$$m(\bar{\Omega}) \leqslant \sum_{k=k_0+1}^{N} \sum_{i\geqslant 1} (\tfrac{3}{2})^{k-k_0} |Q_i^k|$$
$$\leqslant \sum_{k\geqslant k_0} 2^{-(k-k_0)} (\tfrac{3}{2})^{k-k_0} 2^{-k_0} \|f\|_{WH^1} \leqslant \frac{C}{\alpha} \|f\|_{WH^1},$$

the theorem is proven.

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Received August 23, 1984
Revised version September 27, 1985 (1994)

STUDIA MATHEMATICA, T. LXXXV. (1987)

On regular generators of Z^2-actions in exhaustive partitions

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Abstract. It is shown that for every totally ergodic Z^2-action with finite entropy there exists a regular generator in a given exhaustive partition and the set of regular generators is dense in the set of all generators.

1. Introduction. Let $(X, \mathcal{B}, \mu)$ be a Lebesgue probability space, $\mathcal{M}$ the set of all measurable partitions of X and $\mathcal{Z}$ the subset of $\mathcal{M}$ consisting of partitions with finite entropy.

All relations between measurable partitions are to be taken mod 0.

Let ϱ be the metric on $\mathcal{Z}$ defined by the formula

$$\varrho(P, Q) = H(P|Q)+H(Q|P), \qquad P, Q \in \mathcal{Z}.$$

We denote by ε the measurable partition of X into single points and by ν the measurable trivial partition whose only element is X.

Let T be an automorphism of $(X, \mathcal{B}, \mu)$. For $P \in \mathcal{M}$ we define

$$P_T^- = \bigvee_{n=1}^{\infty} T^{-n}P, \qquad P_T = \bigvee_{n=-\infty}^{+\infty} T^n P.$$

If $P_T = \varepsilon$ we say that P is a *generator* of (X, T).

A partition $\zeta \in \mathcal{M}$ is said to be *T-perfect* if

$$T^{-1}\zeta \leqslant \zeta, \quad \zeta_T = \varepsilon, \quad \bigwedge_{n=0}^{\infty} T^{-n}\zeta = \pi(T) \quad \text{and} \quad h(\zeta, T) = h(T)$$

where $\pi(T)$ and $h(T)$ denote the Pinsker partition and the entropy of T respectively.

Rokhlin and Sinai showed in [9] that for every automorphism T there exists a T-perfect partition. If T is aperiodic with $h(T) < \infty$ then for every generator P of (X, T) the partition $\zeta = P \vee P_T^-$ is T-perfect. Rokhlin [7] proved that if $h(T) < \infty$ and ζ is T-perfect then there exists a generator P such that $\zeta = P \vee P_T^-$, i.e. ζ is a past of the process (P, T).

Now, let G be an abelian free group of rank 2 of automorphisms of $(X, \mathcal{B}, \mu)$. We denote by $b(G)$ the set of all ordered pairs of independent generators of G.

The quadruple $(X, \mathscr{B}, \mu, G)$ is said to be a *two-dimensional dynamical system* (*Z^2-action*) and is shortly denoted by (X, G).

The entropy theory for such systems has been developed by Conze [1], Katznelson and Weiss [5], and the theory of invariant partitions by the author [3].

Let Z^2 denote the two-dimensional integers and $\prec$ the lexicographical order in Z^2.

We put $\Pi = \{(i, j) \in Z^2;\ (i, j) \prec (0, 0)\}$. Let $(T, S) \in b(G)$. For $P \in \mathscr{M}$ we define

$$P_G^- = \bigvee_{(k,l)\in\Pi} T^k S^l P, \qquad P_G = \bigvee_{(k,l)\in Z^2} T^k S^l P.$$

A partition $P \in \mathscr{M}$ is said to be a *generator* for (X, G) if $P_G = \varepsilon$.

Now, let G be aperiodic and $h(G) < \infty$. Following Conze [1] we denote by Γ_G the set of all $P \in \mathscr{Z}$ with $h(P, G) = h(G)$ and by B_G the set of all generators of (X, G) with finite entropy. It is proved in [1] that $B_G \neq \emptyset$ and B_G is a dense subset of Γ_G.

In [3] the following two-dimensional analogue of the notion of perfect partition mentioned above is introduced. A partition $\zeta \in \mathscr{M}$ is said to be *(T, S)-exhaustive* if

(i) $T^k S^l \zeta \leqslant \zeta$ for $(k, l) \in \Pi$,

(ii) $\zeta_G = \varepsilon$,

(iii) $\bigwedge_{n=0}^{\infty} S^{-n}\zeta = T^{-1}\zeta_S$.

If ζ also satisfies

(iv) $\bigwedge_{(k,l)\in Z^2} T^k S^l \zeta = \pi(G)$,

(v) $h(G) = H(\zeta|\zeta_G^-) = H(\zeta|S^{-1}\zeta)$,

where $\pi(G)$ and $h(G)$ mean the Pinsker partition and the entropy of G respectively, then it is called *(T, S)-perfect*.

It is clear that conditions (i) and (iv) are equivalent to the following:

(i′) $S^{-1}\zeta \leqslant \zeta, \quad T^{-1}\zeta_S \leqslant \zeta$,

(iv′) $\bigwedge_{n=0}^{\infty} T^{-n}\zeta_S = \pi(G)$.

It is shown in [3] that for every $(T, S) \in b(G)$ there exists a (T, S)-perfect partition.

For $P \in B_G$ we define $\zeta_P = P \vee P_G^-$. In [4] we investigated the following question: is the partition ζ_P (T, S)-perfect for any $P \in B_G$? As we have seen above, the analogue of this question for single automorphisms has the positive answer. Our question is equivalent to the following: is the equality

$$\bigwedge_{n=0}^{\infty} (S^{-n} P_S^- \vee (P_S)_T^-) = (P_S)_T^-$$

satisfied for any $P \in B_G$? It turned out [4] that in general the answer to this question is negative. A generator satisfying the above equality was called in [4] *(T, S)-regular*. We denote the set of all (T, S)-regular generators of (X, G) by $B_{T,S}$.

It is worth noting that Weizsäcker [11] considered a more general problem in probability theory.

Using the relative version of the Kolmogorov zero-one law one can show that the zero time partition in the two-dimensional Bernoulli dynamical system is regular with respect to the pair of the shifts. A more general example is given in [4].

In this paper, using relative versions of some results of the ergodic theory of single automorphisms, we show that if G is totally ergodic then for any $(T, S) \in b(G)$ and for any (T, S)-exhaustive partition ζ there exists $P \in B_{T,S}$ with $P \leqslant \zeta$. Moreover, the set $B_{T,S}$ is dense in B_G. It appears that by the use of regular generators it is possible to characterize the groups with zero entropy in a manner similar to that for single automorphisms.

I am grateful to J. P. Thouvenot for suggesting the possibility of using relative generator theorems for a solution of the question stated above.

2. Some results of the relative ergodic theory. In the sequel we denote by Z the set of integers and by N the set of positive integers.

Let T be an automorphism of $(X, \mathscr{B}, \mu)$ and let $\sigma \in \mathscr{M}$ be such that $T\sigma = \sigma$. For $P \in \mathscr{Z}$ we put

$$h(P, T|\sigma) = H(P|P_T^- \vee \sigma).$$

We define the *σ-relative entropy* $h(T|\sigma)$ of T by the formula

$$h(T|\sigma) = \sup h(P, T|\sigma)$$

where the supremum is taken over all $P \in \mathscr{Z}$.

It is clear that $h(T|\sigma) \leqslant h(T)$. There is a simple formula connecting $h(T)$ with $h(T|\sigma)$.

PROPOSITION 1. $h(T) = h(T|\sigma) + h(T_\sigma)$ *where T_σ denotes the factor automorphism of T on X/σ.*

Proof. Let $P_k, Q_l \in \mathscr{Z}$, $k, l \in N$ be such that $P_k \nearrow \sigma$ and $Q_l \nearrow \varepsilon$. From the Pinsker formula and simple properties of the conditional entropy easily follow the inequalities:

$$h(P_k \vee Q_l, T) \geqslant h(P_k, T) + h(Q_l, T|\sigma),$$

$$h(Q_l, T) \leqslant h(P_k, T) + H(Q_l|\,(Q_l)_T^- \vee (P_k)_T), \qquad k, l \in N.$$

Applying to both the inequalities the well-known limit properties of entropy we obtain the desired equality.

We shall use in the sequel the following result given in [8] (lemma 10.2).

LEMMA 1. *For all* $P, Q \in \mathcal{M}$ *such that* $P \geqslant Q$ *and* $H(P|Q) < \infty$ *there exists* $R \in \mathcal{Z}$ *with* $P = Q \vee R$ *and* $H(R) < H(P|Q) + 3\sqrt{H(P|Q)}$.

The main tool to obtain our main result is a relative version of the well-known Rokhlin generator theorem (cf. [6]). Since the proof runs in a similar way to that of Rokhlin we give only a sketch of it below.

For $n \in N$, $B \in \mathcal{B}$ and a partition $P = (P_i,\ i \in N)$ we define the following partitions:

$$P_T^n = \bigvee_{k=-n+1}^{n-1} T^k P, \qquad P \cap B = (P_i \cap B,\ X \setminus B;\ i \in N).$$

Let $\sigma, \tau \in \mathcal{M}$ be such that $T\sigma = \sigma$, $T\tau = \tau$ and $\sigma \leqslant \tau$.

LEMMA 2. *If* T *is aperiodic with* $h(T|\sigma) < \infty$ *then for all* $P, Q \in \mathcal{Z}$ *and* $\delta > 0$ *there exists a partition* $R \in \mathcal{Z}$ *such that* $R_T \geqslant P_T$ *and* $H(R|TQ_T^- \vee \sigma) \leqslant h(T|\sigma) - h(Q, T|\sigma) + \delta$.

Sketch of proof. Let $\delta > 0$ be arbitrary and $n \in N$ be such that

$$\frac{1}{2n-1} H\big((P \vee Q)_T^n|\sigma\big) - h(P \vee Q, T|\sigma) < \frac{\delta}{3}.$$

We choose $\lambda > 0$ satisfying the condition

$$H(P \cap B) < \frac{\delta}{3} \quad \text{for } \mu(B) < \lambda.$$

The Rokhlin tower theorem implies there exists a set $C \in \mathcal{B}$ such that the sets $C, TC, \ldots, T^{n-1}C$ are pairwise disjoint and $\mu(D) < \lambda$ where $D = X \setminus (C \cup TC \cup \ldots \cup T^{n-1}C)$.

There exists $0 \leqslant k \leqslant n-1$ with

$$H(P_T^n \cap T^k C|TQ_T^- \vee \sigma) \leqslant h(T|\sigma) - h(Q, T|\sigma) + \tfrac{2}{3}\delta.$$

The partition $R = P_T^n \cap T^k C \vee P \cap D$ satisfies the desired properties: $R_T \geqslant P_T$ and

$$\begin{aligned} H(R|TQ_T^- \vee \sigma) &\leqslant H(P_T^n \cap T^k C|TQ_T^- \vee \sigma) + H(P \cap D) \\ &\leqslant h(T|\sigma) - h(Q, T|\sigma) + \delta. \end{aligned}$$

RELATIVE GENERATOR THEOREM. *If* $\tau \in \mathcal{M}$ *is such that the factor automorphism* T_τ *is aperiodic with* $h(T_\tau|\sigma) < \infty$ *then there exists* $P \in \mathcal{Z}$ *such that* $P \leqslant \tau$ *and* $P_T \vee \sigma = \tau$. *Moreover, the set of* $P \in \mathcal{Z}$, $P \leqslant \tau$, $P_T \vee \sigma = \tau$ *is dense in the set of* $P \in \mathcal{Z}$, $P \leqslant \tau$ *and* $h(P, T|\sigma) = h(T_\tau|\sigma)$.

Sketch of proof. We may suppose $\tau = \varepsilon$. Let $\delta > 0$ be arbitrary and $Q \in \mathcal{Z}$ be such that

$$h(T|\sigma) - h(Q, T|\sigma) < \frac{\delta^2}{2^7}.$$

We take a sequence (Q_n) of partitions in $\mathcal{Z}$ with $Q_0 = Q$, $Q_n \nearrow \varepsilon$ and

$$h(T|\sigma) - h(Q_k, T|\sigma) < \frac{\delta^2}{2^{2k+7}}, \qquad k \in N \cup \{0\}.$$

Using Lemma 2 we may choose a sequence (R_k) in $\mathcal{Z}$ with $(R_k)_T \geqslant (Q_k)_T$ and

$$H\big(R_k|\ (Q_{k-1})_T \vee \sigma\big) = H\big((Q_{k-1})_T \vee R_k \vee \sigma\big|(Q_{k-1})_T \vee \sigma\big) < \frac{\delta^2}{2^{2k+4}}, \qquad k \in N.$$

Now Lemma 1 implies there exists a sequence (P_k) in $\mathcal{Z}$ such that

$$(Q_{k-1})_T \vee R_k \vee \sigma = (Q_{k-1})_T \vee P_k \vee \sigma, \qquad H(P_k) < \frac{\delta}{2^k}, \qquad k \in N.$$

This equality gives

$$\Big(Q \vee \bigvee_{k=1}^{n} P_k\Big)_T \vee \sigma \geqslant (Q_n)_T \vee \sigma, \qquad n \in N.$$

Therefore putting $P = Q \vee \bigvee_{k=1}^{\infty} P_k$ we have $P_T \vee \sigma = \varepsilon$,

$$H(P) \leqslant H(Q) + \sum_{k=1}^{\infty} H(P_k) < H(Q) + \delta < \infty$$

and $\varrho(P, Q) < \delta$, which completes the proof.

We denote by $\pi(T, \tau|\sigma)$ the join of all $P \in \mathcal{Z}$ with $P \leqslant \tau$, $h(P, T|\sigma) = 0$ and call it the *σ-relative Pinsker partition* of T_τ. We shall write $\pi(T|\sigma)$ instead of $\pi(T, \varepsilon|\sigma)$. The concept of σ-relative Pinsker partition was introduced in [2] in the case $\tau = \varepsilon$ and called there the Pinsker closure of σ.

It is clear that $\pi(T, \tau|\sigma) \geqslant \sigma$. If $\pi(T, \tau|\sigma) = \sigma$ then we say that T_τ is a *K-automorphism relative to σ*. Let us remark that T is a K-automorphism relative to σ iff for any $\zeta \in \mathcal{M}$ with $T\zeta = \zeta$, $\zeta \geqslant \sigma$, $h(T_\zeta|\sigma) = 0$ we have $\zeta = \sigma$. Thouvenot also defined (cf. [10]) a concept of relative K-automorphism. Using Proposition 1 and the above remark one can easily check that in the case $h(T) < \infty$ both concepts coincide.

Some properties of relative Pinsker partitions:

(a) $T\pi(T, \tau|\sigma) = \pi(T, \tau|\sigma)$.

(b) $h(T_{\pi(T,\tau|\sigma)}|\sigma) = 0$.

(c) *If* S *is an automorphism of* $(X, \mathcal{B}, \mu)$ *commuting with* T *then*

$$S\pi(T, \tau|\sigma) = \pi(T, S\tau|S\sigma).$$

(d) *If* $\tau_i, \sigma_i \in \mathcal{M}$, $T\tau_i = \tau_i$, $T\sigma_i = \sigma_i$, $i = 1, 2$, $\sigma_1 \leqslant \tau_1 \leqslant \tau_2$, $\sigma_1 \leqslant \sigma_2 \leqslant \tau_2$ *then*

$$\pi(T, \tau_1|\sigma_1) \leqslant \pi(T, \tau_2|\sigma_2).$$

(e) T_τ *is a* K*-automorphism relative to* $\pi(T, \tau|\sigma)$.

(f) *If* $\sigma \in \mathcal{M}$, $T\sigma = \sigma$ *and* $\zeta \in \mathcal{M}$ *is such that* $\sigma \leqslant T^{-1}\zeta \leqslant \zeta$ *then*

$$\bigwedge_{n=0}^{\infty} T^{-n}\zeta \geqslant \pi(T, \zeta_T|\sigma).$$

(g) *If* $\sigma \in \mathcal{M}$, $T\sigma = \sigma$ *then for every* $P \in \mathcal{Z}$

$$\bigwedge_{n=0}^{\infty} (T^{-n}P_T^- \vee \sigma) = \pi(T, P_T \vee \sigma|\sigma).$$

(h) *If* $\tau_n \in \mathcal{M}$, $n \in N$, *are such that* $T\tau_n = \tau_n$, $\sigma \leqslant \tau_n \leqslant \tau_{n+1}$ *and* T_{τ_n} *is a* K*-automorphism relative to* σ, $n \in N$, *then* T_τ *is a* K*-automorphism relative to* σ, *where* $\tau = \bigvee_{n=1}^{\infty} \tau_n$.

Proof. Properties (a)–(d) are easy consequences of the definition. The proofs of (f) and (h) are similar to the proofs of Theorems 12.1 and 13.4 of [9] respectively and we omit them.

To prove (e) let us suppose $P \in \mathcal{Z}$, $P \leqslant \tau$ and

$$h(P, T|\ \pi(T, \tau|\sigma)) = 0.$$

Let $Q_n \in \mathcal{Z}$, $n \geqslant 1$, and $Q_n \nearrow \pi(T, \tau|\sigma)$. Using the relative version of the Pinsker formula (cf. [1]) and simple properties of the conditional entropy we have

$$\begin{aligned} h(P \vee Q_n, T|\sigma) & \\ &= h(P, T|\sigma) + H(Q_n|\ (Q_n)_T^- \vee P_T \vee \sigma) \\ &= H(P|\ P_T^- \vee (Q_n)_T^- \vee \sigma) + H(Q_n|\ (Q_n)_T^- \vee P \vee P_T^- \vee \sigma), \qquad n \in N. \end{aligned}$$

Therefore the choice of P and Q_n implies

$$h(P, T|\sigma) = H(P|\ P_T^- \vee (Q_n)_T^- \vee \sigma), \qquad n \in N.$$

Taking the limit as $n \to \infty$ we have

$$h(P, T|\sigma) = H(P|\ P_T^- \vee \pi(T, \tau|\sigma)) = 0,$$

i.e. $P \leqslant \pi(T, \tau|\sigma)$ and (e) is proved.

In order to check (g) let us observe that the inequality

$$\bigwedge_{n=0}^{\infty} (T^{-n}P_T^- \vee \sigma) \geqslant \pi(T, P_T \vee \sigma|\sigma)$$

is an easy consequence of (f). To prove the converse inequality we take $Q \in \mathcal{Z}$ and $Q \leqslant \bigwedge_{n=0}^{\infty} (T^{-n}P_T^- \vee \sigma)$. Hence $Q \leqslant P_T \vee \sigma$ and

$$H(Q|Q_T^- \vee T^{-n}P_T^- \vee \sigma) = 0, \qquad n \in N.$$

Taking the limit as $n \to \infty$ we obtain $H(Q|Q_T^- \vee \sigma) = 0$. This means that $Q \leqslant \pi(T, P_T \vee \sigma|\sigma)$ which proves (g).

3. Existence of regular generators. Let (X, G) be a two-dimensional dynamical system and let $(T, S) \in b(G)$. In order to prove our result we shall need the following.

LEMMA 2. *If* $\zeta \in \mathcal{M}$ *is* (T, S)*-exhaustive then* S *is a* K*-automorphism relative to* $T^n\zeta_S$, $n \in \boldsymbol{Z}$.

Proof. Let $k \in N$ and $P, Q \in \mathcal{Z}$ satisfy the following conditions:

$$P \leqslant T^k S^l \zeta \quad \text{for some } l \in N \quad \text{and} \quad Q \leqslant \pi(S, T^k\zeta_S|T^{-1}\zeta_S).$$

Hence

$$Q \leqslant T^k\zeta_S, \quad h(Q, S|T^{-1}\zeta_S) = 0. \tag{1}$$

Let $m \in N$ be arbitrary. The relative Pinsker formula implies

$$\begin{aligned} h(P \vee Q, S^m|T^{-1}\zeta_S) &= h(P, S^m|T^{-1}\zeta_S) + H(Q|Q_{S^m}^- \vee P_{S^m} \vee T^{-1}\zeta_S) \\ &= h(Q, S^m|T^{-1}\zeta_S) + H(P|P_{S^m}^- \vee Q_{S^m} \vee T^{-1}\zeta_S). \end{aligned}$$

Hence by (1) we have

$$h(P, S^m|T^{-1}\zeta_S) = H(P|P_{S^m}^- \vee Q_{S^m} \vee T^{-1}\zeta_S).$$

Therefore

$$\begin{aligned} H(P|Q \vee T^{-1}\zeta_S) &\geqslant H(P|P_{S^m}^- \vee Q_{S^m} \vee T^{-1}\zeta_S) \\ &= H(P|P_{S^m}^- \vee T^{-1}\zeta_S) \geqslant H(P|S^{-m+1}P_S^- \vee T^{-1}\zeta_S) \\ &\geqslant H(P|S^{-m+l+1}T^k\zeta). \end{aligned} \tag{2}$$

Taking the limit in (2) as $m \to \infty$ and using (iii) we obtain

$$H(P|Q \vee T^{-1}\zeta_S) \geqslant H(P|T^{k-1}\zeta_S). \tag{3}$$

Since P runs through a dense subset of the set $\{R \in \mathcal{Z};\ R \leqslant T^k\zeta_S\}$ we conclude that (3) is valid for any $P \in \mathcal{Z}$ with $P \leqslant T^k\zeta_S$. Assuming $P = Q$ we get $Q \leqslant T^{k-1}\zeta_S$ and so

$$\pi(S, T^k\zeta_S|T^{-1}\zeta_S) \leqslant T^{k-1}\zeta_S.$$

Therefore

$$\pi(S, T^k\zeta_S|T^{-1}\zeta_S) \leqslant \pi(S, T^{k-1}\zeta_S|T^{-1}\zeta_S)$$

and thus

$$\pi(S, T^k\zeta_S|T^{-1}\zeta_S) = T^{-1}\zeta_S, \qquad k \in N.$$

Now (ii) and (h) give $\pi(S|T^{-1}\zeta_S) = T^{-1}\zeta_S$ and so by (c) we obtain the result.

Now, let G be aperiodic with $h(G) < \infty$.

COROLLARY 1. *A generator* $P \in B_{T,S}$ *iff* S *is a* K*-automorphism relative to* $(P_S)_T^-$.

Proof. If $P \in B_{T,S}$ then the partition $\zeta = P \vee P_G^-$ is (T, S)-exhaustive and so, by Lemma 2, S is a K-automorphism relative to $T^{-1}\zeta_S = (P_S)_T^-$.

Now, let S be a K-automorphism relative to $(P_S)_T^-$. From this and (g) it follows that

$$\bigwedge_{n=0}^{\infty} (S^{-n} P_S^- \vee (P_S)_T^-) = \pi(S,\, T(P_S)_T^- |\, (P_S)_T^-) \leqslant \pi(S|\, (P_S)_T^-) = (P_S)_T^-,$$

i.e. $P \in B_{T,S}$.

From Corollary 1 and (e) we obtain at once

Corollary 2. *If $P \in B_G$ and $Q \in \mathscr{Z}$ is such that $(Q_S)_T^- = \pi(S|(P_S)_T^-)$ then $Q \in B_{T,S}$.*

In the theorem below we prove that for a wide class of groups G and for any generator P of (X, G) such a generator Q exists.

Definition. The group G is said to be *totally ergodic* if every automorphism $\varphi \in G$ different from the identity transformation of X is ergodic.

Theorem. *If G is totally ergodic with $h(G) < \infty$, $P \in B_G$ and $(T, S) \in b(G)$ then for every $\varepsilon > 0$ there exists $Q \in B_{T,S}$ such that $P \leqslant Q$ and $\varrho(P, Q) < \varepsilon$.*

Proof. Let us suppose G is totally ergodic, $(T, S) \in b(G)$, $P \in B_G$ and $\varepsilon > 0$ is arbitrary.

By our assumption, the factor automorphism $S_{\pi(S|(P_S)_T^-)}$ is ergodic and since $P \in B_G$, the factor measure induced by μ on $X/\pi(S|(P_S)_T^-)$ is continuous. Therefore the above factor automorphism is aperiodic. Now property (b) implies

$$h(T^{-1}P,\, S|(P_S)_T^-) = h(S_{\pi(S|(P_S)_T^-)}|(P_S)_T^-) = 0.$$

It follows from the Relative Generator Theorem that there exists $R \in \mathscr{Z}$ such that

$$R \leqslant \pi(S|(P_S)_T^-), \quad R_S \vee (P_S)_T^- = \pi(S|(P_S)_T^-) \quad \text{and} \quad \varrho(T^{-1}P, R) < \varepsilon.$$

Putting $Q = P \vee TR$ we have

$$P \leqslant Q, \quad (Q_S)_T^- = \pi(S|(P_S)_T^-) \quad \text{and} \quad \varrho(P, Q) \leqslant \varrho(T^{-1}P, R) < \varepsilon.$$

By Corollary 2 we see that Q satisfies all desired properties.

Since B_G is a dense subset of Γ_G the theorem above implies at once

Corollary 1. *If G is totally ergodic with $h(G) < \infty$ then for every $(T, S) \in b(G)$ the set $B_{T,S}$ is dense in Γ_G.*

Now, let ζ be (T, S)-exhaustive.

Corollary 2. *If G is totally ergodic with $h(G) < \infty$ then for every $(T, S) \in b(G)$ there exists $P \in B_{T,S}$ with $P \leqslant \zeta$.*

Proof. Let $Q \in B_G$ with $Q \leqslant \zeta$. The existence of such a generator Q may be proved by the same method as that used by Rokhlin in [7]. Let $\bar{Q} \in B_G$ be such that $(\bar{Q}_S)_T^- = \pi(S|(Q_S)_T^-)$. As we already know, $\bar{Q} \in B_{T,S}$. It follows from Lemma 2 that

$$T^{-1}\bar{Q} \leqslant (\bar{Q}_S)_T^- = \pi(S|(Q_S)_T^-) \leqslant \pi(S|T^{-1}\zeta_S) = T^{-1}\zeta_S.$$

Hence $P = T^{-1}\bar{Q} \in B_{T,S}$ and $P \leqslant \zeta$.

In the ergodic theory of single automorphisms the following characterization of automorphisms with zero entropy is well known. Namely, an automorphism T has zero entropy iff any generator P of (X, T) is strong, i.e. $P_T^- = \varepsilon$.

It appears that it is possible to obtain a two-dimensional analogue of this result by the use of regular generators. First we define the concept of two-dimensional strong generator with respect to the lexicographical order.

Definition. A partition $P \in \mathscr{Z}$ is said to be a (T, S)-*strong generator* of (X, G) if $\bigvee_{(k,l)\in\Pi} T^k S^l P = \varepsilon$.

Proposition 2. *A totally ergodic group G has zero entropy iff every generator $P \in B_{T,S}$ is (T, S)-strong, $(T, S) \in b(G)$.*

Proof. The sufficiency is obvious. Let us suppose $h(G) = 0$, $(T, S) \in b(G)$ and $P \in B_{T,S}$. Since $H(P|P_S^- \vee (P_S)_T^-) = 0$ we have $P \leqslant S^{-n}P_S^- \vee (P_S)_T^-$, $n \in N$, and so $P \leqslant (P_S)_T^-$ by the regularity of P. Therefore $P_G = (P_S)_T^-$ and thus P is (T, S)-strong.

Remark. The conditions $h(G) = 0$, $P \in B_G$ do not imply that P is (T, S)-strong, $(T, S) \in b(G)$.

Example. Let $(Y, \mathscr{F}, \lambda)$ be a Lebesgue probability space and S_0 an aperiodic automorphism of Y with $h(S_0) = 0$. We denote by $(X, \mathscr{B}, \mu)$ the product space $\prod_{i=-\infty}^{+\infty} (Y_i, \mathscr{F}_i, \lambda_i)$, where $Y_i = Y$, $\mathscr{F}_i = \mathscr{F}$, $\lambda_i = \lambda$. Let T, S be automorphisms of $(X, \mathscr{B}, \mu)$ defined by the formulas

$$(Tx)(n) = x(n+1), \quad (Sx)(n) = S_0 x(n), \quad n \in Z,$$

and let G be the automorphism group generated by T and S. It is clear that G is totally ergodic. It is shown in [1] that $h(G) = h(S_0) = 0$. Let $\alpha = \{A_0, A_1\}$ be a generator of (Y, S_0). The partition $P = \{C(0, A_0), C(0, A_1)\}$ where $C(0, A_i) = \{x \in X;\ x(0) \in A_1\}$, $i = 1, 2$, is a generator of (X, G). We shall check that P is not (T, S)-strong. Let us suppose $P_S^- \vee (P_S)_T^- = \varepsilon$. Since μ is a product measure the partitions P and $P_S^- \vee (P_S)_T^-$ are independent. Hence P and also α are trivial partitions, which is impossible.

Remark. Let G be totally ergodic, $(T, S) \in b(G)$ and let ζ be a (T, S)-perfect partition. It would be interesting to know whether ζ may be represented as the past of a certain two-dimensional process (P, G), i.e. ζ

$= P \vee P_G^-$, $P \in B(G)$. This question has a positive answer if $h(G) = 0$, because in this case every perfect partition is the partition into points and it is sufficient to use Corollary 2 and Proposition 2. We have been unable to decide whether this question has a positive answer in the general case.

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Received January 28, 1985 (2029)
Revised version September 16, 1985

STUDIA MATHEMATICA, T. LXXXV. (1987)

On drop property

by

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Abstract. Let $(X, \| \|)$ be a Banach space. We say that the norm $\| \|$ has the drop property if for each closed set C dosjoint with the closed unit ball $B = \{x: \|x\| \leqslant 1\}$, there is a point $a \in C$ such that $\operatorname{conv}(a \cup B) \cap C = \{a\}$.

We say that a Banach space $(X, \| \|)$ has the drop property if there is a norm $\| \|_1$ equivalent to the given one such that $\| \|_1$ has the drop property.

In the paper it is shown that each superreflexive space has the drop property and each space X which has the drop property is reflexive.

Let $(X, \| \|)$ be a Banach space. Let B denote the unit ball in X. By a *drop* induced by a point $a \notin B$ we mean the set

$$D(a, B) = \operatorname{conv}(a, B). \tag{1}$$

Daneš [3] proved the following

THEOREM 1. (Drop theorem). *Let C be a closed set such that*

$$\inf\{\|x\|: x \in C\} = R > 1. \tag{2}$$

Then there is a point $a \in C$ such that

$$D(a, B) \cap C = \{a\}. \tag{3}$$

The drop theorem was used in various situations (see [1], [2], [4], [5], [10]).

Recently Penot [9] discussed the relations between the drop theorem and Ekeland's variational principle [7].

It is a natural question to ask when we can replace in the drop theorem assumption (2) by the weaker assumption that C is disjoint with B.

We shall say that the norm $\| \|$ has the *drop property* if the drop theorem holds under this weaker assumption. If there is a norm $\| \|_1$ equivalent to the norm $\| \|$ and having the drop property, then we say that the space X has the *drop property*.

In this paper we shall show that the uniformly convex norms have the drop property and that the spaces X with the drop property are reflexive.

Let $(X, \| \|)$ be a Banach space. We recall that the space $(X, \| \|)$ is called *uniformly convex* if there is an increasing positive function $\delta(\varepsilon)$ defined