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On a singular integral

by

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Abstract. In this paper, we establish that the following maximal singular operator:

$$T^*f(x) = \sup_{0<\varepsilon<\infty} \left| \int_{|y|>\varepsilon} h(y) \frac{\Omega(y)}{|y|^n} f(x-y)\,dy \right|$$

is bounded in $L^p(\boldsymbol{R}^n)$ for all $1 < p < \infty$, $n \geqslant 2$, where $h(x)$ is any bounded radial function, and Ω is in $L^q(S^{n-1})$ for $q > 1$.

Introduction. Suppose $K_\varepsilon(x) = \dfrac{\Omega(x)}{|x|^n}\chi_{|x|>\varepsilon}$ is a truncated Calderón–Zygmund kernel on $\boldsymbol{R}^n$, $n \geqslant 2$ (this means that Ω is homogeneous of degree 0 on $\boldsymbol{R}^n$ and Ω is in $L^q(S^{n-1})$ for some q, $q > 1$, and $\int_{S^{n-1}} \Omega\,d\sigma = 0$) where $\chi_{|x|>\varepsilon}$ is the characteristic function of $\{x \in \boldsymbol{R}^n\colon |x| > \varepsilon\}$. We multiply $K_\varepsilon(x)$ by an arbitrary bounded radial function, say $h(x)$, and call the new kernel $\tilde{K}_\varepsilon(x)$.

In this paper, we will prove the maximal singular operator $f \to \sup_{0<\varepsilon<\infty} |\tilde{K}_\varepsilon * f|$ is bounded on $L^p(\boldsymbol{R}^n)$ for all p such that $1 < p < \infty$. This result does not hold for the case when $n = 1$. For example, let $\tilde{K}_0(x) = \sin|x|/x$; then the Fourier transform of $\tilde{K}_0(x)$ is unbounded. It is already known that the principal value singular operator (i.e., $f \to \text{p.v.}\,\tilde{K}_0 * f$) is bounded on $L^p(\boldsymbol{R}^n)$ for all $1 < p < \infty$, $n \geqslant 2$ (see R. Fefferman [4], D. Namazi [5]).

Here we should remark that in this work we were very much motivated by the work of E. M. Stein (see [7]) and some ideas from R. Fefferman's and D. Namazi's papers (see [4], [5]). The author would like to express his sincere appreciation to professor Robert Fefferman for his encouragement, sustained guidance, and, in particular, for his careful review and invaluable discussion of this paper.

We will use $\hat{}$ and $\check{}$ to denote the Fourier and inverse Fourier transforms, respectively. Throughout this paper, x will denote a point in $\boldsymbol{R}^n$, $|x|$ is the length of x, $x' = x/|x|$, and C represents a constant depending on n, p, although different in different places.

THEOREM 1. *If $n \geqslant 2$, let Ω be in $L^q(S^{n-1})$ for some $q > 1$ and suppose that*

Ω is homogeneous of degree zero and satisfies the cancellation property (i.e., $\int_{S^{n-1}} \Omega\, d\sigma = 0$). Let $h(x)$ be any radial bounded function. For $1 < p < \infty$ and $f \in L^p(\boldsymbol{R}^n)$, let

$$T_\varepsilon f(x) = \int_{|y|>\varepsilon} h(y)\frac{\Omega(y)}{|y|^n} f(x-y)\,dy.$$

Then

$$\| \sup_{0<\varepsilon<\infty} |T_\varepsilon f(x)| \|_p \leqslant A_p \|h\|_\infty \|f\|_p.$$

To prove this result, we introduce a family of operators $\{T_\varepsilon^\alpha\}_\alpha$ such that the mapping $\alpha \to \int T_\varepsilon^\alpha f(x)\, g(x)\,dx$ is analytic in the interior of the strip $\{\alpha: |\mathrm{Re}\,\alpha| \text{ very small}\}$ for f, g in the Schwartz class. We define

$$(T_\varepsilon^\alpha f)\hat{}\,(x) = m_\varepsilon^\alpha(x)\hat{f}(x),$$

where α is a complex number and

$$m_\varepsilon^\alpha(x) = \int_\varepsilon^\infty \int_{S^{n-1}} \Omega(\xi)\, e^{2\pi i r|x|x'\cdot\xi}\, d\sigma(\xi)\, h(r)\frac{dr}{r^{1+\alpha}}|x|^{-\alpha}$$

for $x \in \boldsymbol{R}^n$. Let us write

$$w(r, x) = \int_{S^{n-1}} \Omega(\xi)\, e^{2\pi i r|x|x'\cdot\xi}\, d\sigma(\xi)\, h(r)\frac{1}{r^\alpha}|x|^{-\alpha},$$

$$l(r, x') = \int_{S^{n-1}} \Omega(\xi)\, e^{2\pi i r x'\cdot\xi}\, d\sigma(\xi).$$

Therefore,

$$m_\varepsilon^\alpha(x) = \int_\varepsilon^\infty w(r, x)\frac{dr}{r}.$$

It is clear that $T_\varepsilon^0 f(x) = T_\varepsilon f(x)$. We will prove the following two theorems, via a convexity argument involving complex α (see [8], p. 280). Using Stein's interpolation theorem, we deduce that $\sup_{0<\varepsilon<\infty} |T_\varepsilon f(x)|$ is bounded on all spaces $L^p(\boldsymbol{R}^n)$, $n \geqslant 2$.

Remark. In this paper, "$|\mathrm{Re}\,\alpha|$ is very small" means "$|\mathrm{Re}\,\alpha| < (q-1)/2q$".

THEOREM 2. *If $|\mathrm{Re}\,\alpha|$ is very small and the hypotheses are as in Theorem 1, then*

$$\| \sup_{0<\varepsilon<\infty} |T_\varepsilon^\alpha f(x)| \|_2 \leqslant A\|h\|_\infty \|f\|_2,$$

where A only depends on n, $\mathrm{Re}\,\alpha$ and the norm of Ω.

THEOREM 3. *If $\mathrm{Re}\,\alpha > 0$, $\mathrm{Re}\,\alpha$ very small and the hypotheses are as in Theorem 1, then*

$$\| \sup_{0<\varepsilon<\infty} |T_\varepsilon^\alpha f(x)| \|_p \leqslant A_p \|h\|_\infty \|f\|_p,$$

where A_p is dominated by $|\Gamma(\frac{1}{2}(n-\alpha))/\Gamma(\frac{1}{2}\alpha)|$.

The key step in the proof of Theorem 2 is to consider a g-function. We fix a function $p(x)$ which is a radial smooth function in $\boldsymbol{R}^n$ and $\hat{p}(x) = 1$ if $|x| \leqslant 1$, $\hat{p}(x) = 0$ if $|x| \geqslant 2$. Set $p_\varepsilon(x) = \varepsilon^{-n} p(x/\varepsilon)$. Define $g_\alpha(f)(x)$ by

$$g_\alpha(f)(x) = \left(\int_0^\infty |T_\varepsilon^\alpha f(x) - p_\varepsilon * T_0^\alpha f(x)|^2 \frac{d\varepsilon}{\varepsilon} \right)^{1/2}.$$

LEMMA 1 (D. Namazi). *If $n \geqslant 2$ and $\Omega \in L^q(S^{n-1})$ for some $1 < q$, then*

$$\|l(\cdot, x')\|_{L^p((1,\infty),dr)} \leqslant C_p \|\Omega\|_{L^q(S^{n-1})}$$

for any $p > 2q/(q-1)$ and C_p does not depend on $x' \in S^{n-1}$.

Proof. This lemma originally appeared in reference [5]. For the purpose of self-containment, I include the proof in here. It is sufficient to assume $x' = (1, 0, \ldots, 0)$. Let us make a change of variable: $l(r, x')$ becomes

$$l(r, x') = \int_{-1}^1 \int_{\xi' \in S^{n-2}} e^{2\pi i r s}(1-s^2)^{(n-3)/2}\, \Omega(s, \sqrt{1-s^2}\,\xi')\, d\sigma_{n-2}(\xi')\, ds$$

where $d\sigma_{n-2}(\xi')$ is the unit of surface area on the sphere S^{n-2}. Let

$$a(s) = (1-s^2)^{(n-3)/2} \chi_{|s|<1}(s) \int_{\xi' \in S^{n-2}} \Omega(s, \sqrt{1-s^2}\,\xi')\, d\sigma_{n-2}(\xi').$$

Hence

$$l(r, x') = \int_{-\infty}^\infty e^{2\pi i r s} a(s)\, ds.$$

If $1/p + 1/p' = 1$, $p \geqslant 2$, then we apply the Hausdorff–Young inequality and Hölder's inequality twice. We have

$$\begin{aligned}
\|l(r, x')\|^{p'}_{L^p((1,\infty),dr)} &\leqslant \|a(s)\|^{p'}_{L^{p'}(\boldsymbol{R})} \\
&= \int_{-1}^1 \left[(1-s^2)^{(n-3)/2} \left| \int_{\xi' \in S^{n-2}} \Omega(s, \sqrt{1-s^2}\,\xi')\, d\sigma_{n-2}(\xi') \right| \right]^{p'} ds \\
&\leqslant C \int_{-1}^1 (1-s^2)^{(n-3)p'/2} \left(\int_{\xi' \in S^{n-2}} |\Omega(s, \sqrt{1-s^2}\,\xi')|^q\, d\sigma_{n-2}(\xi') \right)^{p'/q} ds \\
&= C \int_{-1}^1 (1-s^2)^{(n-3)(1-1/q)p'/2} \left[(1-s^2)^{(n-3)/2}\right.
\end{aligned}$$

$$\cdot \int_{\xi' \in S^{n-2}} |\Omega(s, \sqrt{1-s^2}\,\xi')|^q \, d\sigma_{n-2}(\xi') \Big]^{p'/q} ds$$

$$\leqslant C \Big\{ \int_{-1}^{1} (1-s^2)^{(n-3)(1-1/q)p'q/(2(q-p'))} \, ds \Big\}^{(q-p')/q} \cdot \Big\{ \int_{-1}^{1} (1-s^2)^{(n-3)/2} \cdot \int_{\xi' \in S^{n-2}} |\Omega(s, \sqrt{1-s^2}\,\xi')|^q \, d\sigma_{n-2}(\xi') \, ds \Big\}^{p'/q}$$

$$\leqslant C \|\Omega\|^{p'}_{Lq(S^{n-1})} \quad (\text{since } p > 2q/(q-1)).$$

Lemma 1 is proved.

LEMMA 2. *If* $|\mathrm{Re}\,\alpha|$ *is very small, then*

$$\Big\| \sup_{0<S<\infty} \frac{1}{S} \int_0^S |T_\varepsilon^\alpha f(x)| \, d\varepsilon \Big\|_2 \leqslant C \|h\|_\infty \|f\|_2.$$

Proof. Note that

$$\frac{1}{S} \int_0^S |T_\varepsilon^\alpha f(x)| \, d\varepsilon \leqslant \frac{1}{S} \int_0^S |T_\varepsilon^\alpha f(x) - p_\varepsilon * T_0^\alpha f(x)| \, d\varepsilon + \frac{1}{S} \int_0^S |p_\varepsilon * T_0^\alpha f(x)| \, d\varepsilon.$$

Therefore,

$$\sup_{0<S<\infty} \frac{1}{S} \int_0^S |T_\varepsilon^\alpha f(x)| \, d\varepsilon$$

is dominated by

$$g_\alpha(f)(x) + M(T_0^\alpha f(x)),$$

where $M(\cdot)$ is the Hardy–Littlewood maximal operator. By the fact that the H–L maximal operator is bounded in $L^2(\boldsymbol{R}^n)$ and using the Plancherel theorem, the L^2-norm of the second term is

$$\|M(T_0^\alpha f(x))\|_2^2 \leqslant \|(T_0^\alpha f)\hat{}(x)\|_2^2 = \int_{\boldsymbol{R}^n} \Big\{ \Big| \int_0^\infty \int_{S^{n-1}} \Omega(\xi) e^{2\pi i r |x| x' \cdot \xi} \, d\sigma(\xi) \, h(r) \frac{|x|^{-\alpha}}{r^\alpha} \frac{dr}{r} \Big| \Big\}^2 |\hat{f}(x)|^2 \, dx.$$

So, we only need show that the term in parentheses is bounded for all x in $\boldsymbol{R}^n$. By applying Lemma 1 and the cancellation of Ω for the second inequality below, we have

$$\Big| \int_0^\infty \int_{S^{n-1}} \Omega(\xi) e^{2\pi i r |x| x' \cdot \xi} \, d\sigma(\xi) \, h(r) \frac{|x|^{-\alpha}}{r^\alpha} \frac{dr}{r} \Big| \leqslant \|h\|_\infty \int_0^\infty \Big| \int_{S^{n-1}} \Omega(\xi) e^{2\pi i r x' \cdot \xi} \, d\sigma(\xi) \Big| \frac{dr}{r^{1+\mathrm{Re}\,\alpha}}$$

$$\leqslant \|h\|_\infty \int_0^1 \Big| \int_{S^{n-1}} \Omega(\xi) (e^{2\pi i r x' \cdot \xi} - 1) \, d\sigma(\xi) \Big| \frac{dr}{r^{1+\mathrm{Re}\,\alpha}} + \|h\|_\infty \int_1^\infty \Big| \int_{S^{n-1}} \Omega(\xi) e^{2\pi i r x' \cdot \xi} \, d\sigma(\xi) \Big| \frac{dr}{r^{1+\mathrm{Re}\,\alpha}}$$

$$\leqslant C \|h\|_\infty \|\Omega\|_{L^1(S^{n-1})} + C' \|h\|_\infty \|\Omega\|_{Lq(S^{n-1})}.$$

Hence,

$$\|M(T_0^\alpha f(x))\|_2^2 \leqslant C \|h\|_\infty^2 \|f\|_2^2.$$

On the other hand,

$$\|g_\alpha(f)(x)\|_2^2 = \int_{\boldsymbol{R}^n} \int_0^\infty |(T_\varepsilon^\alpha f)\hat{}(x) - (p_\varepsilon * T_0^\alpha f)\hat{}(x)|^2 \frac{d\varepsilon}{\varepsilon} dx = \int_{\boldsymbol{R}^n} \int_0^\infty |m_\varepsilon^\alpha(x) - \hat{p}(\varepsilon|x|) \, m_0^\alpha(x)|^2 \frac{d\varepsilon}{\varepsilon} |\hat{f}(x)|^2 \, dx.$$

For the purpose of proving that $g_\alpha(f)(x)$ is bounded in $L^2(\boldsymbol{R}^n)$, it is sufficient to show that

$$\int_0^\infty |m_\varepsilon^\alpha(x) - \hat{p}(\varepsilon|x|) \, m_0^\alpha(x)|^2 \frac{d\varepsilon}{\varepsilon}$$

is bounded for all $x \in \boldsymbol{R}^n$. By definition, this expression equals

$$\int_0^\infty \Big| \int_\varepsilon^\infty \int_{S^{n-1}} \Omega(\xi) e^{2\pi i r |x| x' \cdot \xi} \, d\sigma(\xi) \, h(r) \frac{|x|^{-\alpha}}{r^{1+\alpha}} dr - \hat{p}(\varepsilon|x|) \int_0^\infty \int_{S^{n-1}} \Omega(\xi) e^{2\pi i r |x| x' \cdot \xi} \, d\sigma(\xi) \, h(r) \frac{|x|^{-\alpha}}{r^{1+\alpha}} dr \Big|^2 \frac{d\varepsilon}{\varepsilon}$$

$$= \int_0^\infty \Big| \int_\varepsilon^\infty \int_{S^{n-1}} \Omega(\xi) e^{2\pi i r x' \cdot \xi} \, d\sigma(\xi) \, h\Big(\frac{r}{|x|}\Big) \frac{dr}{r^{1+\alpha}} - \hat{p}(\varepsilon) \int_0^\infty \int_{S^{n-1}} \Omega(\xi) e^{2\pi i r x' \cdot \xi} \, d\sigma(\xi) \, h\Big(\frac{r}{|x|}\Big) \frac{dr}{r^{1+\alpha}} \Big|^2 \frac{d\varepsilon}{\varepsilon}$$

$$= \int_0^1 \cdots \frac{d\varepsilon}{\varepsilon} + \int_1^2 \cdots \frac{d\varepsilon}{\varepsilon} + \int_2^\infty \cdots \frac{d\varepsilon}{\varepsilon} = I_A + I_B + I_C.$$

By the cancellation of $\Omega(\xi)$ and the definition of $p(x)$ (i.e., $\hat{p}(\varepsilon) = 1$ if $\varepsilon \leqslant 1$, $\hat{p}(\varepsilon) = 0$ if $\varepsilon \geqslant 2$), we can write

$$I_A = \int_0^1 \Big| \int_0^\varepsilon \int_{S^{n-1}} \Omega(\xi) (e^{2\pi i r x' \cdot \xi} - 1) \, d\sigma(\xi) \, h\Big(\frac{r}{|x|}\Big) \frac{dr}{r^{1+\alpha}} \Big|^2 \frac{d\varepsilon}{\varepsilon}$$

$$\leqslant C\|h\|_\infty^2 \|\Omega\|_{L^1(S^{n-1})}^2 \int_0^1 \left| \int_0^\varepsilon \frac{dr}{r^{\mathrm{Re}\alpha}} \right|^2 \frac{d\varepsilon}{\varepsilon}.$$

This is bounded if $|\mathrm{Re}\,\alpha|$ is very small. The third term is

$$I_C = \int_2^\infty \left| \int_\varepsilon^\infty \int_{S^{n-1}} \Omega(\xi) e^{2\pi i r x'\cdot\xi} d\sigma(\xi) h\left(\frac{r}{|x|}\right) \frac{dr}{r^{1+\alpha}} \right|^2 \frac{d\varepsilon}{\varepsilon}$$
$$= \int_2^\infty \left| \int_\varepsilon^\infty l(r, x') h\left(\frac{r}{|x|}\right) \frac{dr}{r^{1+\alpha}} \right|^2 \frac{d\varepsilon}{\varepsilon}.$$

We fix any p such that $\infty > p > 2q/(q-1)$. By applying the Hölder inequality and Lemma 1, I_C is bounded by

$$\|h\|_\infty^2 \int_2^\infty \left\{ \int_1^\infty |l(r, x')|^p dr \right\}^{2/p} \left\{ \int_\varepsilon^\infty \frac{dr}{r^{(1+\mathrm{Re}\alpha)p'}} \right\}^{2/p'} \frac{d\varepsilon}{\varepsilon} \leqslant C\|h\|_\infty^2 \|\Omega\|_{L^q(S^{n-1})}^2,$$

if $|\mathrm{Re}\,\alpha|$ very small. Finally the second term is

$$I_B = \int_1^2 |\ldots| \frac{d\varepsilon}{\varepsilon}$$
$$\leqslant C \left\{ \int_0^\infty \left| \int_{S^{n-1}} \Omega(\xi) e^{2\pi i r x'\cdot\xi} d\sigma(\xi) \right| \left| h\left(\frac{r}{|x|}\right) \right| \frac{dr}{r^{1+\mathrm{Re}\alpha}} \right\}^2$$
$$= C \left\{ \int_0^1 \ldots \frac{dr}{r^{1+\mathrm{Re}\alpha}} + \int_1^\infty \ldots \frac{dr}{r^{1+\mathrm{Re}\alpha}} \right\}^2.$$

By the cancellation of Ω and, once more, the application of Lemma 1, we have

$$I_B \leqslant (C\|h\|_\infty \|\Omega\|_{L^1(S^{n-1})} + C'\|h\|_\infty \|\Omega\|_{L^q(S^{n-1})})^2.$$

Lemma 2 is proved.

Proof of Theorem 2. By the definition of $T_\varepsilon^\alpha f(x)$ (i.e., $(T_\varepsilon^\alpha f)\hat{}(x) = m_\varepsilon^\alpha(x)\hat{f}(x)$, $m_\varepsilon^\alpha(x) = \int_\varepsilon^\infty w(r, x)\,dr/r$) and the following computation:

$$\frac{1}{S}\int_0^S m_\varepsilon^\alpha(x)\,d\varepsilon = \frac{1}{S}\int_0^S \int_\varepsilon^\infty w(r, x)\frac{dr}{r}d\varepsilon$$
$$= \frac{1}{S}\int_0^S \int_0^r w(r, x)\,d\varepsilon \frac{dr}{r} + \frac{1}{S}\int_S^\infty \int_0^S w(r, x)\,d\varepsilon \frac{dr}{r}$$
$$= \frac{1}{S}\int_0^S w(r, x)\,dr + \int_S^\infty w(r, x)\frac{dr}{r}$$
$$= \frac{1}{S}\int_0^S w(r, x)\,dr + m_S^\alpha(x),$$

we get the equality

$$T_S^\alpha f(x) = \frac{1}{S}\int_0^S T_\varepsilon^\alpha f(x)\,d\varepsilon - \frac{1}{S}\int_0^S (w(r, \cdot)\hat{f}(\cdot))\check{}(x)\,dr.$$

Hence,

$$\sup_{0<S<\infty} |T_S^\alpha f(x)| \leqslant \sup_{0<S<\infty} \frac{1}{S}\int_0^S |T_\varepsilon^\alpha f(x)|\,d\varepsilon + \left(\int_0^\infty |(w(r, \cdot)\hat{f}(\cdot))\check{}(x)|^2 \frac{dr}{r} \right)^{1/2}.$$

By Lemma 2, the first term on the right-hand side of the above inequality is bounded in $L^2(\boldsymbol{R}^n)$. Applying the Plancherel theorem, the L^2-norm of the second term is equal to

$$\left\| \left(\int_0^\infty |w(r, x)\hat{f}(x)|^2 \frac{dr}{r} \right)^{1/2} \right\|_2.$$

Employing the same argument as that used in the proof of Lemma 2, we can write

$$\int_0^\infty |w(r, x)|^2 \frac{dr}{r} = \int_0^\infty \left| \int_{S^{n-1}} \Omega(\xi) e^{2\pi i r|x|x'\cdot\xi} d\sigma(\xi) h(r) \frac{|x|^{-\alpha}}{r^\alpha} \right|^2 \frac{dr}{r}$$
$$\leqslant \|h\|_\infty^2 \int_0^\infty \left| \int_{S^{n-1}} \Omega(\xi) e^{2\pi i r x'\cdot\xi} d\sigma(\xi) \right|^2 \frac{dr}{r^{1+2\mathrm{Re}\alpha}}$$
$$\leqslant \|h\|_\infty^2 \int_0^1 \left| \int_{S^{n-1}} \Omega(\xi)(e^{2\pi i r x'\cdot\xi} - 1)\,d\sigma(\xi) \right|^2 \frac{dr}{r^{1+2\mathrm{Re}\alpha}}$$
$$+ \|h\|_\infty^2 \int_1^\infty \left| \int_{S^{n-1}} \Omega(\xi) e^{2\pi i r x'\cdot\xi} d\sigma(\xi) \right|^2 \frac{dr}{r^{1+2\mathrm{Re}\alpha}}$$
$$\leqslant C\|h\|_\infty^2 \|\Omega\|_{L^1(S^{n-1})}^2 + C'\|h\|_\infty^2 \int_1^\infty |l(r, x)|^2 \frac{dr}{r^{1+2\mathrm{Re}\alpha}}.$$

Applying Lemma 1 and Hölder's inequality, the second term is bounded by

$$C\|h\|_\infty^2 \|\Omega\|_{L^q(S^{n-1})}^2.$$

Theorem 2 is proved.

Remark. By the same proof as in [4] or [5], we can conclude that if we let

$$\tilde{L}_0(x) = \frac{\Omega(x)h(x)}{|x|^{n+\alpha}}, \quad \mathrm{Re}\,\alpha > 0,$$

then

$$\tilde{L}_0 * \frac{1}{|x|^{n-\alpha}}$$

satisfies Hörmander's condition. And by Theorem 2, the Fourier transform of $\tilde{L}_0 * 1/|x|^{n-\alpha}$ is bounded by $\|\Omega\|_{L^q(S^{n-1})}$ if $q > 1$. Therefore,

$$\|(\tilde{L}_0 * 1/|x|^{n-\alpha}) * f\|_p \leqslant C\|h\|_\infty \|f\|_p$$

for any $1 < p < \infty$, where C depends on n, p and the norm of Ω.

LEMMA 3. *Suppose Ω in $L^1(S^{n-1})$ is homogeneous of degree zero. Let*

$$H(x) = \frac{\chi_{|x|<1}|\Omega(x)|}{|x|^{n+\alpha}}, \qquad \mathrm{Re}\,\alpha > 0,$$

and $H_\varepsilon(x) = \varepsilon^{-n} H(x/\varepsilon)$. If $f \in L^p(\boldsymbol{R}^n)$ for some $1 < p < \infty$, then

$$\| \sup_{0<\varepsilon<\infty} |H_\varepsilon * f(x)| \|_p \leqslant C_p \|\Omega\|_{L^1(S^{n-1})} \|f\|_p.$$

Proof. If $\xi \in S^{n-1}$, $f \in L^p(\boldsymbol{R}^n)$, we let

$$M_\xi f(x) = \sup_{0<a<\infty} \frac{1}{a} \int_0^a |f(x-r\xi)|\, dr.$$

Then we have the following two observations.

(1) $\|M_\xi f(\cdot)\|_{L^p(\boldsymbol{R}^n)} \leqslant C_p \|f\|_{L^p(\boldsymbol{R}^n)}$ for all $\xi \in S^{n-1}$, where C_p depends on n, p.

(2) $\varepsilon^{\mathrm{Re}\alpha} \int_\varepsilon^\infty |f(x-r\xi)| \frac{dr}{r^{1+\mathrm{Re}\alpha}} \leqslant C M_\xi f(x)$, where C depends on $\mathrm{Re}\,\alpha$.

The first one is a well-known result (see [2]). The second is easy to show, since

$$\begin{aligned}
\varepsilon^{\mathrm{Re}\alpha} \int_\varepsilon^\infty |f(x-r\xi)| \frac{dr}{r^{1+\mathrm{Re}\alpha}} &= \sum_{k=0}^\infty \varepsilon^{\mathrm{Re}\alpha} \int_{2^k\varepsilon}^{2^{k+1}\varepsilon} |f(x-r\xi)| \frac{dr}{r^{1+\mathrm{Re}\alpha}} \\
&\leqslant \sum_{k=0}^\infty \frac{1}{2^{k\mathrm{Re}\alpha}} \cdot \frac{1}{2^k \varepsilon} \int_0^{2^{k+1}\varepsilon} |f(x-r\xi)|\, dr \\
&\leqslant C M_\xi f(x).
\end{aligned}$$

Let us write

$$\begin{aligned}
|H_\varepsilon * f(x)| &= \left| \frac{\varepsilon^\alpha \chi_{|x|>\varepsilon} |\Omega(x)|}{|x|^{n+\alpha}} * f(x) \right| \\
&\leqslant \int_{S^{n-1}} |\Omega(\xi)|\, \varepsilon^{\mathrm{Re}\alpha} \int_\varepsilon^\infty |f(x-r\xi)| \frac{dr}{r^{1+\mathrm{Re}\alpha}}\, d\sigma(\xi) \\
&\leqslant C \int_{S^{n-1}} |\Omega(\xi)|\, M_\xi f(x)\, d\sigma(\xi).
\end{aligned}$$

By Minkowski's integral inequality, applied to (1), we obtain

$$\| \sup_{0<\varepsilon<\infty} |H_\varepsilon * f(x)| \|_p \leqslant C \|\Omega\|_{L^1(S^{n-1})} \|f\|_p.$$

Lemma 3 is proved.

Proof of Theorem 3. If $\mathrm{Re}\,\alpha > 0$ then

$$T_\varepsilon^\alpha f(x) = C_\alpha \left(\frac{\Omega(x) h(|x|)}{|x|^{n+\alpha}} \chi_{|x|>\varepsilon} * \frac{1}{|x|^{n-\alpha}} \right) * f(x),$$

where $C_\alpha = \dfrac{\pi^{(\alpha-n)/2}\, \Gamma((n-\alpha)/2)}{\Gamma(\alpha/2)}$. Let ψ be a positive smooth function with compact support in $\{x \in \boldsymbol{R}^n\colon |x| < 1\}$, and $\int \psi = 1$. Also let

$$L_\varepsilon(x) = \frac{\Omega(x) h(|x|)}{|x|^{n+\alpha}} \chi_{|x|>\varepsilon}, \qquad L_1(x) = \frac{\Omega(x) h(\varepsilon|x|)}{|x|^{n+\alpha}} \chi_{|x|>1},$$

$$L_0(x) = \frac{\Omega(x) h(\varepsilon|x|)}{|x|^{n+\alpha}}, \qquad \tilde{L}_0(x) = \frac{\Omega(x) h(|x|)}{|x|^{n+\alpha}}, \qquad |x| > 0.$$

Define

$$\Phi(x) = L_1 * \frac{1}{|x|^{n-\alpha}} - \psi * \left(L_0 * \frac{1}{|x|^{n-\alpha}} \right),$$

and dilating set

$$(**) \qquad \Phi_\varepsilon(x) = \frac{1}{\varepsilon^n} \Phi\left(\frac{x}{\varepsilon}\right) = L_\varepsilon * \frac{1}{|x|^{n-\alpha}} - \psi_\varepsilon * \left(\tilde{L}_0 * \frac{1}{|x|^{n-\alpha}} \right).$$

We claim that if $f \in L^p(\boldsymbol{R}^n)$, $\Omega \in L^q(S^{n-1})$ for $1 < q \leqslant \infty$, then $\| \sup_{0<\varepsilon<\infty} |\Phi_\varepsilon * f| \|_p \leqslant C\|h\|_\infty \|f\|_p$ for all $1 < p < \infty$, where C depends on n, p, and the norm of Ω.

If we can show that this claim is true, then

$$\left| \left(L_\varepsilon * \frac{1}{|x|^{n-\alpha}} \right) * f \right| \leqslant \left| \left(\psi_\varepsilon * \left(\tilde{L}_0 * \frac{1}{|x|^{n-\alpha}} \right) \right) * f \right| + |\Phi_\varepsilon * f|.$$

Therefore, by our remark and the above claim, we get

$$\left\| \sup_{0<\varepsilon<\infty} \left| L_\varepsilon * \frac{1}{|x|^{n-\alpha}} * f \right| \right\|_p \leqslant C_p \|h\|_\infty \|f\|_p$$

for all $1 < p < \infty$.

Now it suffices to prove the claim. If we can show $\Phi(x)$ is dominated by

$$\|h\|_\infty \|\Omega\|_{L^1(S^{n-1})} \left(C\chi_{|x|\leqslant 6} + \frac{C'}{|x|^{n+1-\mathrm{Re}\alpha}} \chi_{|x|>6} \right)$$

$$+ \|h\|_\infty \left(H * \frac{\chi_{|x|<2}}{|x|^{n-\mathrm{Re}\alpha}} + \psi * H * \frac{\chi_{|x|<3}}{|x|^{n-\mathrm{Re}\alpha}} + H * \frac{\chi_{|x|>2}}{|x|^{n+1-\mathrm{Re}\alpha}} \right),$$

where $H(x) = \chi_{|x|>1}|\Omega(x)|/|x|^{n+\mathrm{Re}\alpha}$, then our claim follows by applying Lemma 3. First consider $|x| \leqslant 6$; we estimate the right-hand side of (**):

$$\left|L_1 * \frac{1}{|x|^{n-\alpha}}\right| = \left|\int_{|z|>1} \frac{\Omega(z)\,h(\varepsilon|z|)}{|z|^{n+\alpha}} \cdot \frac{1}{|x-z|^{n-\alpha}}\,dz\right|$$

$$\leqslant C\|h\|_\infty \int_{\substack{|x-z|>2\\ |z|>1}} \frac{|\Omega(z)|}{|z|^{n+\mathrm{Re}\alpha}}\,dz$$

$$+\|h\|_\infty \int_{\substack{|x-z|<2\\ |z|>1}} \frac{|\Omega(z)|}{|z|^{n+\mathrm{Re}\alpha}} \cdot \frac{1}{|x-z|^{n-\mathrm{Re}\alpha}}\,dz$$

$$\leqslant C\|h\|_\infty \|\Omega\|_{L^1(S^{n-1})} + C'\|h\|_\infty H * \frac{\chi_{|x|<2}}{|x|^{n-\mathrm{Re}\alpha}}.$$

On the other hand,

$$\psi * \left(L_0 * \frac{1}{|x|^{n-\alpha}}\right) = \int_{|y|<1} \psi(y) \int_{|z|<1} \frac{\Omega(z)\,h(\varepsilon|z|)}{|z|^{n+\alpha}} \cdot \frac{1}{|x-y-z|^{n-\alpha}}\,dz\,dy$$

$$+ \int_{|y|<1} \psi(y) \int_{|z|>1} \frac{\Omega(z)\,h(\varepsilon|z|)}{|z|^{n+\alpha}} \cdot \frac{1}{|x-y-z|^{n-\alpha}}\,dz\,dy.$$

By the same method as in the above proof, the second term is bounded by

$$\|h\|_\infty \|\Omega\|_{L^1(S^{n-1})} + C'\|h\|_\infty \psi * H * \frac{\chi_{|x|<2}}{|x|^{n-\mathrm{Re}\alpha}}.$$

The first term, by Fubini's theorem and the cancellation of Ω, equals

$$\int_{|z|<1} \frac{\Omega(z)\,h(\varepsilon|z|)}{|z|^{n+\alpha}} \left\{\int_{|y|<1} \psi(y) \frac{1}{|x-y-z|^{n-\alpha}}\,dy - \int_{|y|<1} \psi(y) \frac{1}{|x-y|^{n-\alpha}}\,dy\right\} dz.$$

But, after a change of variables, the term in parentheses is

$$\int_{\mathbb{R}^n} |\psi(x-y-z) - \psi(x-y)| \frac{1}{|y|^{n-\mathrm{Re}\alpha}}\,dy \leqslant C|z| \int_{|y|<8} \frac{1}{|y|^{n-\mathrm{Re}\alpha}}\,dy$$

$$\leqslant c|z| \quad \text{(if } \mathrm{Re}\alpha > 0 \text{ and very small).}$$

Therefore, the first term is bounded by $\|h\|_\infty \|\Omega\|_{L^1(S^{n-1})}$. Finally, when $|x| \geqslant 6$,

$$\Phi(x) = L_1 * \frac{1}{|x|^{n-\alpha}} - \psi * \left(L_0 * \frac{1}{|x|^{n-\alpha}}\right)$$

$$= \int_{|z|>1} \frac{\Omega(z)\,h(\varepsilon|z|)}{|z|^{n+\alpha}} \cdot \frac{1}{|x-z|^{n-\alpha}}\,dz$$

$$- \int_{|y|<1} \psi(y) \int_{\mathbb{R}^n} \frac{\Omega(z)\,h(\varepsilon|z|)}{|z|^{n+\alpha}} \cdot \frac{1}{|x-y-z|^{n-\alpha}}\,dz\,dy$$

$$= \int_{|y|<1} \psi(y) \int_{|z|>1} \frac{\Omega(z)\,h(\varepsilon|z|)}{|z|^{n+\alpha}} \left(\frac{1}{|x-z|^{n-\alpha}} - \frac{1}{|x-y-z|^{n-\alpha}}\right) dz\,dy$$

$$- \int_{|y|<1} \psi(y) \int_{|z|<1} \frac{\Omega(z)\,h(\varepsilon|z|)}{|z|^{n+\alpha}} \cdot \frac{1}{|x-y-z|^{n-\alpha}}\,dz\,dy.$$

By the cancellation of $\Omega(x)$, the second term is equal to

$$\left|\int_{|y|<1} \psi(y) \int_{|z|<1} \frac{\Omega(z)\,h(\varepsilon|z|)}{|z|^{n+\alpha}} \left(\frac{1}{|x-y-z|^{n-\alpha}} - \frac{1}{|x-y|^{n-\alpha}}\right) dz\,dy\right|$$

$$\leqslant C\|h\|_\infty \|\Omega\|_{L^1(S^{n-1})} \frac{1}{|x|^{n+1-\mathrm{Re}\alpha}} \quad \text{(since } |x| > 6\text{)}.$$

Next, we separate the first term into

$$\int_{|y|<1} \int_{\substack{|x-z|>2\\ |z|>1}} \ldots dz\,dy + \int_{|y|<1} \int_{\substack{|x-z|<2\\ |z|>1}} \ldots dz\,dy = \mathrm{I} + \mathrm{II},$$

$$|\mathrm{I}| = \left|\int_{|y|<1} \psi(y) \int_{\substack{|x-z|>2\\ |z|>1}} \frac{\Omega(z)\,h(\varepsilon|z|)}{|z|^{n+\alpha}} \left(\frac{1}{|x-z|^{n-\alpha}} - \frac{1}{|x-y-z|^{n-\alpha}}\right) dz\,dy\right|$$

$$\leqslant C\|h\|_\infty \int_{\substack{|x-z|>2\\ |z|>1}} \frac{|\Omega(z)|}{|z|^{n+\mathrm{Re}\alpha}} \cdot \frac{1}{|x-z|^{n+1-\mathrm{Re}\alpha}}\,dz$$

$$= C\|h\|_\infty H * \frac{\chi_{|x|>2}}{|x|^{n+1-\mathrm{Re}\alpha}},$$

$$|\mathrm{II}| = \left|\int_{|y|<1} \psi(y) \int_{\substack{|x-z|<2\\ |z|>1}} \frac{\Omega(z)\,h(\varepsilon|z|)}{|z|^{n+\alpha}} \left(\frac{1}{|x-z|^{n-\alpha}} - \frac{1}{|x-y-z|^{n-\alpha}}\right) dz\,dy\right|$$

$$\leqslant C\|h\|_\infty \left(H * \frac{\chi_{|x|<2}}{|x|^{n-\mathrm{Re}\alpha}} + \psi * H * \frac{\chi_{|x|<3}}{|x|^{n-\mathrm{Re}\alpha}}\right).$$

We have proved our claim; thus Theorem 3 is proved.

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On subsequences of the Haar basis in $H^1(\delta)$ and isomorphism between H^1-spaces

by

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Abstract. We classify and characterize the subspaces of $H^1(\delta)$ spanned by subsequences of the Haar basis. l_1 and $(\sum H^1_n)_{l^1}$ and $H^1(\delta)$ are the only isomorphic types which occur in this way. We also give a necessary and sufficient condition on an increasing sequence of fields $(\mathscr{F}_n)$ for $H^1((\mathscr{F}_n))$ to be linearly isomorphic to $H^1(\delta)$, thus verifying a conjecture of B. Maurey.

Introduction. To the pair (n, i), $n \in \mathbf{N}$, $0 \leqslant i \leqslant 2^n-1$, we associate the dyadic interval $(ni) = (2^{-n}i, 2^{-n}(i+1)]$ and the Haar function h_{ni} which is 1 on the left half of $(2^{-n}i, 2^{-n}(i+1)]$, -1 on the right half and zero elsewhere. The σ-algebra generated by the sets $\{(2^{-n}i, 2^{-n}(i+1)]\colon 0 \leqslant i \leqslant 2^n-1\}$ is denoted by $\mathscr{E}_n$. Dyadic intervals are nested in the sense that if $I \cap J \neq \emptyset$ then either $I \subset J$ or $J \subset I$.

We will work in the following setting: Given $f = \sum_{(ni)} a_{ni} h_{ni}$ in $L^1(0, 1]$, we write

$$S(f) = \Big(\sum_{(ni)} a_{ni}^2 h_{ni}^2\Big)^{1/2} \quad \text{and} \quad \|f\|_{H^1(\delta)} = \int S(f),$$

$$H^1(\delta) = \{f \in L^1\colon \|f\|_{H^1(\delta)} < \infty\}.$$

H^1_n denotes the subspace of $H^1(\delta)$ which is spanned by $\{h_{mj}\colon m \leqslant n, 0 \leqslant j \leqslant 2^m-1\}$, and

$$\Big(\sum H^1_n\Big)_{l^1} = \{(f_n)_{n \in \mathbf{N}}\colon f_n \in H^1_n \text{ and } \sum \|f_n\| < \infty\}.$$

Given $f \in L^1(0, 1]$ and a dyadic interval I we write $f_I = |I|^{-1} \int_I f$ and

$$\|f\|_{\mathrm{BMO}(\delta)} = \sup\{(|I|^{-1} \int_I |f-f_I|^2)^{1/2}\colon I \text{ a dyadic interval}\},$$

$$\mathrm{BMO}(\delta) = \{f \in L^1\colon \int f = 0 \wedge \|f\|_{\mathrm{BMO}(\delta)} < \infty\}.$$

The connection between $\mathrm{BMO}(\delta)$ and $H^1(\delta)$ is given by the following formula:

$$\|f\|_{H^1(\delta)} = \sup\{|\int fg|\colon \|g\|_{\mathrm{BMO}} = 1 \wedge g \in L^\infty\}.$$