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On basic Hahn–Banach extensions

by

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Abstract. A criterion is derived for the existence of a basic sequence of Hahn–Banach extensions of the coefficient functionals of a basic sequence of codimension one in a Banach space. Using this criterion and renorming results which are shown to characterize the usual basis for c_0, a negative answer is given to a question of Retherford concerning the existence of such extensions.

A problem of Retherford concerning the existence of norm-preserving extensions of coefficient functionals is the following (see [7, p. 66], or [6, p. 84]).

Given a basic sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space X having coefficient functionals $\{x_n^*\}_{n=1}^{\infty}$ in $[x_n]^*$, does there exist a sequence of Hahn–Banach extensions of the functionals $\{x_n^*\}_{n=1}^{\infty}$ which is a basic sequence in X^*?

i.e. does there exist a sequence $\{g_n\}_{n=1}^{\infty}$ in X^* for which $\|g_n\| = \|x_n^*\|$ for all n and for which $\{x_n, g_n\}_{n=1}^{\infty}$ is a bi-basic system (see [2] and [6, p. 85])?

In a recent paper [7] Terenzi has given a partial answer to this question by showing that there always exists *some* basic sequence $\{g_n\}_{n=1}^{\infty}$ in X^* which is biorthogonal to $\{x_n\}_{n=1}^{\infty}$, but it may not be that $\|g_n\| = \|x_n^*\|$ for all n. In fact, the proof does not even guarantee that $\sup_n \|x_n\| \, \|g_n\| < +\infty$.

The purpose of this paper is to give a negative answer to the question of Retherford as an outgrowth of a general study of the problem of existence of basic Hahn–Banach extensions of coefficient functionals in the simplest possible case, where $\operatorname{codim}[x_n]_{n=1}^{\infty} = 1$. This result is a consequence of a general existence criterion for such Hahn–Banach extensions (Theorem 1) and of related results which essentially show that the guaranteed existence of such extensions characterizes the unit vector basis $\{e_n\}_{n=1}^{\infty}$ for c_0 (Proposition 1 and Theorem 3). We begin with general discussion of Hahn–Banach extensions which will culminate in the first of these results.

Suppose $\{x_n\}_{n=1}^{\infty}$ is a basic sequence in X for which $M = [x_n]_{n=1}^{\infty}$ is of codimension one in X. Then $\{x_n\}_{n=1}^{\infty}$ is a basis for M having coefficient

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functionals $\{x_n^*\}_{n=1}^{\infty}$ in M^*, and if x_0 is any vector not in M the sequence $\{x_n\}_{n=0}^{\infty}$ is a basis for X whose sequence of coefficient functionals $\{f_n\}_{n=1}^{\infty}$ forms a basic sequence in X^*. Since $\langle f_n, x_m\rangle = \delta_{nm}$ for all n and m it must be that $M^{\perp} = [f_0]$ and $f_n|_M = x_n^*$ for $n \geqslant 1$, so if $\{g_n\}_{n=1}^{\infty} \subset X^*$ is any sequence of extensions of $\{x_n^*\}_{n=1}^{\infty}$ to X then $g_n|_M = f_n|_M = x_n^*$, from which it follows that $g_n - f_n \in M^{\perp} = [f_0]$, and hence that $g_n = f_n - \lambda_n f_0$ for some scalars $\{\lambda_n\}_{n=1}^{\infty}$. Conversely, any sequence in X^* of the form $\{f_n - \lambda_n f_0\}_{n=1}^{\infty}$ is clearly a sequence of extensions of $\{x_n^*\}_{n=1}^{\infty}$ to X. Therefore a sequence in X^* is a sequence of Hahn–Banach extensions of $\{x_n^*\}$ $\Leftrightarrow$ it is of the form $\{f_n - \lambda_n f_0\}_{n=1}^{\infty}$, where $\|f_n - \lambda_n f_0\| = \|x_n^*\| = \|f_n|_M\|$.

But it is well known that if $h \in X^*$ then

$$\|h|_M\| = \inf_{f \in M^{\perp}} \|h - f\| \qquad [4, \text{ p. } 121],$$

so

$$\|f_n - \lambda_n f_0\| = \|f_n|_M\| \Leftrightarrow \|f_n - \lambda_n f_0\| = \inf_{\lambda} \|f_n - \lambda f_0\| = \operatorname{dist}(f_n, [f_0]).$$

That is, *there exists a basic sequence of Hahn–Banach extensions of* $\{x_n^*\}_{n=1}^{\infty}$ $\Leftrightarrow$ *there is a sequence of scalars* $\{\lambda_n\}_{n=1}^{\infty}$ *for which* $\|f_n - \lambda_n f_0\| = \operatorname{dist}(f_n, [f_0])$ *and* $\{f_n - \lambda_n f_0\}_{n=1}^{\infty}$ *is a basic sequence in* X^*.

Now in a previous paper [3] we showed that a sequence of the form $\{f_n - \lambda_n f_0\}_{n=1}^{\infty}$ is a basic sequence in X^* $\Leftrightarrow$ it has codimension one in $[f_n]_{n=0}^{\infty}$. Hence $\{f_n - \lambda_n f_0\}_{n=1}^{\infty}$ is basic in X^* $\Leftrightarrow \exists G \neq 0$ in $([f_n]_{n=0}^{\infty})^*$ for which $\langle G, f_n\rangle = \lambda_n \langle G, f_0\rangle$ for all $n \geqslant 1$, and hence $\Leftrightarrow \exists F \in ([f_n]_{n=0}^{\infty})^*$ for which $\langle F, f_0\rangle = 1$ and $\langle F, f_n\rangle = \lambda_n$ for all $n \geqslant 1$. Consequently, if $\{f_n - \lambda_n f_0\}_{n=1}^{\infty}$ is basic in X^* then

$$\sup_N \Big\|\sum_{n=1}^{N} \lambda_n x_n\Big\| = \sup_N \Big\|\sum_{n=1}^{N} \langle F, f_n\rangle x_n\Big\|$$

(where $F \in ([f_n]_{n=0}^{\infty})^*$ is as above), and where this last is $\leqslant K\|F\| < +\infty$ for some K independent of F [5, p. 126]. Conversely, if $\sup_N \big\|\sum_{n=1}^{N} \lambda_n x_n\big\| < +\infty$ then setting $\lambda_0 = 1$ we have $\sup_N \big\|\sum_{n=0}^{N} \lambda_n x_n\big\| < +\infty$ and there is an $F \in ([f_n]_{n=0}^{\infty})^*$ for which $\langle F, f_n\rangle = \lambda_n$ for all $n \geqslant 0$ [5, p. 126], hence for which $\langle F, f_n\rangle = \lambda_n \langle F, f_0\rangle$ for all $n \geqslant 1$, so $\{f_n - \lambda_n f_0\}_{n=1}^{\infty}$ is of codimension one in $[f_n]_{n=0}^{\infty}$ and is therefore basic in X^* by the above. That is, we have proved:

THEOREM 1. *Let* $\{x_n\}_{n=1}^{\infty}$ *be a basic sequence in* X *for which* $\operatorname{codim}[x_n]_{n=1}^{\infty} = 1$ *and having coefficient functionals* $\{x_n^*\}_{n=1}^{\infty}$ *in* $[x_n]^*$. *There exists a sequence of Hahn–Banach extensions of* $\{x_n^*\}$ *which is a basic sequence in* X^* $\Leftrightarrow$ *for any* $x_0 \notin [x_n]_{n=1}^{\infty}$ *the coefficient functionals* $\{f_n\}_{n=0}^{\infty}$ *of the basis* $\{x_n\}_{n=0}^{\infty}$ *for* X *have the property that there is a sequence of scalars* $\{\lambda_n\}_{n=1}^{\infty}$ *for which*

$$\|f_n - \lambda_n f_0\| = \inf_{\lambda} \|f_n - \lambda f_0\| \quad \textit{for all } n \geqslant 1$$

and for which $\sup_N \big\|\sum_{n=1}^{N} \lambda_n x_n\big\| < +\infty$.

This result (Theorem 1) is central to all we do in this paper. Using it we could now easily give an example of a basic sequence whose coefficient functionals admit no basic Hahn–Banach extensions, thereby answering negatively the question of Retherford. However, we prefer to give a more comprehensive discussion of what is intrinsically involved in all such examples, eventually obtaining a theorem which has as a consequence the existence of infinitely many nonequivalent basic sequences with this property (Theorem 3). We begin with a pair of positive results concerning basic Hahn–Banach extensions.

THEOREM 2. *Let* $\{x_n\}_{n=1}^{\infty}$ *be a basic sequence in* X *for which* $\operatorname{codim}[x_n]_{n=1}^{\infty} = 1$ *and having coefficient functionals* $\{x_n^*\}_{n=1}^{\infty}$ *in* $[x_n]^*$. *Then there is an equivalent norm on* X *for which the sequence* $\{x_n^*\}_{n=1}^{\infty}$ *has a basic sequence of Hahn–Banach extensions in* X^*.

Proof. Let $\|\cdot\|$ denote both the original norm on X and the dual norm on X^*. If $x_0 \notin [x_n]_{n=1}^{\infty}$ then $\{x_n\}_{n=0}^{\infty}$ is a basis for $(X, \|\cdot\|)$ with coefficient functionals $\{f_n\}_{n=0}^{\infty}$ in $(X^*, \|\cdot\|)$. Since there is an equivalent norm on X under which $\{x_n\}_{n=0}^{\infty}$ is a normalized monotone basis [5, p. 250], *we may assume* $\{x_n\}_{n=0}^{\infty}$ *is a monotone basis for* $(X, \|\cdot\|)$ *with* $\|x_n\| = 1$ *for all* $n \geqslant 0$. Consequently we will have that $\{f_n\}_{n=0}^{\infty}$ is a monotone basic sequence in $(X, \|\cdot\|)^* = (X^*, \|\cdot\|)$ [5, p. 251], and the canonical embedding of X into $([f_n]_{n=0}^{\infty}, \|\cdot\|)^*$ is an isometry [5, p. 115]. It follows that if we define an equivalent norm on $[f_n]_{n=0}^{\infty}$, say $|||\cdot|||$, then the expression $|||x||| = \sup\{|\langle f, x\rangle|\ |||f||| \leqslant 1, f \in [f_n]_{n=0}^{\infty}\}$ defines a norm on X equivalent to $\|\cdot\|$. Moreover, if the basis $\{f_n\}_{n=0}^{\infty}$ for $([f_n]_{n=0}^{\infty}, |||\cdot|||)$ is still monotone then for any $f \in [f_n]_{n=0}^{\infty}$ we have $|||f||| = \sup\{|\langle f, x\rangle|\ |||x||| = 1, x \in X\}$ [5, p. 115], so $([f_n]_{n=0}^{\infty}, |||\cdot|||)$ will be embedded isometrically in $(X, |||\cdot|||)^*$. That is, such a renorming of $[f_n]_{n=0}^{\infty}$ induces an equivalent renorming of X with the property that the new dual norm on X^* agrees with the newly defined norm on $[f_n]_{n=0}^{\infty}$.

With this in mind we define on the space $[f_n]_{n=0}^{\infty}$ the norm

$$|||f||| = \Big|\Big|\Big|\sum_{n=0}^{\infty} c_n f_n\Big|\Big|\Big| = \|c_0 f_0\| + \Big\|\sum_{n=1}^{\infty} c_n f_n\Big\|.$$

Obviously $|||\cdot|||$ is equivalent to $\|\cdot\|$ on $[f_n]_{n=0}^{\infty}$, and in this new norm $\{f_n\}_{n=0}^{\infty}$ is still monotone. Therefore, by the above, if we define a new norm on X by $|||x||| = \sup\{|\langle f, x\rangle|\ |||f||| \leqslant 1, f \in [f_n]_{n=0}^{\infty}\}$ then $|||\cdot|||$ is equivalent to $\|\cdot\|$ and $([f_n]_{n=0}^{\infty}, |||\cdot|||) \subset (X, |||\cdot|||)^*$ (isometrically). But for any $n \geqslant 1$,

$$\inf_{\lambda} |||f_n - \lambda f_0||| = \inf_{\lambda} [\|-\lambda f_0\| + \|f_n\|] = \|f_n\|,$$

and this inf is attained only when $\lambda = \lambda_n = 0$. Hence by Theorem 1 it follows that the coefficient functionals $\{x_n^*\}_{n=1}^{\infty}$ of the basic sequence $\{x_n\}_{n=1}^{\infty}$ in $(X, |||\cdot|||)$ have Hahn–Banach extensions which are a basic sequence in $(X, |||\cdot|||)^*$, and the proof is complete.

Theorem 2 shows that one can (at least) equivalently renorm X to obtain a basic sequence of Hahn–Banach extensions for the coefficient functionals of a basic sequence of codimension one. Our next result shows that in the case of one particular type of basic sequence no renorming is necessary, even when $\operatorname{codim}[x_n] = +\infty$.

PROPOSITION 1. *Let $\{x_n\}_{n=1}^{\infty}$ be a basic sequence in X which is equivalent to the usual basis $\{e_n\}_{n=1}^{\infty}$ for c_0. Then there is a basic sequence in X^* of Hahn–Banach extensions of the coefficient functionals for $\{x_n\}_{n=1}^{\infty}$.*

Proof. If $\{x_n\}_{n=1}^{\infty}$ is equivalent to the basis $\{e_n\}_{n=1}^{\infty}$ for c_0 then, in particular, $0 < \delta = \inf_n \|x_n\| \leqslant \sup_n \|x_n\| \leqslant M < +\infty$ for some δ and M. Suppose $\{x_n^*\}_{n=1}^{\infty} \subset [x_n]^*$ is biorthogonal to $\{x_n\}_{n=1}^{\infty}$, and let $\{f_n\}_{n=1}^{\infty} \subset X^*$ be any sequence of Hahn–Banach extensions of $\{x_n^*\}_{n=1}^{\infty}$. Then $\{f_n\}_{n=1}^{\infty}$ is biorthogonal to $\{x_n\}_{n=1}^{\infty}$ and $\|f_n\| = \|x_n^*\|$ for all n, so $\sup_n \|f_n\| = \sup_n \|x_n^*\| < +\infty$, since $\inf_n \|x_n\| = \delta > 0$. Hence for any constants $\{c_n\}_{n=1}^{N}$ we have

$$(\sup_n \|f_n\|) \sum_{n=1}^{N} |c_n| \geqslant \Big\| \sum_{n=1}^{N} c_n f_n \Big\| \geqslant \sup_{\|\sum_{n=1}^{\infty} a_n x_n\| = 1} \Big| \Big\langle \sum_{n=1}^{N} c_n f_n, \sum_{n=1}^{\infty} a_n x_n \Big\rangle \Big|$$

$$= \sup_{\|\sum_{n=1}^{\infty} a_n x_n\| = 1} \Big| \sum_{n=1}^{N} a_n c_n \Big|.$$

But since $\{x_n\}_{n=1}^{\infty}$ is equivalent to $\{e_n\}_{n=1}^{\infty}$ in c_0 there is an $\varepsilon > 0$ (independent of $\{c_n\}$) for which this last is

$$\geqslant \varepsilon \sup_{|\varepsilon_n| = 1} \Big| \sum_{n=1}^{N} \varepsilon_n c_n \Big| = \varepsilon \sum_{n=1}^{N} |c_n|.$$

That is, the mapping $T\colon l^1 \to X^*$ defined by $T(e_n) = f_n$ is an isomorphism, implying that $\{f_n\}_{n=1}^{\infty}$ is a basic sequence in X^* (which is, in fact, equivalent to the basis $\{e_n\}_{n=1}^{\infty}$ for l^1) and is therefore the desired sequence of extensions.

Now it follows from Proposition 1 that if $\{x_n\}_{n=1}^{\infty}$ is a basic sequence in X which is equivalent to the basis $\{e_n\}_{n=1}^{\infty}$ for c_0, then no matter how X is equivalently renormed there will still always exist a basic sequence of Hahn–Banach extensions for the coefficient functionals of $\{x_n\}_{n=1}^{\infty}$. We now show that, at least for basic sequences of codimension one, this property characterizes the basis $\{e_n\}_{n=1}^{\infty}$ for c_0, thereby not only providing numerous

examples of basic sequences whose coefficient functionals do not admit basic Hahn–Banach extensions, but also completing the circle of ideas concerning the existence and stability of such basic sequences inherent in earlier parts of this paper.

THEOREM 3. *Let $\{x_n\}_{n=1}^{\infty}$ be a bounded basic sequence in a Banach space X for which* $\operatorname{codim}[x_n]_{n=1}^{\infty} = 1$, *and suppose $\{x_n\}_{n=1}^{\infty}$ is not equivalent to the basis $\{e_n\}_{n=1}^{\infty}$ for c_0. Then there is an equivalent norm on X for which no basic sequence of Hahn–Banach extensions of the coefficient functionals for $\{x_n\}_{n=1}^{\infty}$ exists.*

Proof. As in the proof of Theorem 2 let $\|\cdot\|$ denote the original norm on X and let $x_0 \notin [x_n]_{n=1}^{\infty}$, so that $\{x_n\}_{n=0}^{\infty}$ is a basis for $(X, \|\cdot\|)$ which is not equivalent to the basis $\{e_n\}_{n=1}^{\infty}$ for c_0 and which may be assumed to be normalized and monotone. If $\{f_n\}_{n=0}^{\infty} \subset X^*$ is biorthogonal to $\{x_n\}_{n=0}^{\infty}$ then, just as in the proof of Theorem 2, defining an equivalent norm $|||\cdot|||$ on $[f_n]_{n=0}^{\infty}$ in which $\{f_n\}_{n=0}^{\infty}$ is still monotone will result in the expression $|||x||| = \sup\{\langle f, x\rangle|\ |||f||| \leqslant 1, f \in [f_n]_{n=0}^{\infty}\}$ defining an equivalent norm on X whose dual norm agrees with $|||\cdot|||$ on $[f_n]_{n=0}^{\infty}$. Consequently, to prove the theorem *we need only* (according to Theorem 1) *define an equivalent norm $|||\cdot|||$ on $[f_n]_{n=0}^{\infty}$ in which $\{f_n\}_{n=0}^{\infty}$ is still monotone and so that whenever $|||f_n - \lambda_n f_0||| = \inf_\lambda |||f_n - \lambda f_0|||$ for all $n \geqslant 1$, then $\sup_N \|\sum_{n=1}^{N} \lambda_n x_n\| = +\infty$.*

To define such a norm we first note that the assumption that $\{x_n\}_{n=0}^{\infty}$ is not equivalent to $\{e_n\}_{n=1}^{\infty}$ in c_0 implies $\sum_{n=0}^{\infty} x_n$ is not weakly unconditionally Cauchy [1], and hence there is $h_0 \in X^*$ for which $\|h_0\| = 1$ and $\sum_{n=0}^{\infty} |\langle h_0, x_n\rangle| = +\infty$ [5, p. 434].

Now define a new norm $|||\cdot|||$ on $[f_n]_{n=0}^{\infty}$ by the expression

$$|||f||| = \Big|\Big|\Big| \sum_{n=0}^{\infty} c_n f_n \Big|\Big|\Big| = \Big[\|c_0 f_0\|^2 + \Big\| \sum_{n=1}^{\infty} c_n f_n \Big\|^2\Big]^{1/2} + \sup_{n \geqslant 1} |c_0\, \|f_0\| + \varepsilon_n c_n \|f_n\||,$$

where

$$\varepsilon_n = \begin{cases} 1 & \text{if } \langle h_0, x_n\rangle \geqslant 0, \\ -1 & \text{if } \langle h_0, x_n\rangle < 0. \end{cases}$$

It is trivial to check that $|||\cdot|||$ is, indeed, a norm on $[f_n]_{n=0}^{\infty}$ which is equivalent to $\|\cdot\|$, and it is obvious that since $\{f_n\}_{n=0}^{\infty}$ is monotone in the norm $\|\cdot\|$ it will also be monotone in $|||\cdot|||$. Moreover, for any $n \geqslant 1$ we have

$$\inf_\lambda |||f_n - \lambda f_0||| = \inf_\lambda \big([\|\lambda f_0\|^2 + \|f_n\|^2]^{1/2} + \sup\{\|\lambda f_0\|, |-\lambda \|f_0\| + \varepsilon_n \|f_n\||\}\big),$$

and this inf will be attained only when λ has the sign of ε_n. Since $\|x_n\| = 1$ for all n implies $\|f_n\| \geq 1$, it now follows easily from elementary calculus that $\inf_\lambda \|f_n - \lambda f_0\|$ is attained only when $\lambda = \lambda_n = \frac{\varepsilon_n \|f_n\|}{2\|f_0\|}$. However, we then have

$$\sup_N \Big\|\sum_{n=0}^{N} \lambda_n x_n\Big\| = \tfrac{1}{2}\sup_N \Big\|\sum_{n=0}^{N} \varepsilon_n \frac{\|f_n\|}{\|f_0\|} x_n\Big\| \geq \tfrac{1}{2}\sup_N \sum_{n=0}^{N} \varepsilon_n \frac{\|f_n\|}{\|f_0\|} \langle h_0, x_n\rangle$$

(since $\|h_0\| = 1$)

$$\geq \frac{1}{2\|f_0\|} \sup_N \sum_{n=0}^{N} |\langle h_0, x_n\rangle|$$

(by the definition of ε_n and the fact that $\|f_n\| \geq 1$). But $\sup_N \sum_{n=0}^{N} |\langle h_0, x_n\rangle| = +\infty$, so according to our previous remarks the proof is complete.

In particular, then, there is an equivalent norm $|||\cdot|||_p$ on the space l^p $(1 \leq p < +\infty)$ so that the coefficient functionals for the basic sequence $\{e_n\}_{n=2}^{\infty}$ in $(l^p, |||\cdot|||_p)$ admit no basic Hahn–Banach extensions. Therefore the basic sequences $\{e_n\}_{n=2}^{\infty}$ in the spaces $\{l^p, |||\cdot|||_p\}$ provide an infinite number of (nonequivalent) counterexamples to the question mentioned at the beginning of the paper.

Remarks. Although we did not check the details it appears to be only a technical exercise to extend Theorems 2 and 3 to the case where $\{x_n\}_{n=1}^{\infty}$ is of arbitrary *finite* codimension in X. A more serious problem related to these results is:

PROBLEM 1. Are Theorems 2 and 3 valid when $\operatorname{codim}[x_n]_{n=1}^{\infty} = +\infty$?

We should also note the following problem to which we referred earlier, and which would seem to be fundamental in regard to questions of the sort treated in this paper:

PROBLEM 2. If $\{x_n\}_{n=1}^{\infty}$ is a normalized basic sequence in X for which $\operatorname{codim}[x_n]_{n=1}^{\infty} = +\infty$, does there exist a *bounded* basic sequence $\{f_n\}_{n=1}^{\infty}$ in X^* which is biorthogonal to $\{x_n\}_{n=1}^{\infty}$?

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