L^2H(D)$ is dense in $L^1H(D)$. Even more is true: the space $H(\bar D)$ of functions holomorphic in a neighborhood of $\bar D$ is dense in $L^1H(D)$. This was proved by Kerzman in [8] (see also [13]). However, we shall briefly outline the proof, because we shall need the same construction in the sequel.

For every $z\in\partial D$ there exists a neighborhood V_z in $\bar D$ such that for small ε, $V_z\Subset D+\varepsilon\eta_z$, where η_z is the outer normal to ∂D at z. Since ∂D is compact, we can find open sets $U_0, U_1,\ldots,U_k$ in C^n, points $z_i\in U_i\cap\partial D$ for $i=1,\ldots,k$ and $\varepsilon_0>0$ such that $\bar D\subset\bigcup_{i=0}^k U_i$, $U_0\Subset D$ and $U_i\cap D\Subset D+\varepsilon\eta_{z_i}$ for $0<\varepsilon<\varepsilon_0$, $i=1,\ldots,k$. Let $f\in L^1H(D)$. The functions $f(w-\varepsilon\eta_{z_i})$ tend to f in $L^1(U_i\cap D)$ as $\varepsilon\to0$, by the Lebesgue theorem. Each such function is defined and holomorphic on some open neighborhood of $\overline{U_i\cap D}$. Thus for each ε we can find functions $f_0^\varepsilon=f, f_1^\varepsilon,\ldots,f_k^\varepsilon$ with

$$f_i^\varepsilon\in L^1H(V_i),\qquad D\cap U_i\Subset V_i,\qquad\int_{D\cap U_i}|f_i^\varepsilon-f|<\varepsilon.$$

Let φ_i be a partition of unity corresponding to the covering $\{U_i\}_{i=0}^k$. We have

$$\Big\|f-\sum_i\varphi_if_i^\varepsilon\Big\|_{L^1(D)}\leqslant c\varepsilon,$$

where c is independent of ε and f. We can also find $\delta>0$ (depending on ε) such that

$$\Big\|\bar\partial\sum\varphi_if_i^\varepsilon\Big\|_{L^1_{\langle0,1\rangle}(D_\delta)}\leqslant2c\varepsilon,$$

where $D_\delta=\{z:\varrho(z)<\delta\}$. Since the Henkin operators T_δ solving the $\bar\partial$-problem are continuous from $L^1_{\langle0,1\rangle}(D_\delta)$ into $L^1(D_\delta)$ (see [8]) and the norms of T_δ are uniformly bounded if $0\leqslant\delta\leqslant\delta_0$ for δ_0 sufficiently small, we have

$$\Big\|T_\delta\big(\bar\partial\sum\varphi_if_i^\varepsilon\big)\Big\|_{L^1(D_\delta)}<c_1\varepsilon,$$

where c_1 is independent of ε. Thus the sequence of functions

$$h_\varepsilon=\sum_i\varphi_if_i^\varepsilon-T_\delta\big(\bar\partial\sum_i\varphi_if_i^\varepsilon\big)\to f\quad\text{in } L^1(D).$$

It follows from the above construction that the h_ε are in $H(\bar D)$.

Let now φ be a continuous functional on $L^1 H(D)$. It can be extended by the Hahn–Banach theorem to a continuous functional $\tilde{\varphi}$ on $L^1(D)$ and its extension is represented by a bounded function $m_{\tilde{\varphi}}$. Put $T(\varphi) = B(m_{\tilde{\varphi}})$. It is clear that T does not depend on the choice of the extension of φ and $m_{\tilde{\varphi}}$. Proposition 1 implies that T maps $(L^1 H(D))^*$ into $\mathrm{Bl}H(D)$. Since $L^2 H(D)$ is dense in $L^1 H(D)$ it follows that T is one-to-one, and onto $\mathrm{Bl}H(D)$. The inverse mapping T^{-1} is given by $T^{-1}(h)(u) = \langle u, h\rangle_1 = \langle u, L^1 h\rangle$. Since

$$|T^{-1}(h)(u)| \leqslant \|u\|_{L^1(D)} \|L^1 h\|_\infty \leqslant \|u\|_{L^1(D)} \|h\|_{\mathrm{Bl}(D)},$$

T^{-1} is continuous. By the open mapping theorem, T^{-1} is an isomorphism between $\mathrm{Bl}H(D)$ and $(L^1 H(D))^*$.

Proof of Theorem 1. To prove Theorem 1 we must first prove that if D is a bounded domain with boundary of class C^4 in $\boldsymbol{R}^m$ then $L^2\,\mathrm{Harm}(D)$ is dense in $L^1\,\mathrm{Harm}(D)$.

We proceed in the same way as in the proof of Proposition 2 and construct an open covering $U_0, \ldots, U_k$ of $\bar{D}$ such that for every $f \in L^1\,\mathrm{Harm}(D)$ and $\varepsilon > 0$ there exist f_i^ε harmonic on an open neighborhood of $\overline{U_i \cap D}$ such that $\int_{D\cap U_i} |f_i^\varepsilon - f| < \varepsilon$. Let again $\{\varphi_i\}$ be a partition of unity corresponding to the covering $\{U_i\}$. We have $\|f - \sum_i \varphi_i f_i^\varepsilon\|_{L^1(D)} \leqslant c\varepsilon$, where c is independent on ε and f. Denote by u_ε the function $\sum_i \varphi_i f_i^\varepsilon$.

Let $G(x, y)$ denote the Green function of the domain D,

$$G(x, y) = \frac{1}{c_m |x-y|^{m-2}} - G_1(x, y),$$

where G_1 is harmonic with respect to x and y and smooth on $\bar{D}\times\bar{D}\setminus\{(x, x)\colon x \in \partial D\}$. Let

$$w_\varepsilon(y) = \int_D \Delta u_\varepsilon(x)\, G(x, y)\, dV_x.$$

We have $\Delta w_\varepsilon = \Delta u_\varepsilon$ and

$$\Delta u_\varepsilon = \Delta(u_\varepsilon - f) = \sum_i \Delta\big(\varphi_i (f_i^\varepsilon - f)\big)$$
$$= \sum_i \left[\Delta\varphi_i (f_i^\varepsilon - f) + 2\sum_{j=1}^m \frac{\partial\varphi_i}{\partial x_j}\frac{\partial(f_i^\varepsilon - f)}{\partial x_j}\right].$$

Since $G(x, y) \equiv 0$ on $\partial D \times D$ we can integrate by parts and get the following expression for w_ε:

$$w_\varepsilon(y) = \int_D \sum_i \left[\Delta\varphi_i (f_i^\varepsilon - f) + 2\sum_{j=1}^m \frac{\partial\varphi_i}{\partial x_j}\frac{\partial(f_i^\varepsilon - f)}{\partial x_j}\right]\cdot\left[\frac{1}{c_m|x-y|^{m-2}} - G_1(x, y)\right] dV_x$$
$$= \left\{-\sum_i \int_D \Delta\varphi_i (f_i^\varepsilon - f)\frac{1}{c_m|x-y|^{m-2}} dV_x + (m-2)\sum_{i,j}\int_D \frac{\partial\varphi_i}{\partial x_j}(f_i^\varepsilon - f)\frac{x_j - y_j}{c_m|x-y|^m} dV_x\right\}$$
$$+\left\{+\sum_i\int_D \Delta\varphi_i (f_i^\varepsilon - f)\, G_1(x, y)\, dV_x - \sum_{i,j}\int_D \frac{\partial\varphi_i}{\partial x_j}(f_i^\varepsilon - f)\frac{\partial G_1(x, y)}{\partial x_j} dV_x\right\}$$
$$= v_\varepsilon(y) + h_\varepsilon(y).$$

The function $h_\varepsilon(y)$ is harmonic since $G_1(x, y)$ and $\partial G_1(x, y)/\partial x_j$ are harmonic with respect to y. Thus $\Delta u_\varepsilon = \Delta v_\varepsilon$. Since the singular integral operators with kernels $(x_j - y_j)/|x-y|^m$ and $1/|x-y|^{m-2}$ map continuously $L^1(D)$ into $L^1(D)$ we have $\|v_\varepsilon\|_{L^1} < c\varepsilon$, c independent of ε. We have also $v_\varepsilon \in C^\infty(\bar{D})$, $u_\varepsilon \in C^\infty(\bar{D})$. Thus

$$\lim_{\varepsilon\to 0}(u_\varepsilon - v_\varepsilon) = f \quad \text{in } L^1 H(D), \qquad u_\varepsilon - v_\varepsilon \in C^\infty(\bar{D}) \cap L^2\,\mathrm{Harm}(D).$$

Now, let $h \in L^2\,\mathrm{Harm}(D)$. Proposition 2 yields that

$$\|Bh\|_{L^1(D)} \leqslant c \sup_{\substack{g\in \mathrm{Bl}H(D)\\ \|g\|_{\mathrm{Bl}H(D)}\leqslant 1}} |\langle Bh, g\rangle_1| = c \sup_{\substack{g\in \mathrm{Bl}H(D)\\ \|g\|_{\mathrm{Bl}H(D)}\leqslant 1}} |\langle Bh, g\rangle|$$
$$= c \sup_{\substack{g\in \mathrm{Bl}H(D)\\ \|g\|_{\mathrm{Bl}H(D)}\leqslant 1}} |\langle h, g\rangle| = c \sup_{\substack{g\in \mathrm{Bl}H(D)\\ \|g\|_{\mathrm{Bl}H(D)}\leqslant 1}} |\langle h, L^1 g\rangle|$$
$$\leqslant c \sup_{\substack{g\in \mathrm{Bl}H(D)\\ \|g\|_{\mathrm{Bl}H(D)}\leqslant 1}} \|h\|_{L^1(D)} \|L^1 g\|_{L^\infty(D)} \leqslant c_1 \|h\|_{L^1(D)}.$$

We have used here the fact that $L^1 g - g$ is orthogonal to harmonic functions and that $\|L^1 g\|_{L^\infty} \lesssim \|g\|_{\mathrm{Bl}H(D)}$.

Thus B is bounded in the L^1 norm on harmonic functions and since $L^2\,\mathrm{Harm}(D)$ is dense in $L^1\,\mathrm{Harm}(D)$, it can be extended to a continuous projection from $L^1\,\mathrm{Harm}(D)$ onto $L^1 H(D)$.

Proof of Proposition 3. Since D is bounded, $\mathrm{Bl}(D) \subset \bigcap_p L^p(D)$ and for every p the L^p norm is weaker than the Bloch norm. Moreover, it is easy to prove that for every $g \in \mathrm{Bl}(D)$, $\varrho g \in \Lambda_\alpha(D)$ for every $0 < \alpha < 1$ and $\varrho g = 0$ on ∂D. Recall that ϱ denotes a defining function for D.

We shall now proceed as in the proof of Proposition 1. We have $B = G\big(I - (G^* - G)\big)^{-1}$. Krantz's estimates [11] applied in the same manner as in [17] yield that the operator $G^* - G$ maps continuously $L^p(D)$ into $\Lambda_{1/2-(2n+2)/(2p)}(D)$, $p > 2n+2$. Thus $G^* - G$ maps continuously $\mathrm{Bl}(D)$ into $\Lambda_\alpha(D)$ for every $0 < \alpha < \frac{1}{2}$.

Thus $I - (G^* - G)$ is a Fredholm operator on $\mathrm{Bl}(D)$. It now suffices to prove that G maps $\mathrm{Bl}(D)$ onto $\mathrm{Bl}\,H(D)$. The operator G is a singular integral operator with kernel

$$(*)\qquad G(w,z)=\frac{nc\det\begin{bmatrix}-\varrho(z) & \dfrac{\partial\varrho}{\partial z_j}(z)\\ -\dfrac{\partial\varrho}{\partial\bar z_j}(z) & \dfrac{\partial^2\varrho}{\partial z_i\,\partial\bar z_j}(z)\end{bmatrix}+O(|z-w|)}{(F(w,z)-\varrho(z))^{n-1}}$$

$$+\text{nonsingular terms},$$

$$Gf(w)=\int_D f(z)\,G(w,z)\,dV_z.$$

We must estimate as before

$$\operatorname{grad} Gf(w)=\int_D f(z)\operatorname{grad}_w G(w,z)\,dV_z\quad\text{ for } f\in \mathrm{Bl}(D).$$

Krantz's estimates yield that if suffices to estimate

$$\int_D f(z)\frac{\dfrac{\partial}{\partial w_j}(F(w,z)-\varrho(z))\,l(z)}{(F(w,z)-\varrho(z))^{n+2}}\,dV_z,$$

where

$$l(z)=\det\begin{bmatrix}-\varrho(z) & \dfrac{\partial\varrho}{\partial z_i}(z)\\ -\dfrac{\partial\varrho}{\partial\bar z_j}(z) & \dfrac{\partial^2\varrho}{\partial z_i\,\partial\bar z_j}(z)\end{bmatrix}.$$

We have $\dfrac{\partial}{\partial w_j}F(w,z)=\dfrac{\partial\varrho}{\partial z_j}(z)+O(|z-w|)$ and thus it remains to estimate

$$\int_D\frac{f(z)\dfrac{\partial\varrho}{\partial z_j}(z)\,l(z)}{(F(w,z)-\varrho(z))^{n+2}}\,dV_z.$$

Since $f\varrho$ vanishes on ∂D, we can integrate by parts and get

$$\int_D\frac{f(z)\dfrac{\partial\varrho}{\partial z_j}(z)\,l(z)}{(F(w,z)-\varrho(z))^{n+2}}\,dV_z=-\int_D\frac{\varrho\dfrac{\partial f}{\partial z_j}l}{(F(w,z)-\varrho(z))^{n+2}}\,dV_z$$

$$+(n+1)\int_D\frac{\varrho f l\dfrac{\partial}{\partial z_j}(F(w,z)-\varrho(z))}{(F(w,z)-\varrho(z))^{n+3}}\,dV_z$$

$$-\int_D\frac{\varrho f\dfrac{\partial l}{\partial z_j}}{(F(w,z)-\varrho(z))^{n+2}}\,dV_z.$$

The first term in the above expression is dominated by

$$\int_D\frac{\|f\|_{\mathrm{Bl}(D)}}{[(-\varrho(w)-\varrho(z))/2+c\,|w-z|^2]^{n+2}}\,dV_z.$$

Since we have $f\varrho\in\Lambda_\alpha$ for every $0<\alpha<1$, $f\varrho=0$ on ∂D, $\|f\varrho\|_{\Lambda_\alpha}\leqslant c_\alpha\|f\|_{\mathrm{Bl}}$ and $(\partial/\partial z_i)(F(w,z)-\varrho(z))=O(|z-w|)$, it follows that the second term is dominated by

$$\int_D\frac{\|f\|_{\mathrm{Bl}(D)}\,c_\alpha|\varrho|^\alpha|z-w|}{[(-\varrho(w)-\varrho(z))/2+c\,|z-w|^2]^{n+3}}\,dV_z$$

$$\leqslant\int_D\frac{\|f\|_{\mathrm{Bl}(D)}\,c_\alpha}{[(-\varrho(w)-\varrho(z))/2+c\,|z-w|^2]^{n+2+1/2-\alpha}}\,dV_z$$

If we take $\alpha\geqslant\frac{1}{2}$ then this is dominated by

$$\int_D\frac{c_\alpha\|f\|_{\mathrm{Bl}(D)}}{[(-\varrho(w)-\varrho(z))/2+|w-z|^2]^{n+2}}\,dV_z.$$

For the same reason the third term is dominated by the same expression. Thus, the last part of the proof of Proposition 1 implies that $|\operatorname{grad}_w Gf(w)|\leqslant c/|\varrho(w)|$ and thus Gf belongs to $\mathrm{Bl}H(D)$. The closed graph theorem implies that G is continuous from $\mathrm{Bl}(D)$ onto $\mathrm{Bl}H(D)$. This ends the proof of Proposition 3.

Proof of Theorems 2 and 3. We begin with the following observation: the space $(H^1(\partial D))^*$ is equal to the image of $L^\infty(\partial D)$ under the Szegö projection S. This follows from the fact that $H^2(\partial D)$ is dense in $H^1(\partial D)$. (Romanov [21] and Henkin [6] proved that $H(\bar D)$, the space of functions holomorphic in a neighborhood of $\bar D$, is dense in $H^p(\partial D)$ for every $1\leqslant p<\infty$.) Every functional φ from $(H^1(\partial D))^*$ extends to a continuous functional on $L^1(\partial D)$ and therefore can be represented by a bounded function m on ∂D. Thus the mapping $\varphi\to S(m)$ is a well-defined one-to-one correspondence between $(H^1(\partial D))^*$ and $S(L^\infty(\partial D))$. If we equip the space $S(L^\infty(\partial D))$ with the norm

$$\|f\|^*=\sup_{\substack{g\in H^2(D)\\ \|g\|_{H^1(D)}\leqslant 1}}|\langle g,f\rangle_{H^2}|$$

we get a Banach space isomorphic to $(H^1(\partial D))^*$. Thus we can say that S maps continuously $L^\infty(\partial D)$ onto $(H^1(\partial D))^*$.

The Poisson formula gives an isomorphism between $L^\infty(\partial D)$ and $\mathrm{Harm}^\infty(D)$, the space of bounded harmonic functions. Denote this isomorphism by P. We are now going to prove that $S-BP$ is a compact operator which maps $L^\infty(\partial D)$ continuously into $\Lambda_{1/4}(D)$.

Since for every bounded harmonic function u there exists $v\in L^\infty(\partial D)$ such that $u=Pv$ (by the Fatou theorem), it follows that $B(\mathrm{Harm}^\infty(D))$

$= S(L^\infty(\partial D)) = (H^1(\partial D))^*$. We have as before

$$B = G\big(I-(G^*-G)\big)^{-1} = G+G(G^*-G)\big(I-(G^*-G)\big)^{-1}.$$

By the Kerzman–Stein result [9]

$$S = H\big(I-(H^*-H)\big)^{-1} = H+H(H^*-H)\big(I-(H^*-H)\big)^{-1}.$$

Recall that

$$Hv(w) = \int_{\partial D} N(w, z)v(z)\,d\sigma_z + \int_{\partial D} R(w, z)v(z)\,d\sigma_z$$

where R is a nonsingular correction term added to make $H(w, z)$ holomorphic in w (compare the proof of Proposition 1). It follows from the estimates of [7] and [11] that H^*-H and G^*-G map continuously $L^\infty(\partial D)$ and $L^\infty(D)$ into $\Lambda_{1/2}(D)$ and both H and G map continuously $\Lambda_\alpha(D)$ into $\Lambda_{\alpha/2}(D)$ for all $0<\alpha<1$. (If the boundary of D is of class C^k, $k \geqslant 5$, then H and G map $\Lambda_\alpha(D)$ into $\Lambda_\alpha(D)$ for $0<\alpha<1$.) Let, as before,

$$l(z) = \det\begin{bmatrix} -\varrho(z) & \dfrac{\partial\varrho}{\partial z_i} \\ -\dfrac{\partial\varrho}{\partial\bar z_j} & \dfrac{\partial^2\varrho}{\partial\bar z_j\,\partial z_i}\end{bmatrix}.$$

We have

$$Hv(w) = c\int_{\partial D}\frac{\big(l(z)+O(|z-w|)\big)v(z)}{|\nabla\varrho(z)|\big(F(w, z)-\varrho(z)\big)^n}\,d\sigma_z + \text{nonsingular operator},$$

$$Gu(w) = nc\int_D\frac{\big(l(z)+O(z-w|)\big)u(z)}{\big(F(w, z)-\varrho(z)\big)^{n+1}}\,dV_z + \text{nonsingular operator}.$$

Below, the symbol $\nabla\varrho$ will denote $\operatorname{grad}\varrho$ extended from ∂D to a nonvanishing function on D.

In order to prove that $S-BP$ is a compact operator on $L^\infty(\partial D)$ it suffices to show that

$$H_1 v - G_1 u \text{ is a compact operator if } v = Pu.$$

H_1 denotes here the integral operator acting on ∂D with kernel $\dfrac{l(z)\,d\sigma_z}{|\nabla\varrho(z)|\big(F(w, z)-\varrho(z)\big)^n}$ and G_1 denotes the operator with kernel $\dfrac{nl(z)\,dV_z}{\big(F(w, z)-\varrho(z)\big)^{n+1}}$.

Let as before $v \in L^\infty(\partial D)$ and $u = Pv$ be the harmonic extension of v. Since

$$d\sigma = \sum_{i=1}^n (-1)^{i-1}\,d\bar z_1 \wedge \ldots \wedge \widehat{d\bar z_i} \wedge \ldots \wedge d\bar z_n \wedge dz\,\frac{\partial\varrho}{\partial z_i}\,\frac{1}{|\nabla\varrho|^2}$$

we have

$$H_1 v(w) = \int_{\partial D}\sum_i (-1)^{i-1}\frac{l(z)\dfrac{\partial\varrho}{\partial z_i}v(z)}{|\nabla\varrho(z)|^2\big(F(w, z)-\varrho(z)\big)^n}\,d\bar z_1 \wedge \ldots \wedge \widehat{d\bar z_i} \wedge \ldots \wedge d\bar z_n \wedge dz$$

$$= -n\int_D\sum_i\frac{l(z)\dfrac{\partial\varrho}{\partial z_i}\dfrac{\partial}{\partial\bar z_i}\big(F(w, z)-\varrho(z)\big)u(z)}{|\nabla\varrho(z)|^2\big(F(w, z)-\varrho(z)\big)^{n+1}}\,dV_z$$

$$+\int_D\sum_i\frac{l(z)\dfrac{\partial\varrho}{\partial z_i}\dfrac{\partial u}{\partial\bar z_i}}{|\nabla\varrho(z)|^2\big(F(w, z)-\varrho(z)\big)^n}\,dV_z$$

$$+\int_D\sum_i\frac{\partial}{\partial\bar z_i}\left(\frac{l(z)(\partial\varrho/\partial z_i)}{|\nabla\varrho(z)|^2}\right)\frac{u(z)}{\big(F(w, z)-\varrho(z)\big)^n}\,dV_z.$$

Since $(\partial/\partial\bar z_i)\big(F(w, z)-\varrho(z)\big) = -\partial\varrho/\partial\bar z_i + O(|w-z|)$, the first term on the right is equal to $G_1 u + K_1$ where K_1 maps $L^\infty(D)$ into $\Lambda_{1/2}(D)$. The last term on the right is also a compact operator which maps $L^\infty(D)$ into $\Lambda_\alpha(D)$ for every $0<\alpha<1$. Thus it suffices to estimate the second term. We have

$$\int_D\sum_i\frac{l(z)\dfrac{\partial\varrho}{\partial z_i}(z)\dfrac{\partial u}{\partial\bar z_i}(z)}{|\nabla\varrho(z)|^2\big(F(w, z)-\varrho(z)\big)^n}\,dV_z$$

$$= -\int_D\sum_i\frac{\partial}{\partial z_i}\left(\frac{l}{|\nabla\varrho|^2}\right)\frac{\varrho\dfrac{\partial u}{\partial\bar z_i}}{\big(F(w, z)-\varrho(z)\big)^n}\,dV_z + n\int_D\sum_i\frac{l\varrho\dfrac{\partial u}{\partial\bar z_i}\dfrac{\partial}{\partial z_i}\big(F(w, z)-\varrho(z)\big)}{|\nabla\varrho|^2\big(F(w, z)-\varrho(z)\big)^{n+1}}\,dV_z$$

$$= \int_D\tfrac{1}{2}\Delta\left(\frac{l}{|\nabla\varrho|^2}\right)\frac{\varrho u}{\big(F(w, z)-\varrho(z)\big)^n}\,dV_z + \int_D\sum_i\frac{\partial}{\partial z_i}\left(\frac{l}{|\nabla\varrho|^2}\right)\frac{\dfrac{\partial\varrho}{\partial\bar z_i}u}{\big(F(w, z)-\varrho(z)\big)^n}\,dV_z$$

$$-n\int_D\sum_i\frac{\partial}{\partial z_i}\left(\frac{l}{|\nabla\varrho|^2}\right)\frac{\varrho u\dfrac{\partial}{\partial\bar z_i}\big(F(w, z)-\varrho(z)\big)}{\big(F(w, z)-\varrho(z)\big)^{n+1}}\,dV_z$$

$$-n\int_D\sum_i\frac{\partial}{\partial\bar z_i}\left[\frac{l(\partial/\partial z_i)\big(F(w, z)-\varrho(z)\big)}{|\nabla\varrho|^2}\right]\frac{u\varrho}{\big(F(w, z)-\varrho(z)\big)^{n+1}}\,dV_z$$

$$-n\int_D\sum_i\frac{lu\frac{\partial\varrho}{\partial z_i}\frac{\partial}{\partial\bar z_i}(F(w,z)-\varrho(z))}{(F(w,z)-\varrho(z))^{n+1}}dV_z$$
$$+n(n+1)\int_D\sum_i\frac{lu\varrho\frac{\partial}{\partial z_i}(F(w,z)-\varrho(z))\frac{\partial}{\partial\bar z_i}(F(w,z)-\varrho(z))}{(F(w,z)-\varrho(z))^{n+2}}dV_z.$$

We have twice integrated by parts and used the fact that $\sum_i\partial^2u/\partial z_i\partial\bar z_i=0$ since u is harmonic. Since $(\partial/\partial z_i)(F(w,z)-\varrho(z))=O(|z-w|)$ and

$$\left|\frac{\varrho(z)}{(F(w,z)-\varrho(z))^m}\right|\lesssim\frac{1}{|(-\varrho(z)-\varrho(w))/2+c|z-w|^2|^{m-1}},$$

for all kernels $G_i'(w,z)$ which arise in the last sum we have

$$|\mathrm{grad}_w G_i'(w,z)|\lesssim\frac{1}{|(-\varrho(z)-\varrho(w))/2+c|z-w|^2|^{n+1+1/2}}.$$

Thus the integral operators corresponding to these kernels map continuously $L^\infty(D)$ into $\Lambda_{1/2}(D)$. This ends the proof of Theorem 2.

To prove Theorem 3 we must observe that the above-given description of $S-BP$ implies that $S-BP$ maps continuously $L^\infty(\partial D)$ into $\Lambda_{1/4-(n+1)/(2p)}$ for $p>2n+2$ (see the proof of Proposition 3 and observe that the $L^p(\partial D)$ norm is stronger that the $L^p(D)$ norm of the harmonic extension of a function from $L^p(\partial D)$). It was proved by Kerzman and Stein [9] that the Szegö projection maps continuously $L^p(\partial D)$ onto $H^p(\partial D)$ (they stated this fact for C^∞-smooth domains but their proof remains valid in our case). This implies that B maps continuously $\mathrm{Harm}^p(\partial D)$ onto $H^p(\partial D)$ for $p>2n+2$. On the other hand, the $L^2(\partial D)$ norm on $\mathrm{Harm}^2(\partial D)$ is equivalent to the Sobolev norm $\|\ \|_{1/2}$. From the estimates due to Greene and Krantz [5] it follows that if D is a bounded strictly pseudoconvex domain with boundary of class C^{s+3} then B maps continuously the Sobolev space W^s into itself (this follows as usual from the estimates for the $\bar\partial$-Neumann problem, see [5], § 3, Propositions 3.2.23 and 3.2.25). Thus in our case B maps continuously $L^2(D)$ into itself and $W^1(D)$ into itself. Interpolation shows that it must also map $W^{1/2}(D)$ into itself and therefore $\mathrm{Harm}^2(\partial D)$ onto $H^2(\partial D)$. Hence, again by interpolation B maps $\mathrm{Harm}^p(\partial D)$ onto $H^p(\partial D)$ for every $2\leqslant p<\infty$.

Proof of Theorems 4 and 5. It follows from Theorem 0, Theorem 2 and Remark 3 of [19] that $L^2PH(D)=\mathrm{Re}L^2H(D)\otimes\boldsymbol{C}\oplus E$ where $E\perp\mathrm{Re}L^2H(D)\otimes\boldsymbol{C}$, $\dim_{\boldsymbol C}E\leqslant\dim_{\boldsymbol R}H^1(D,\boldsymbol R)<\infty$ ($H^1(D,\boldsymbol R)$ is the first de Rham cohomology group of D) and E consists of functions belonging to

$\Lambda_{1/2}(D)$. Thus $Q=S+R$ where R is the finite-dimensional orthogonal projection from $L^2(D)$ onto E. This implies that Theorem 4 is a direct consequence of Theorem 5. To prove Theorem 5 we shall proceed in exactly the same way as in the proofs of Theorem 2 and Remark 3 in [19]. For every real $u\in L^2(D)$ we have

$$S_r(u)=2\mathrm{Re}Bu-\mathrm{Re}B\left(\overline{S_r^0(u)}\,L^1(1)\right)$$

where $S_r^0(u)$ is the function from $L^2H(D)$ such that $S_ru=\mathrm{Re}S_r^0(u)$ and $\int_D\mathrm{Im}S_r^0(u)=0$.

In order to prove that S_r maps $L_r^\infty(D)$ onto $\mathrm{Re\,Bl}H(D)$ and $\mathrm{Bl}_r(D)$ onto $\mathrm{Re\,Bl}H(D)$ we shall repeat the procedure from [19], Remark 3. There it was proved that $L^1(1)u$ maps $L^p\mathrm{Harm}(D)$ into $\mathring{W}_p^1(D)$. By the Sobolev imbedding theorem $\mathring{W}_p^1(D)$ is continuously imbedded in $L^q(D)$ for $q=2np/(2n-p)$ if $p<2n$ and in $\Lambda_\alpha(D)$ for $\alpha=1-2n/p$ if $p>2n$. Moreover, the Bergman projection B maps continuously $L^p(D)$ into $L^p(D)$, and $\Lambda_\alpha(D)$ into $\Lambda_{\alpha/2}(D)$ for $0<\alpha<1$ (see [17] and [19], Remark 3).

Now, let $u\in L_r^\infty(D)$ or $u\in\mathrm{Bl}_r(D)$. Then $Bu\in\mathrm{Bl}H(D)$ and

$$\overline{S_r^0(u)}\,L^1(1)\in\mathring{W}_2^1(D)\subset L^q(D),\qquad q=\frac{2n}{n-1}.$$

Thus $S_r(u)\in L^q(D)$ since B maps $L^q(D)$ into $L^q(D)$. Hence

$$\overline{S_r^0u}\,L^1(1)\in\mathring{W}_q^1(D)\subset L^{q_1}(D),\qquad q_1=\frac{2n}{n-2}.$$

and as before $S_ru\in L^{q_1}(D)$ and

$$\overline{S_r^0u}\,L^1(1)\in\mathring{W}_{q_1}^1(D)\subset L^{q_2}(D),\qquad q_2=\frac{2n}{n-3}.$$

After $n-1$ such steps we get $\overline{S_r^0u}\,L^1(1)\in\mathring{W}_{2n}^1(D)\subset L^q(D)$ for every $1<q<\infty$. Thus $S_ru\in L^q(D)$ for every q and hence $\overline{S_r^0u}\,L^1(1)$ belongs to $\Lambda_\alpha(D)$ for each $0<\alpha<1$. This implies that $B(\overline{S_r^0u}\,L^1(1))$ maps continuously $L_r^\infty(D)$ (and $\mathrm{Bl}_r(D)$) into $\Lambda_\alpha H(D)$ for $0<\alpha<\frac{1}{2}$. Thus S_r maps continuously $L_r^\infty(D)$ and $\mathrm{Bl}_r(D)$ onto $\mathrm{Re\,Bl}H(D)$ since $\mathrm{Re}Bu$ maps, by Propositions 1 and 3, $L^\infty(D)$ onto $\mathrm{Re\,Bl}(D)$ and $\mathrm{Bl}(D)$ onto $\mathrm{Re\,Bl}H(D)$.

We have proved parts (a1) and (a2) of Theorem 5. Part (c) follows from (a2) in the same manner as Proposition 2 from Proposition 1, and (a3) follows from (c) and (a2) in the same way as Theorem 1 from Propositions 1 and 2.

To prove (a4) we shall proceed in the following way. Let S denote as before the Szegö projection and let R denote the orthogonal projection from

$L^2(\partial D)$ onto $\operatorname{Re} H^2(\partial D)$. If u is real, then

$$2\operatorname{Re} Su = Ru + \operatorname{Re} S(\overline{R^0 u}),$$

where $R^0 u$ is the holomorphic function from $H^2(\partial D)$ such that $Ru = \operatorname{Re} R^0 u$ and $\int\limits_D \operatorname{Im} R^0 u = 0$.

We are now going to prove that $R(L^\infty(\partial D)) = S_r(\mathrm{Harm}_r^\infty(D))$. We have

$$Ru = 2\operatorname{Re} Su - \operatorname{Re}\big(B(\overline{R^0 u}) - K(\overline{R^0 u})\big)$$

where $K = S - BP$ is the compact operator described in detail in the proof of Theorem 2 (P denotes here, as before, the harmonic extension of functions from ∂D to D).

Since $S_r u = 2\operatorname{Re} Bu - \operatorname{Re} B(\overline{S_r^0(u)}\, L^1(1))$, for every $u \in L^\infty(\partial D)$ we have

$$(*)\quad S_r(Pu) - Ru$$
$$= 2\operatorname{Re}(BPu - Su) - \operatorname{Re} B(\overline{S_r^0 Pu}\, L^1(1)) - \operatorname{Re}\big(B(\overline{R^0 u}) - K(\overline{R^0 u})\big).$$

In the proof of Theorem 3 it was observed that the operator $BPu - Su$ maps continuously $L^\infty(\partial D)$ into $\Lambda_{1/4-(n+1)/(2p)}(D)$, $p > 2n+2$. The operator $B(\bar{f}) = B(\bar{f} L^1(1))$ maps $L^p H(D)$ into $L^q H(D)$, $q = 2np/(2n-p)$, for $p < 2n$, and $L^p H(D)$ into $\Lambda_{1-2n/p} H(D)$ for $p > 2n$.

The operator $S(\bar{f})$ maps $H^p(\partial D)$ into $H^q(\partial D)$ for every $q < (2n+2)p/(2n+2-p)$, $p < 2n+2$, and $H^p(\partial D)$ into $\Lambda_{1/2-(n+1)/p}$ if $p > 2n+2$. This can be proved in the following way. The operator $H - H^*$ maps $L^p(\partial D)$ into $L^q(\partial D)$ for $p < 2n+2$ and $L^p(\partial D)$ into $\Lambda_{1/2-(n+1)/2}$ for $p > 2n+2$. This follows from the estimates in [7], [21], and [12]. It then suffices to prove that H maps $H^p(\partial D)$ into $L^q(\partial D)$ and $H^p(\partial D)$ into Λ_α for suitable p, q, α (see the proof of Theorems 2 and 3). We have

$$H\bar{f}(w) = \int\limits_{\partial D} \frac{(l(z) + O(|z-w|))\bar{f}(z)}{|\nabla\varrho(z)|^2 (F(w,z) - \varrho(z))^n} d\sigma_z$$
$$= \int\limits_{\partial D} \frac{(l(z) + O(|z-w|))\bar{f}(z)}{|\nabla\varrho(z)|^2 (F(w,z) - \varrho(z))^n} \sum_i \frac{\partial\varrho}{\partial\bar{z}_i} d\bar{z} \wedge dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge dz_n$$
$$= \sum_i \int\limits_D \frac{\partial}{\partial z_i}\left(\frac{\partial\varrho}{\partial\bar{z}_i} \frac{l(z) + O(|z-w|)}{|\nabla\varrho(z)|^2}\right) \frac{\bar{f}(z)}{(F(w,z) - \varrho(z))^n} dV_z$$
$$+ n\sum_i \int\limits_D \frac{\partial\varrho}{\partial\bar{z}_i} \frac{l(z) + O(|z-w|)}{|\nabla\varrho(z)|^2} \frac{\bar{f}(z)\frac{\partial}{\partial z_i}(F(w,z) - \varrho(z))}{(F(w,z) - \varrho(z))^{n+1}} dV_z$$

since $\partial\bar{f}(z)/\partial z_i = 0$. Note that by the definition of $F(w, z)$

$$\frac{\partial}{\partial z_i}(F(w,z) - \varrho(z)) = O(|w-z|)$$

and thus $H\bar{f}$ maps $L^{2n+2}(\partial D)$ into $H^q(\partial D)$ for every $0 < q < \infty$ and $L^p(\partial D)$ into $\Lambda_\alpha(D)$, $\alpha = \frac{1}{2} - (n+1)/p$. Thus $S\bar{f}$ has the desired property for $p > 2n+2$. We now have the duality relation

$$\langle h, S\bar{f}\rangle_{L^2(\partial D)} = \langle f, S\bar{h}\rangle_{L^2(\partial D)}.$$

By this relation $S\bar{f}$ maps $H^{1+\varepsilon}(\partial D)$ into $H^{(2n+2)/(2n+1)}(\partial D)$ for every $\varepsilon > 0$. The Riesz–Thorin theorem yields that $S\bar{f}$ maps $H^p(\partial D)$ into $H^q(\partial D)$ for every $q < (2n+2)p/(2n+2-p)$. We can now write $Ru = 2\operatorname{Re} Su - \operatorname{Re} S(\overline{R^0 u})$ and use the same procedure as for the operators S_r and B to prove that R maps $L^p(\partial D)$ into $L^p(\partial D)$ for every $2 \leqslant p < \infty$. The above estimates for K, $B\bar{f}$ and $S\bar{f}$ and $(*)$ imply that the operator $S_r u - Ru$ maps continuously $L^\infty(\partial D)$ into $\Lambda_\alpha(D)$, $\alpha = \frac{1}{4} - (n+1)/(2p)$, $p > 2n+2$. Thus $S_r u - Ru = \operatorname{Re} h$, $h \in \Lambda_\alpha(D)$. Since $\operatorname{Re} h$ belongs to $R(L^\infty(\partial D)) \cap S_r(\mathrm{Harm}_r^\infty(D))$ we have $S_r(\mathrm{Harm}_r^\infty(D)) = R(L^\infty(\partial D))$.

The space $\operatorname{Re} H^1(\partial D)$ is not a closed subspace of $L^1(\partial D)$. However, we can treat it as a normed subspace of $L^1(\partial D)$ and consider the adjoint space $(\operatorname{Re} H^1(\partial D))^*$. It is clear that $(\operatorname{Re} H^1(\partial D))^* = (\overline{\operatorname{Re} H^1(\partial D)})^*$, where the closure is taken in $L^1(\partial D)$. Since $H(\bar{D})$ is dense in $H^1(\partial D)$, $\operatorname{Re} H(\bar{D})$ is dense in $\operatorname{Re} H^1(\partial D)$ and thus $\operatorname{Re} H^2(\partial D)$ is dense in $\operatorname{Re} H^1(\partial D)$. Each element φ from $(\operatorname{Re} H^1(\partial D))^*$ can be extended to a continuous functional on $L^1(\partial D)$ and therefore represented by some function $m \in L^\infty(\partial D)$. Just as in the proof of Theorem 2, the function $R(m)$ represents φ as a functional on $(\operatorname{Re} H^1(\partial D))^*$ and we have a one-to-one correspondence between $(\operatorname{Re} H^1(\partial D))^*$ and $R(L^\infty(\partial D))$. Thus we have

$$S_r(\mathrm{Harm}_r^\infty(D)) = (\operatorname{Re} H^1(\partial D))^* = \operatorname{Re}(H^1(\partial D))^*.$$

The last equality follows from the fact that $(H^1(\partial D))^* = S(L^\infty(\partial D))$ and $Ru - 2\operatorname{Re} Su \in \operatorname{Re}\Lambda_\alpha H(D)$ if $u \in L^\infty(\partial D)$ and $\alpha = \frac{1}{4} - (n+1)/(2p)$, $p > 2n+2$.

Part (a5) of Theorem 5 can be proved in exactly the same way as above using the fact that

$$\|f\|_{H^p(\partial D)} \leqslant \|\operatorname{Re} f\|_{L^p(\partial D)}$$

if $\int\limits_D \operatorname{Im} f = 0$ and $1 < p < \infty$ (see Stout [23]).

Part (b) of Theorem 5 follows from part (a) via complexification.

Problem. I. Lieb and M. Range in [14], [15] and [16] used integral formulae to get Hölder and C^k estimates for the operator $\bar{\partial}^* N\alpha$, where α is a (p, q)-form and N is the operator solving the $\bar{\partial}$-Neumann problem $\Box\varphi = \alpha$. Does the operator $\bar{\partial}^* N\alpha$ behave better if α is a form with harmonic coefficients, or at least if α is such that $\bar{\partial}\alpha = 0$ and $\tau\alpha = 0$? (τ is the formal adjoint of $\bar{\partial}$.)

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On functions whose improper Riemann integral is absolutely convergent

by

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Abstract. Absolutely convergent improper Darboux integrable functions on the compact support of a nonnegative Radon measure are introduced.

Introduction. S. Rolewicz deduced in [10] that a consistent definition of the Lebesgue integral is not possible for functions $f\colon [0, 1] \to X$ where X is a non-locally convex linear metric space. Hence, D. Przeworska-Rolewicz and S. Rolewicz [8] and independently B. Gramsch [1] introduced the Riemann integral for that situation. S. Rolewicz and the author [4] defined the Riemann integral for functions $f\colon K \to X$ where K is the compact support of a nonnegative Radon measure μ and where X is a topological linear space.

In [4], [5] of S. Rolewicz and the author, the translation of the classical result – i.e. a bounded function $f\colon [0, 1] \to \boldsymbol{R}$ is Riemann integrable iff f is continuous almost everywhere – was proved for Darboux integrable functions. These functions are characterized by Darboux lower and upper sums resp. by distance sums in the general case. Darboux integrable functions are Riemann integrable but the converse is false in general.

In this paper we characterize the Darboux integrability by a kind of fractional continuity. This allows us to obtain a definition of absolutely convergent improper Darboux integrability with respect to K and μ: Indeed, an unbounded function g is absolutely convergent improper Darboux integrable iff the following holds: (i) g fulfils the fractional continuity property and (ii) g is absolutely $\bar\mu$-Bochner integrable (or equivalently: the improper μ-Riemann integral of g is absolutely convergent).

There are also recent studies on Riemann integrable functions: R. Henstock [2], J. Kurzweil [6] and E. J. McShane [7] deduced that modifications of the Riemann integral on [0, 1] yield the Lebesgue and even the Perron–Ward integral. C. S. Hönig [3] found examples of Hilbert space valued functions on [0, 1] which are Riemann integrable but not measurable with respect to the complete Lebesgue measure. G. C. da Rocha Filho [9] analysed Riemann integration depending on the geometry of Banach spaces.