that $UV \in \mathcal{S}'(\mathbf{R}^n)$, we introduce the notion of *$\mathcal{S}'$-product*, denoted by $U \odot V$. It is one of the limits (i)–(iii), under the assumption that conditions (i)–(vi) hold in the case of $\mathcal{S}'(\mathbf{R}^n)$. So $U \odot V$ is a tempered distribution.

Now we give a generalization of Theorem 0.3 on the exchange formula:

THEOREM 2. *Let $U, V \in \mathcal{S}'(\mathbf{R}^n)$. If the improper $\mathcal{S}'$-convolution of U, V exists then the $\mathcal{S}'$-product of their Fourier transforms exists and*

$$(U \circledast V)^{\wedge} = (2\pi)^{n/2}\, \hat{U} \odot \hat{V}.$$

The same is true for the inverse Fourier transform.

The theorem is a consequence of Theorem 1 and Corollary 3.1 in the case of $\mathcal{S}'(\mathbf{R}^n)$ and the definitions of improper $\mathcal{S}'$-convolution and $\mathcal{S}'$-product. We omit the simple proof.

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INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
Śniadeckich 8, 00-950 Warszawa, Poland

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Nonlinear transformations on spaces of continuous functions

by

HANS VOLKMER (Essen) and HANS WEBER (Potenza)

Abstract. We study nonlinear transformations on spaces of continuous functions with values in a Banach space. The continuous functions are defined on an arbitrary topological space and have totally bounded range in a second Banach space. In particular, we consider transformations which satisfy the Hammerstein property of Batt [1973] and integral operators. Our results, which are obtained by a systematic use of the semivariation, generalize some of the results of Batt. Further, we point out some connections between our results and the theory of locally solid Riesz spaces and abstract integration theory.

0. Introduction. This paper deals with a generalization of the classical Riesz representation theorem to nonlinear transformations. In [VW2] we have given a new approach to the representation of continuous linear operators $T: \mathscr{C}(\Omega, E) \to F$, where Ω is an arbitrary topological space, E and F are Banach spaces and $\mathscr{C}(\Omega, E)$ denotes the space of all E-valued continuous functions on Ω with totally bounded range endowed with the topology of uniform convergence. J. Batt asked whether it is possible to obtain with our method [VW2] his integral representation theorem for certain nonlinear transformations [B2]. Batt investigated in [B2], for a compact Hausdorff space Ω, transformations $T: \mathscr{C}(\Omega, E) \to F$ which are uniformly continuous on bounded sets and satisfy $T0 = 0$ and the "Hammerstein property"

$$T(f+f_1+f_2)+T(f) = T(f+f_1)+T(f+f_2)$$

for all $f, f_1, f_2 \in \mathscr{C}(\Omega, E)$ with f_1 and f_2 having disjoint supports. For a short description of the relation with earlier research of Drewnowski–Orlicz, Mizel–Sundaresan, Chacon–Friedman, Friedman–Katz and Friedman–Tong, see [B1].

Our approach to the representation of nonlinear transformations $T: \mathscr{C}(\Omega, E) \to F$ is based on a systematic study of the semivariation of T (= modulus of continuity) in Section 1. The semivariation is used to obtain a continuous extension of T in Section 2. In Theorem (4.3)(a) we show the following result, which may be considered as the kernel of a representation theorem: Let Ω be an arbitrary (not necessarily compact) topological space.

Denote by A the algebra generated by the cozero sets $\{\varphi>0\}$ $(:=\{x\in\Omega\colon \varphi(x)>0\})$, $\varphi\in\mathscr{C}(\Omega,\boldsymbol{R})$, and by $\mathfrak{M}(A,E)$ the uniform closure of the space $\mathfrak{E}(A,E)$ of A-simple functions with values in E. Then the restriction $S\mapsto S|\mathscr{C}(\Omega,E)$ defines an algebraic isomorphism between the space of all transformations $S\colon \mathfrak{M}(A,E)\to F$ which are uniformly continuous on bounded sets and "regular" and the space of all transformations $T\colon \mathscr{C}(\Omega,E)\to F$ with these properties. This result also makes transparent the role of the Hammerstein property for an integral representation $Tf=\int f\,d\mu$ $(f\in\mathscr{C}(\Omega,E))$ for certain transformations $T\colon \mathscr{C}(\Omega,E)\to F$; the Hammerstein property guarantees that a transformation $S\colon \mathfrak{M}(A,E)\to F$ can be considered as an integral $Sf=\int f\,d\mu$ with respect to the set function μ defined by $\mu(A)y:=S(\chi_A y)$ (cf. Sections 3 and 4).

In Sections 5 and 6 we examine various properties of transformations $T\colon \mathscr{C}(\Omega,E)\to F$, namely s-boundedness, weak compactness, σ-smoothness and τ-smoothness, further, the connection between these properties of the represented transformation and the representing content. For the transformations considered, weak compactness implies s-boundedness, and s-boundedness implies regularity, hence representability in the sense mentioned above. If Ω is compact, the transformations considered are always τ-smooth, hence σ-smooth. Our approach is related to an abstract integration theory. Combining both we immediately get a characterization of s-bounded σ-smooth transformations $T\colon \mathscr{C}(\Omega,E)\to F$ with the Hammerstein property, which generalizes a result of Brooks–Lewis [BL] for T linear and Ω compact.

1. The semivariation. Throughout this paper, Ω is a topological space and $(E,|\ |)$, $(F,|\ |)$ are Banach spaces over the real or complex field. In this section let

$$T\colon \mathscr{F}\to F$$

be a nonlinear transformation defined on a linear subspace $\mathscr{F}$ of the space $\mathfrak{B}(\Omega,E)$ of all bounded functions on Ω with values in E. If not explicitly stated otherwise we consider $\mathfrak{B}(\Omega,E)$ and its subspaces as normed spaces with the sup-norm $\|f\|_\infty:=\sup\{|f(x)|\colon x\in\Omega\}$. The function space $\mathscr{F}$ will mostly be one of the following spaces:

(i) The space $\mathscr{C}(\Omega,E)$ of all continuous functions on Ω with totally bounded range in E.

(ii) The space $\mathfrak{E}(A,E)$ of A-simple functions on Ω with values in E where A is an algebra of subsets of Ω. More precisely, $\mathfrak{E}(A,E)$ is the linear span of $\{\chi_A y\colon A\in A,\ y\in E\}$ where χ_A denotes the characteristic function of A.

(iii) The space $\mathfrak{M}(A,E)$ of totally A-measurable functions on Ω with values in E. This space is defined as the topological closure of $\mathfrak{E}(A,E)$ in $\mathfrak{B}(\Omega,E)$.

We denote the corresponding vector lattices of real-valued functions by $\mathfrak{B}(\Omega)$, $\mathscr{C}(\Omega)$, $\mathfrak{E}(A)$, $\mathfrak{M}(A)$.

We shall use Riesz pseudonorms on these vector lattices to generate topologies. A *Riesz pseudonorm* (see [AB, p. 39]) is a map p defined on a vector lattice L such that

(i) $p\colon L\to[0,\infty[$.

(ii) $p(\varphi+\psi)\leqslant p(\varphi)+p(\psi)$ for all $\varphi,\psi\in L$.

(iii) $p(\lambda\varphi)\to 0$ as $\lambda\to 0$, for each $\varphi\in L$.

(iv) $p(\varphi)\leqslant p(\psi)$ whenever $|\varphi|\leqslant|\psi|$ holds in L.

Our pseudonorms will be defined in terms of the semivariation of the given transformation T. The *semivariation* of T with respect to $\alpha>0$,

$$\|\ \|_{T,\alpha}\colon \mathfrak{B}(\Omega)\to[0,\infty],$$

is defined by

$$\|\varphi\|_{T,\alpha}:=\sup\{|Tf-Tg|\colon f,g\in\mathscr{F}_\alpha,\ |f-g|\leqslant|\varphi|\}$$

where

$$\mathscr{F}_\alpha:=\{h\in\mathscr{F}\colon \|h\|_\infty\leqslant\alpha\},\qquad |h|(x):=|h(x)|\quad \text{for } h\colon \Omega\to E.$$

Obviously, the semivariation satisfies property (iv) of a Riesz pseudonorm. The transformation T is uniformly continuous on $\mathscr{F}_\alpha$ if and only if condition (iii) holds. In this case the range of T on $\mathscr{F}_\alpha$ is bounded and, therefore, the range of $\|\ \|_{T,\alpha}$ is also bounded, hence (i) holds. In general, the semivariation does not satisfy (ii) on the whole lattice $\mathfrak{B}(\Omega)$. The following lemma is useful in this context.

(1.1) LEMMA. *Let $\mathscr{F}$, $\mathscr{F}'$ be one of the pairs of spaces $\mathscr{C}(\Omega,E)$, $\mathscr{C}(\Omega)$ or $\mathfrak{E}(A,E)$, $\mathfrak{E}(A)$ or $\mathfrak{M}(A,E)$, $\mathfrak{M}(A)$. Let $f,g\in\mathscr{F}_\alpha$, $\varphi\in\mathfrak{B}(\Omega)$, $\psi\in\mathscr{F}'$ and assume that $|f-g|\leqslant|\varphi+\psi|$. Then there are sequences k_n, l_n in $\mathscr{F}_\alpha$ such that $|f-k_n|\leqslant|\psi|$, $|l_n-g|\leqslant|\varphi|$ and $|k_n-l_n|\leqslant 1/n$ $(=1/n\cdot\chi_\Omega)$.*

Proof. Let $h=f-g$ and set

$$\varrho:=\frac{|\psi|\wedge|h|}{1/n\vee|h|},\qquad \sigma:=\frac{|h|-|\psi|\wedge|h|}{1/n\vee|h|},$$

$$k_n:=(1-\varrho)f+\varrho g,\qquad l_n:=\sigma f+(1-\sigma)g,$$

where $\wedge$ and $\vee$ denote the pointwise infimum and supremum, respectively. Since $0\leqslant\varrho\leqslant 1$, $0\leqslant\sigma\leqslant 1$, the functions k_n and l_n are in $\mathscr{F}_\alpha$. Moreover, $|f-k_n|=\varrho|h|\leqslant|\psi|$ and $|l_n-g|=\sigma|h|\leqslant|\varphi|$. From $\varrho+\sigma=|h|/(1/n\vee|h|)$ we obtain

$$0\leqslant 1-(\varrho+\sigma)=\frac{1/n\vee|h|-|h|}{1/n\vee|h|}\leqslant\frac{1/n}{1/n\vee|h|}.$$

Hence $|k_n-l_n|=(1-(\varrho+\sigma))|h|\leqslant 1/n$.

(1.2) PROPOSITION. *Let $\mathscr{F}$, $\mathscr{F}'$ be as in (1.1) and assume that T is uniformly continuous on $\mathscr{F}_\alpha$. Then*

$$\|\psi+\varphi\|_{T,\alpha} \leqslant \|\psi\|_{T,\alpha}+\|\varphi\|_{T,\alpha} \quad \text{for all } \varphi\in\mathfrak{B}(\Omega) \text{ and } \psi\in\mathscr{F}'.$$

In particular, the restriction $\| \ \|_{T,\alpha}|\mathscr{F}'$ of the semivariation $\| \ \|_{T,\alpha}$ to $\mathscr{F}'$ is a Riesz pseudonorm.

Proof. Let $f, g\in\mathscr{F}_\alpha$ and assume that $|f-g| \leqslant |\varphi+\psi|$. Choose k_n and l_n according to Lemma (1.1). Then

$$\begin{aligned}|Tf-Tg| &\leqslant |Tf-Tk_n|+|Tl_n-Tg|+|Tk_n-Tl_n|\\ &\leqslant \|\psi\|_{T,\alpha}+\|\varphi\|_{T,\alpha}+\|1/n\|_{T,\alpha}\\ &\to \|\psi\|_{T,\alpha}+\|\varphi\|_{T,\alpha} \quad \text{as } n\to\infty\end{aligned}$$

because T is uniformly continuous on $\mathscr{F}_\alpha$. It follows that $|Tf-Tg| \leqslant \|\psi\|_{T,\alpha}+\|\varphi\|_{T,\alpha}$ and this proves the proposition.

We remark that (1.1) and (1.2) are valid for more general spaces $\mathscr{F}$, $\mathscr{F}'$. It suffices to assume that $\mathscr{F}$ is a linear subspace of $\mathfrak{B}(\Omega, E)$, $\mathscr{F}'$ is a vector lattice in $\mathfrak{B}(\Omega)$, $|\mathscr{F}|\subset\mathscr{F}'$, $\mathscr{F}'\cdot\mathscr{F}\subset\mathscr{F}$, $1\in\mathscr{F}'$ and $\varphi/\psi\in\mathscr{F}'$ whenever $\varphi, \psi\in\mathscr{F}'$ are such that $\inf\{\psi(x)\colon x\in\Omega\} > 0$.

If $\mathscr{F}\in\mathscr{C}(\Omega, E)$ then we can prove a further result on the subadditivity of the semivariation. As in [VW2, p. 174] we set

$$\mathfrak{L} := \{\varphi\in\mathfrak{B}(\Omega)\colon \varphi\geqslant 0 \text{ and } \{\varphi>t\}\in \boldsymbol{P} \text{ for all real } t\}$$

where $\boldsymbol{P}$ is the system of cozero sets in Ω. The system of zero sets will be denoted by $\boldsymbol{Z}$. The definitions of $\boldsymbol{P}$ and $\boldsymbol{Z}$ as well as the basic properties of these systems which we need in the sequel can be found in [VW2, pp. 173–174]. The book [GJ, 1.10–1.15] contains a more detailed study of these systems.

(1.3). PROPOSITION. *Let $T\colon \mathscr{C}(\Omega, E)\to F$ be uniformly continuous on $\mathscr{C}(\Omega, E)_\alpha$. Then*

$$\|\varphi_1+\varphi_2\|_{T,\alpha} \leqslant \|\varphi_1\|_{T,\alpha}+\|\varphi_2\|_{T,\alpha}$$

for all $\varphi_1, \varphi_2\in\mathfrak{L}$.

The proof parallels that of [VW2, (1.1.3)] and is omitted. The reference in that proof to [VW1] has to be replaced by Proposition (1.2) of this paper.

If we use the system of open sets instead of the system of cozero sets in the definition of $\mathfrak{L}$ then, in general, Proposition (1.3) is true only if Ω is normal.

2. Extension of transformations on $\mathscr{C}(\Omega, E)$. In this section let $T\colon \mathscr{C}(\Omega, E)\to F$ be a nonlinear transformation which is uniformly continuous on bounded subsets of $\mathscr{C}(\Omega, E)$. For each $\alpha>0$, we define $\| \ \|_\alpha\colon \mathfrak{B}(\Omega)\to[0, \infty[$ by

$$\|\varphi\|_\alpha := \inf\{\|\psi\|_{T,\alpha}\colon |\varphi|\leqslant\psi\in\mathfrak{L}\}.$$

It follows easily from Proposition (1.3) and $\mathfrak{L}+\mathfrak{L}\subset\mathfrak{L}$ that $\| \ \|_\alpha$ is a Riesz pseudonorm. If $\varphi\in\mathfrak{L}$ then $\|\varphi\|_\alpha$ coincides with the semivariation $\|\varphi\|_{T,\alpha}$. The system $(\| \ \|_\alpha)_{\alpha>0}$ generates a vector topology τ' on $\mathfrak{B}(\Omega)$. This topology is locally solid in the sense of [AB, p. 33]. The topology τ' can be generated by a single Riesz pseudonorm, for instance,

$$\|\varphi\| := \sum_{n=1}^{\infty} 2^{-n}(\|\varphi\|_n \wedge 1),$$

but we shall not use this pseudonorm.

Let $\mathfrak{I}(\Omega)$ denote the τ'-closure of $\mathscr{C}(\Omega)$. Then $\mathfrak{I}(\Omega)$ is a vector lattice in $\mathfrak{B}(\Omega)$. For each $\alpha>0$, we define

$$\||\ \||_\alpha\colon \mathfrak{B}(\Omega, E)\to[0, \infty[$$

by $\||f\||_\alpha := \|\,|f|\,\|_\alpha$. The system $(\||\ \||_\alpha)_{\alpha>0}$ generates a vector topology τ on $\mathfrak{B}(\Omega, E)$. Let $\mathfrak{I}(\Omega, E)$ denote the closure of $\mathscr{C}(\Omega, E)$ with respect to this topology. Then $\mathfrak{I}(\Omega, E)$ is a linear subspace of $\mathfrak{B}(\Omega, E)$.

The following lemma is obvious.

(2.1) LEMMA. (a) *The τ'- and τ-topologies are coarser than the topologies generated by the sup-norm $\| \ \|_\infty$ on $\mathfrak{B}(\Omega)$, $\mathfrak{B}(\Omega, E)$, respectively.*

(b) *For all $f, g\in\mathscr{C}(\Omega, E)_\alpha$,*

$$|Tf-Tg| \leqslant \||f-g\||_\alpha.$$

Hence T is uniformly τ-continuous on $\mathscr{C}(\Omega, E)_\alpha$ for each $\alpha>0$.

(2.2) THEOREM. (a) *There exists a uniquely determined extension $\bar{T}\colon \mathfrak{I}(\Omega, E)\to F$ of T which is uniformly τ-continuous on all $\| \ \|_\infty$-bounded subsets of $\mathfrak{I}(\Omega, E)$.*

(b) *This extension satisfies*

$$|\bar{T}f-\bar{T}g| \leqslant \||f-g\||_\alpha$$

for all $f, g\in\mathfrak{I}(\Omega, E)_\alpha$, in particular,

$$\|\varphi\|_{\bar{T},\alpha} \leqslant \|\varphi\|_\alpha$$

for every $\varphi\in\mathfrak{B}(\Omega)$.

Proof. First we prove that $\mathfrak{I}(\Omega, E)_\alpha$ is contained in the τ-closure of $\mathscr{C}(\Omega, E)_\alpha$ for each $\alpha>0$. Let $f\in\mathfrak{I}(\Omega, E)_\alpha$. Then there is a sequence $f_n\in\mathscr{C}(\Omega, E)$ such that f_n is τ-convergent to f. Define a map $\varrho\colon E\to E$ by

$$\varrho(y) := \begin{cases} y & \text{if } |y|\leqslant\alpha,\\ (\alpha/|y|)\,y & \text{if } |y|>\alpha.\end{cases}$$

Then ϱ satisfies the Lipschitz condition

$$|\varrho(y_1)-\varrho(y_2)| \leqslant 2|y_1-y_2|$$

for all $y_1, y_2 \in E$. Hence the functions $g_n := \varrho \circ f_n$ belong to $\mathscr{C}(\Omega, E)_\alpha$ and

$$|f-g_n| = |\varrho \circ f - \varrho \circ f_n| \leqslant 2|f-f_n|.$$

It follows that g_n is τ-convergent to f, consequently, f is in the τ-closure of $\mathscr{C}(\Omega, E)_\alpha$. By this result and by Lemma (2.1)(b), $T|\mathscr{C}(\Omega, E)_\alpha$ can be extended to a uniformly τ-continuous transformation on $\mathfrak{I}(\Omega, E)_\alpha$. Since this is true for all $\alpha > 0$ the statement (a) follows. Part (b) of the theorem is now obvious.

We call a subset M of Ω *T-measurable* if $\chi_M y \in \mathfrak{I}(\Omega, E)$ for all $y \in E$. The system of all T-measurable sets is denoted by $\boldsymbol{M}$.

The next theorem gives some alternative descriptions of T-measurable sets.

(2.3) THEOREM. *The following five statements are equivalent for every cozero set M. If M is an arbitrary subset of Ω then the first three statements are equivalent.*

(i) $M \in \boldsymbol{M}$.

(ii) $\chi_M \in \mathfrak{I}(\Omega)$.

(iii) *For all $\alpha > 0$ and $\varepsilon > 0$, there are sets $Z \in \boldsymbol{Z}$ and $P \in \boldsymbol{P}$ such that $Z \subset M \subset P$ and $\|\chi_{P \setminus Z}\|_{T,\alpha} < \varepsilon$.*

(iv) *The system $\{\psi \in \mathscr{C}(\Omega)\colon 0 \leqslant \psi \leqslant \chi_M\}$ increasingly directed by the usual pointwise-defined order relation is Cauchy in $(\mathfrak{B}(\Omega), \tau')$.*

(v) *The system $\{\psi \in \mathscr{C}(\Omega)\colon 0 \leqslant \psi \leqslant \chi_M\}$ is τ'-convergent to χ_M.*

The proof is similar to that of [VW2, Theorem (1.3.2)] and is omitted. Since $\mathfrak{I}(\Omega)$ is a vector lattice which contains χ_Ω, the equivalence of (i) and (ii) shows that $\boldsymbol{M}$ is an algebra of sets. If there are enough T-measurable sets then T can be extended by continuity to a transformation defined on a space of totally measurable functions:

(2.4) THEOREM. *Assume that all cozero sets in Ω are T-measurable, and let A denote an algebra of subsets of Ω such that $\boldsymbol{P} \subset A \subset \boldsymbol{M}$. Then $\mathscr{C}(\Omega, E)$ is contained in $\mathfrak{M}(A, E)$, and $S := \bar{T}|\mathfrak{M}(A, E)$ is an extension of T. Moreover, S is uniformly continuous on bounded subsets of $\mathfrak{M}(A, E)$ and the following relations hold for the semivariations*:

$$\|\ \|_{T,\alpha} \leqslant \|\ \|_{S,\alpha} \leqslant \|\ \|_{\bar{T},\alpha},$$

$$\|\varphi\|_{T,\alpha} = \|\varphi\|_{S,\alpha} = \|\varphi\|_{\bar{T},\alpha} \quad \textit{for every } \varphi \in \mathfrak{L}.$$

Proof. The inclusion $\mathscr{C}(\Omega, E) \subset \mathfrak{M}(A, E)$ is shown in [VW2, (1.2.4)]. By the definition of $\boldsymbol{M}$, the inclusion $\mathfrak{E}(A, E) \subset \mathfrak{I}(\Omega, E)$ holds. By (2.1)(a), it follows that $\mathfrak{M}(A, E) \subset \mathfrak{I}(\Omega, E)$. Hence we can define $S = \bar{T}|\mathfrak{M}(A, E)$. By

Theorem (2.2), S is uniformly τ-continuous on $\|\ \|_\infty$-bounded subsets of $\mathfrak{M}(A, E)$. Since uniform τ-continuity implies uniform continuity with respect to $\|\ \|_\infty$, S is uniformly continuous on bounded subsets of $\mathfrak{M}(A, E)$, too. Since $T \subset S \subset \bar{T}$, we obtain

$$\|\ \|_{T,\alpha} \leqslant \|\ \|_{S,\alpha} \leqslant \|\ \|_{\bar{T},\alpha}.$$

If $\varphi \in \mathfrak{L}$ then, by (2.2)(b) and the definition of $\|\ \|_\alpha$,

$$\|\varphi\|_{\bar{T},\alpha} \leqslant \|\varphi\|_\alpha = \|\varphi\|_{T,\alpha}.$$

This completes the proof of the theorem.

The s-bounded transformations form an important class of transformations which satisfy the assumption $\boldsymbol{P} \subset \boldsymbol{M}$ of Theorem (2.4). T is called *s-bounded* if, for every $\alpha > 0$ and every sequence $\varphi_n \in \mathscr{C}(\Omega)$ such that $0 \leqslant \varphi_n \leqslant 1$ and $\varphi_n \wedge \varphi_m = 0$ for $m \neq n$, the sequence $\|\varphi_n\|_{T,\alpha}$ converges to zero.

(2.5) THEOREM. *If T is s-bounded then every cozero set in Ω is T-measurable.*

Proof. We assume that there is a set $P \in \boldsymbol{P}$ which is not in $\boldsymbol{M}$. Then, by (2.3) (i) $\Leftrightarrow$ (iii), there are $\alpha > 0$, $\varepsilon > 0$ such that $\|\chi_{P \setminus Z}\|_{T,\alpha} > \varepsilon$ for each zero set $Z \subset P$. Choose $Z_0 \in \boldsymbol{Z}$, $Z_0 \subset P$. Since $\|\chi_{P \setminus Z_0}\|_{T,\alpha} > \varepsilon$, there is $\varphi_1 \in \mathscr{C}(\Omega)$ such that $0 \leqslant \varphi_1 \leqslant \chi_{P \setminus Z_0}$ and $\|\varphi_1\|_{T,\alpha} > \varepsilon$. It follows from the next lemma that we can assume that there is a set $Z_1 \in \boldsymbol{Z}$ such that $\{\varphi_1 \neq 0\} \subset Z_1 \subset P \setminus Z_0$.

We now repeat the previous argument with $Z_0 \cup Z_1$ replacing Z_0. We obtain $\varphi_2 \in \mathscr{C}(\Omega)$ and $Z_2 \in \boldsymbol{Z}$ such that $0 \leqslant \varphi_2 \leqslant 1$, $\|\varphi_2\|_{T,\alpha} > \varepsilon$ and $\{\varphi_2 \neq 0\} \subset Z_2 \subset P \setminus (Z_0 \cup Z_1)$. In this way we construct a sequence $\varphi_n \in \mathscr{C}(\Omega)$ such that $0 \leqslant \varphi_n \leqslant 1$, $\varphi_n \wedge \varphi_m = 0$ for $m \neq n$ and $\|\varphi_n\|_{T,\alpha} > \varepsilon$. Hence T is not s-bounded.

(2.6) LEMMA. *Let $\varphi \in \mathscr{C}(\Omega)$ and $\|\varphi\|_{T,\alpha} > \gamma$ for some $\gamma > 0$. Then there exist $\psi \in \mathscr{C}(\Omega)$ and $Z \in \boldsymbol{Z}$ such that $\|\psi\|_{T,\alpha} > \gamma$, $0 \leqslant \psi \leqslant |\varphi|$ and $\{\psi \neq 0\} \subset Z \subset \{\varphi \neq 0\}$.*

Proof. We can assume that $\varphi \geqslant 0$. Then set $\psi_n := (\varphi - 1/n) \vee 0$. We see that $\psi_n \in \mathscr{C}(\Omega)$, $0 \leqslant \psi_n \leqslant \varphi$ and $\|\psi_n - \varphi\|_\infty \to 0$ as $n \to \infty$. It follows that $\|\psi_n\|_{T,\alpha} \to \|\varphi\|_{T,\alpha}$. Hence $\psi := \psi_n$, $Z := \{\varphi \geqslant 1/n\}$ satisfy the statement of the lemma for sufficiently large n.

We remark that the s-boundedness of T can be reformulated in several ways. For instance, Lemma (2.6) shows that T is s-bounded if and only if $\|\chi_{Z_n}\|_{T,\alpha} \to 0$ as $n \to \infty$ for every $\alpha > 0$ and every sequence Z_n of disjoint zero sets. Our definition of s-boundedness is also equivalent to the pre-Lebesgue property of the locally solid Riesz space $(\mathscr{C}(\Omega), \tau')$ in the sense of [AB, p. 53]. This follows from [AB, Theorem 10.1 (i) $\Leftrightarrow$ (iii)]. The same theorem shows that the pre-Lebesgue property of $(\mathscr{C}(\Omega), \tau')$ is equivalent to the property that every increasingly directed net (φ_i) in $\mathscr{C}(\Omega)$ which satisfies $0 \leqslant \varphi_i \leqslant \varphi$ for some $\varphi \in \mathscr{C}(\Omega)$ and every i, is a τ'-Cauchy net. This yields a second proof of Theorem (2.5) because of the equivalence (i) $\Leftrightarrow$ (iv) of Theorem (2.3). Another equivalent formulation of s-boundedness of T is that, for every $\alpha > 0$ and every sequence $\varphi_n \in \mathscr{C}(\Omega)$ such that

$$\sum_{n=1}^{\infty} |\varphi_n|$$

is bounded, the sequence $\|\varphi_n\|_{T,\alpha}$ converges to zero. This property is called "schwach halbadditiv" in [S, Section 2.5].

3. Transformations on $\mathfrak{E}(A, E)$ or $\mathfrak{M}(A, E)$ and the Hammerstein property. The aim of this section is to introduce the Hammerstein property and an integral for totally measurable functions. We need both concepts for the integral representation presented in the next section. Our notions coincide with those of [B2, Sections 1, 2]. As in [B2, p. 147] we denote by $M(E, F)$ the linear space of all transformations $U: E \to F$ which satisfy $U0 = 0$ and are uniformly continuous on bounded subsets of E. Let $\mathscr{F}$ be a linear subspace of $\mathfrak{B}(\Omega, E)$ and $T \in M(\mathscr{F}, F)$. Then T satisfies the *Hammerstein property* if

$$T(f+f_1+f_2)+Tf = T(f+f_1)+T(f+f_2)$$

for all $f, f_1, f_2 \in \mathscr{F}$ such that $|f_1| \wedge |f_2| = 0$. The space of all $T \in M(\mathscr{F}, F)$ which satisfy the Hammerstein property is denoted by $M_{\mathrm{HP}}(\mathscr{F}, F)$.

If $\mathscr{F} = \mathscr{C}(\Omega, E)$ then Batt uses $\overline{\{f_1 \neq 0\}} \cap \overline{\{f_2 \neq 0\}} = \emptyset$ instead of $|f_1| \wedge |f_2| = 0$ in the definition of the Hammerstein property. We note that both definitions are equivalent. This follows from the continuity of T and the following observation. If $f_1, f_2 \in \mathscr{C}(\Omega, E)$ and $|f_1| \wedge |f_2| = 0$ then the functions

$$f_{in} := \frac{(|f_i| - 1/n) \vee 0}{|f_i| \vee 1/n} f_i \in \mathscr{C}(\Omega, E)$$

satisfy $\|f_{in} - f_i\|_\infty \leqslant 1/n \to 0$ as $n \to \infty$ and $\overline{\{f_{1n} \neq 0\}} \cap \overline{\{f_{2n} \neq 0\}} = \emptyset$ for every n.

We also note that the Hammerstein property for $T \in M(\mathscr{F}, F)$ implies that

$$T(f_1+f_2) = Tf_1 + Tf_2$$

whenever $f_1, f_2 \in \mathscr{F}$ and $|f_1| \wedge |f_2| = 0$.

In general, this property is not equivalent to the Hammerstein property. This is shown by the example [B2, p. 150]

$$Tf = \inf\{|f(x)|: x \in [0, 1]\}$$

and $\mathscr{F} = \mathscr{C}([0, 1])$.

In the rest of this section let A be an algebra of subsets of Ω. A *content*

$$\mu: A \to M(E, F)$$

is a finitely additive set function. The *integral* $\int f\, d\mu$ of an A-simple function $f: \Omega \to E$ with respect to the content μ is defined by

$$\int f\, d\mu = \sum_{k=1}^{n} \mu(A_k) y_k$$

if f has the representation

$$f = \sum_{k=1}^{n} \chi_{A_k} y_k$$

where $y_1, \ldots, y_n \in E$ and $A_1, \ldots, A_n$ are disjoint sets in A. It is easy to see that the integral is the same for every such representation of f.

The *semivariation* of μ is defined as the semivariation of the transformation $S: \mathfrak{E}(A, E) \to F$ where $Sf := \int f\, d\mu$. We sometimes write $\|\ \|_{\mu,\alpha}$ for $\|\ \|_{S,\alpha}$. The transformation S is in $M(\mathfrak{E}(A, E), F)$ if and only if $\lim_{n\to\infty} \|1/n\|_{\mu,\alpha} = 0$ for each $\alpha > 0$. The linear space of all contents μ which satisfy this condition is denoted by $\mathrm{ba}(A, E, F)$.

(3.1) Proposition. (a) *The map $\mu \mapsto \int \cdot\, d\mu$ establishes an algebraic isomorphism between* $\mathrm{ba}(A, E, F)$ *and* $M_{\mathrm{HP}}(\mathfrak{E}(A, E), F)$.

(b) *For every $S \in M(\mathfrak{E}(A, E), F)$, the Hammerstein property is equivalent to*

$$S(f_1+f_2) = Sf_1 + Sf_2$$

for all $f_1, f_2 \in \mathfrak{E}(A, E)$ such that $|f_1| \wedge |f_2| = 0$.

Proof. (a) Obviously, the map $\mu \mapsto S := \int \cdot\, d\mu$ is linear and one-to-one from $\mathrm{ba}(A, E, F)$ to $M(\mathfrak{E}(A, E), F)$. As in [B2, p. 148] we see that S satisfies the Hammerstein property. If $S \in M_{\mathrm{HP}}(\mathfrak{E}(A, E), F)$ then $\mu(A) y = S(\chi_A y)$ defines a content $\mu \in \mathrm{ba}(A, E, F)$ such that $S = \int \cdot\, d\mu$.

(b) This follows from the observation that the last argument of the proof of (a) remains true if $S \in M(\mathfrak{E}(A, E), F)$ satisfies the weaker additivity condition.

For every transformation $S \in M(\mathfrak{E}(A, E), F)$, there is a uniquely determined continuous extension $\bar{S} \in M(\mathfrak{M}(A, E), F)$. Hence there is a natural isomorphism between these spaces.

(3.2) Proposition. *Let S, $\bar{S}$ be as above. Then*

(a) $\|\ \|_{S,\alpha} = \|\ \|_{\bar{S},\alpha}$.

(b) *S satisfies the Hammerstein property if and only if $\bar{S}$ satisfies this property.*

Proof. (a) Let us prove that $\|\ \|_{\bar{S},\alpha} \leqslant \|\ \|_{S,\alpha}$. Let $\varphi \in \mathfrak{B}(\Omega)$ and $f, g \in \mathfrak{M}(A, E)_\alpha$ with $|f-g| \leqslant \varphi$. Since $\overline{\mathfrak{E}(A, E)_\alpha} = \mathfrak{M}(A, E)_\alpha$ there are $f_1, g_1 \in \mathfrak{E}(A, E)_\alpha$ such that $\|f-f_1\|_\infty < \varepsilon$, $\|g-g_1\|_\infty < \varepsilon$ for each $\varepsilon > 0$. Then

$$|\bar{S}f-\bar{S}g| \leqslant |Sf_1-Sg_1|+|\bar{S}f-Sf_1|+|Sg_1-\bar{S}g| \leqslant \|\varphi+2\varepsilon\|_{S,\alpha}+2\|\varepsilon\|_{\bar{S},\alpha}.$$

It follows from Proposition (1.2) that

$$|\bar{S}f-\bar{S}g| \leqslant \|\varphi\|_{S,\alpha}+4\|\varepsilon\|_{\bar{S},\alpha} \to \|\varphi\|_{S,\alpha} \quad \text{as } \varepsilon \to 0.$$

Hence $\|\varphi\|_{\bar{S},\alpha} \leqslant \|\varphi\|_{S,\alpha}$.

(b) It suffices to prove that for every $f \in \mathfrak{M}(A, E)$ there exists a sequence $f_n \in \mathfrak{E}(A, E)$ such that $\|f_n-f\|_\infty \to 0$ as $n \to \infty$ and $\{f_n \neq 0\} \subset \{f \neq 0\}$. Let $f \in \mathfrak{M}(A, E)$, $\varepsilon > 0$. Choose

$$g = \sum_{k=1}^{n} \chi_{A_k} y_k,$$

$y_k \in E$, A_k disjoint sets in A, $\|f-g\|_\infty \leqslant \varepsilon$. We can assume that $A_1, \ldots, A_m \subset \{f \neq 0\}$, $A_{m+1}, \ldots, A_n \not\subset \{f \neq 0\}$. Let $m+1 \leqslant k \leqslant n$, $x_0 \in A_k$, $f(x_0) = 0$. Then $\varepsilon \geqslant |f(x_0)-g(x_0)| = |y_k|$ and, for each $x \in A_k$, $|f(x)| \leqslant |f(x)-g(x)|+|y_k| \leqslant 2\varepsilon$. Now

$$h = \sum_{k=1}^{m} \chi_{A_k} y_k$$

satisfies $\{h \neq 0\} \subset \{f \neq 0\}$ and $\|f-h\|_\infty \leqslant 2\varepsilon$.

If $\mu \in \mathrm{ba}(A, E, F)$ then $S := \int \cdot\, d\mu \in M_{\mathrm{HP}}(\mathfrak{E}(A, E), F)$. Hence we can define the integral

$$\int f\, d\mu := \bar{S}f$$

for all $f \in \mathfrak{M}(A, E)$ and $\bar{S} \in M_{\mathrm{HP}}(\mathfrak{M}(A, E), F)$.

4. Integral representation of transformations on $\mathscr{C}(\Omega, E)$. In this section we continue our theory of the second section in order to obtain integral representations of transformations on $\mathscr{C}(\Omega, E)$ which satisfy the Hammerstein property.

(4.1) Proposition. *Let $T \in M(\mathscr{C}(\Omega, E), F)$ and assume that $T(g_1+g_2) = Tg_1+Tg_2$ for all $g_1, g_2 \in \mathscr{C}(\Omega, E)$ such that $|g_1| \wedge |g_2| = 0$. Define M and $\bar{T}$ as in Section 2. Then $\bar{T}$ satisfies the Hammerstein property on $\mathfrak{M}(M, E)$.*

Proof. Since $\bar{T}$ is continuous with respect to the $\|\ \|_\infty$-norm, Propositions (3.1)(b) and (3.2) show that it suffices to prove $\bar{T}(f_1+f_2) = \bar{T}f_1 + \bar{T}f_2$ for all $f_1, f_2 \in \mathfrak{E}(M, E)$ such that $|f_1| \wedge |f_2| = 0$.

We choose a positive integer α such that $f_1, f_2 \in \mathfrak{E}(M, E)_\alpha$ and write

$$f_1 = \sum_{k=1}^{m} \chi_{M_k} y_k, \quad f_2 = \sum_{k=m+1}^{n} \chi_{M_k} y_k,$$

where $M_1, \ldots, M_n$ are disjoint sets in $\boldsymbol{M}$. By Theorem (2.3), for each $\varepsilon > 0$, there are sets $Z_k \in \boldsymbol{Z}$, $P_k \in \boldsymbol{P}$, $k = 1, \ldots, n$, such that $Z_k \subset M_k \subset P_k$ and $\|\chi_{P_k \setminus Z_k}\|_{T,\alpha} < \varepsilon$. Since $Z_1, \ldots, Z_n$ are disjoint zero sets there are disjoint cozero sets $P'_1, \ldots, P'_n$ such that $Z_k \subset P'_k$ for $k = 1, \ldots, n$. Since we can replace P'_k by $P_k \cap P'_k$, we can assume that $P'_k \subset P_k$.

There are continuous functions $\varphi_k\colon \Omega \to [0, 1]$ such that $\varphi_k(x) = 1$ for $x \in Z_k$ and $\varphi_k(x) = 0$ for $x \in \Omega \setminus P'_k$. Then

$$|||\chi_{M_k} y_k - \varphi_k y_k|||_\alpha \leqslant \alpha \|\chi_{M_k} - \varphi_k\|_\alpha \leqslant \alpha \|\chi_{P_k \setminus Z_k}\|_{T,\alpha} < \alpha\varepsilon.$$

Define

$$g_1 := \sum_{k=1}^{m} \varphi_k y_k, \quad g_2 := \sum_{k=m+1}^{n} \varphi_k y_k.$$

Since $g_1, g_2 \in \mathscr{C}(\Omega, E)$, $|g_1| \wedge |g_2| = 0$, it follows from our assumption that $T(g_1+g_2) = Tg_1+Tg_2$. Hence

$$\begin{aligned}
&|\bar{T}(f_1+f_2) - \bar{T}f_1 - \bar{T}f_2| \\
&\qquad \leqslant |\bar{T}(f_1+f_2) - \bar{T}(g_1+g_2)| + |\bar{T}f_1 - \bar{T}g_1| + |\bar{T}f_2 - \bar{T}g_2| \\
&\qquad \leqslant |||f_1+f_2-g_1-g_2|||_\alpha + |||f_1-g_1|||_\alpha + |||f_2-g_2|||_\alpha \\
&\qquad \leqslant 2\sum_{k=1}^{n} |||\chi_{M_k} y_k - \varphi_k y_k|||_\alpha < 2\alpha n\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ can be made arbitrarily small it follows that $\bar{T}(f_1+f_2) = \bar{T}f_1 + \bar{T}f_2$.

In the sequel let us denote by $\boldsymbol{A}$ the smallest algebra containing $\boldsymbol{P}$. This algebra will be used for the integral representation.

(4.2) Corollary. *Let $T \in M(\mathscr{C}(\Omega, E), F)$ and assume that all cozero sets of Ω are T-measurable. Then the following three statements are equivalent:*

(i) *T satisfies the Hammerstein property.*

(ii) *$T(f_1+f_2) = Tf_1+Tf_2$ for all $f_1, f_2 \in \mathscr{C}(\Omega, E)$ such that $|f_1| \wedge |f_2| = 0$.*

(iii) *There is a content $\mu \in \mathrm{ba}(A, E, F)$ such that $Tf = \int f\, d\mu$ for all $f \in \mathscr{C}(\Omega, E)$.*

Proof. By (3.1) and (3.2), (iii) implies (i). (i) $\Rightarrow$ (ii) is trivial. In order to prove (ii) $\Rightarrow$ (iii) we apply Theorem (2.4). This theorem shows that $S := \bar{T}|\mathfrak{M}(A, E)$ is in $M(\mathfrak{M}(A, E), F)$ and extends T. By Proposition (4.1), S satisfies the Hammerstein property. Hence by (3.1) and (3.2) there is a content $\mu \in \mathrm{ba}(A, E, F)$ such that $Sf = \int f\, d\mu$ for every $f \in \mathfrak{M}(A, E) \supset \mathscr{C}(\Omega, E)$.

In order to simplify the formulation of the next theorem let us introduce the class of "regular" transformations $T\colon \mathscr{F} \to F$ on a linear subspace $\mathscr{F}$ of $\mathfrak{B}(\Omega, E)$. We call T *regular* if the semivariation of T is regular on A, i.e. if for all $A \in A$, $\varepsilon > 0$, $\alpha > 0$ there are sets $P \in \boldsymbol{P}$, $Z \in \boldsymbol{Z}$ such that $Z \subset A \subset P$ and $\|\chi_{P\setminus Z}\|_{T,\alpha} < \varepsilon$. If $\mathscr{F} = \mathscr{C}(\Omega, E)$ then, by Theorem (2.3), T is regular if and only if all cozero sets in Ω are T-measurable. For example, the transformation $Tf := \inf\{|f(x)|\colon x \in [0, 1]\}$ satisfies (ii) but not (i) of Corollary (4.2). Hence T is not a regular transformation.

We denote by $\operatorname{rba}(A, E, F)$ the set of all contents $\mu \in \operatorname{ba}(A, E, F)$ such that the semivariation $\|\ \|_{\mu,\alpha}$ is regular on A for each $\alpha > 0$.

(4.3) THEOREM. (a) *The map $\Psi\colon S \mapsto S|\mathscr{C}(\Omega, E)$ establishes an algebraic isomorphism between the linear space of all regular transformations $S \in M(\mathfrak{M}(A, E), F)$ and the linear space of all regular transformations $T \in M(\mathscr{C}(\Omega, E), F)$.*

(b) *The map Φ defined by*

$$\Phi(\mu)f = \int f\,d\mu, \quad f \in \mathscr{C}(\Omega, E),$$

establishes an algebraic isomorphism between $\operatorname{rba}(A, E, F)$ and the linear space of all regular transformations $T \in M_{\mathrm{HP}}(\mathscr{C}(\Omega, E), F)$.

Proof. (a) Obviously, Ψ is a well-defined linear map. Ψ is onto by Theorem (2.4). If $Sf = 0$ for every $f \in \mathscr{C}(\Omega, E)$ then it follows easily from the regularity of S that $Sg = 0$ for every $g \in \mathfrak{E}(A, E)$ (see [VW2, (1.4.1)]). Hence, by the continuity of S, $Sf = 0$ for every $f \in \mathfrak{M}(A, E)$.

(b) The map Φ is the composition of the following three isomorphisms and, therefore, is itself an isomorphism. By Proposition (3.1), the map $\mu \mapsto \int \cdot\,d\mu$ is an isomorphism between $\operatorname{rba}(A, E, F)$ and the space of regular transformations $S \in M_{\mathrm{HP}}(\mathfrak{E}(A, E), F)$. By Proposition (3.2), the map $S \mapsto \bar{S}$ is an isomorphism between the space of regular transformations $S \in M_{\mathrm{HP}}(\mathfrak{E}(A, E), F)$ and the space of regular transformations $T \in M_{\mathrm{HP}}(\mathfrak{M}(A, E), F)$.

Finally, by Proposition (4.1) and part (a) of this theorem, the map $T \mapsto T|\mathscr{C}(\Omega, E)$ is an isomorphism between the space of regular transformations $T \in M_{\mathrm{HP}}(\mathfrak{M}(A, E), F)$ and the space of regular transformations $U \in M_{\mathrm{HP}}(\mathscr{C}(\Omega, E), F)$.

If $T \in M_{\mathrm{HP}}(\mathscr{C}(\Omega, E), F)$ is regular and $T = \Phi(\mu)$ then we call μ the *representation content* of T. The semivariations of T, $\bar{T}$ and μ are closely connected:

(4.4) LEMMA. *Let $T \in M_{\mathrm{HP}}(\mathscr{C}(\Omega, E), F)$ be regular and μ the representation content of T. Then*

(a) $\|\ \|_{T,\alpha} \leqslant \|\ \|_{\mu,\alpha} \leqslant \|\ \|_{\bar{T},\alpha}$.

(b) $\|\varphi\|_{T,\alpha} = \|\varphi\|_{\mu,\alpha} = \|\varphi\|_{\bar{T},\alpha}$ *for every* $\varphi \in \mathfrak{L}$.

This lemma is a consequence of (2.4) and (3.2)(a).

5. s-Bounded and weakly compact transformations. The aim of this section is to give several conditions for a transformation $T\colon \mathscr{C}(\Omega, E) \to F$ which guarantee that T can be represented as an integral. We shall need the following lemma.

(5.1) LEMMA. *Let $T \in M_{\mathrm{HP}}(\mathscr{C}(\Omega, E), F)$. Then*

$$\|\varphi\|_{T,\alpha} = \sup\{|Tf - Tg|\colon f, g \in \mathscr{C}(\Omega, E)_\alpha,\ |f-g| \leqslant |\varphi|,$$
$$\{f \neq 0\} \cup \{g \neq 0\} \subset \{\varphi \neq 0\}\}$$

for every $\varphi \in \mathfrak{B}(\Omega)_\alpha$.

Proof. The inequality $\geqslant$ is trivial. Now assume that $\|\varphi\|_{T,\alpha} > \gamma$. We shall construct $f, g \in \mathscr{C}(\Omega, E)_\alpha$ such that $|f-g| \leqslant |\varphi|$, $\{f \neq 0\} \cup \{g \neq 0\} \subset \{\varphi \neq 0\}$ and $|Tf - Tg| > \gamma$ which will complete the proof. There is $\psi \in \mathscr{C}(\Omega)$, $0 \leqslant \psi \leqslant |\varphi|$, such that $\|\psi\|_{T,\alpha} > \gamma$. By Lemma (2.6), we can assume that $\{\psi \neq 0\} \subset Z \subset P \subset \{\varphi \neq 0\}$ for some $Z \in \boldsymbol{Z}$, $P \in \boldsymbol{P}$. Choose a continuous function $\varrho\colon \Omega \to [0, 1]$ such that $\varrho(x) = 1$ for $x \in Z$ and $\varrho(x) = 0$ for $x \in \Omega\setminus P$. Since $\|\psi\|_{T,\alpha} > \gamma$ there are $f_1, g_1 \in \mathscr{C}(\Omega, E)_\alpha$ such that $|f_1 - g_1| \leqslant \psi$ and $|Tf_1 - Tg_1| > \gamma$. Now define

$$f := \varrho f_1, \quad g := (1-\varrho)(g_1 - f_1) + \varrho g_1.$$

Since $|f_1 - g_1| \leqslant \psi \leqslant |\varphi| \leqslant \alpha$, we see that $f, g \in \mathscr{C}(\Omega, E)_\alpha$, $|f-g| = |f_1 - g_1| \leqslant |\varphi|$ and $\{f \neq 0\} \cup \{g \neq 0\} \subset P \subset \{\varphi \neq 0\}$. Since T satisfies the Hammerstein property and $|f_1 - g_1| \wedge |1-\varrho| = 0$ we have

$$Tg = T(f_1 + (\varrho - 1)f_1 + (g_1 - f_1)) = Tf + Tg_1 - Tf_1.$$

Hence $|Tf - Tg| = |Tf_1 - Tg_1| > \gamma$.

(5.2) THEOREM. *Let $T \in M_{\mathrm{HP}}(\mathscr{C}(\Omega, E), F)$.*

(a) *If T is weakly compact, i.e. $T(\mathscr{C}(\Omega, E)_\alpha)$ is relatively weakly compact in F for each $\alpha > 0$, then T is s-bounded.*

(b) *If F has no subspace isomorphic to c_0, then T is s-bounded.*

(c) *If T is s-bounded, then T is regular.*

Proof. (a)(b) By Lemma (5.1), it is sufficient to show that $Tf_n - Tg_n$ converges to zero for all sequences $f_n, g_n \in \mathscr{C}(\Omega, E)_\alpha$ such that $|f_n - g_n| \leqslant 1$ and the sets $\{f_n \neq 0\} \cup \{g_n \neq 0\}$ are disjoint. Let f_n, g_n be as above, $\xi \in F'$ a continuous linear functional on F and Q a finite set of positive integers. Then, by the Hammerstein property of T,

$$(*) \qquad \sum_{n\in Q}(Tf_n - Tg_n) = T\Big(\sum_{n\in Q} f_n\Big) - T\Big(\sum_{n\in Q} g_n\Big) \in H$$

where $H := \{Tf - Tg\colon f, g \in \mathscr{C}(\Omega, E)_\alpha\}$.

Since $T(\mathscr{C}(\Omega, E)_\alpha)$ is bounded, the set H is bounded. Hence

$$\Big|\sum_{n\in Q} \xi(Tf_n - Tg_n)\Big| \leqslant c$$

where c is independent of Q. It is well known that this implies

$$(**) \qquad \sum_{n=1}^{\infty} |\xi(Tf_n - Tg_n)| < \infty.$$

By (*) and (**), we see that the sequence $\sum_{n=1}^{N}(Tf_n - Tg_n)$ is contained in H and is weakly Cauchy-convergent. The same is true for all subsequences of f_n and g_n. Hence if T is weakly compact, then H is weakly compact and the series $\sum_{n=1}^{\infty}(Tf_n - Tg_n)$ and all its subseries are weakly convergent in F. By the theorem of Orlicz–Pettis [D, Ch. IV, Th. 1] this implies that the series is also convergent in the norm of F; consequently, $Tf_n - Tg_n$ converges to zero. This completes the proof of (a).

If F contains no subspace isomorphic to c_0 then, for every sequence $z_n \in F$ such that $\sum_{n=1}^{\infty} |\xi z_n| < \infty$ for all $\xi \in F'$, there is a $z \in F$ such that

$$\xi z = \sum_{n=1}^{\infty} \xi z_n \quad \text{for all } \xi \in F'$$

(see [B2, p. 173, line 4]). We can apply this property to $z_n = Tf_n - Tg_n$ because (**) holds. Hence the series $\sum_{n=1}^{\infty}(Tf_n - Tg_n)$ and all its subseries are weakly convergent in F. The rest of the proof is the same as above.

(c) is another formulation of (2.5).

Every functional in $M_{\mathrm{HP}}(\mathscr{C}(\Omega, E), \boldsymbol{K})$ is (weakly) compact where $\boldsymbol{K}$ denotes the scalar field of E. Hence (5.2) yields the following corollary (cf. [B2, Theorem 1]).

(5.3) Corollary. *The map $\mu \mapsto \int \cdot \, d\mu$ defines an algebraic isomorphism from* $\mathrm{rba}(A, E, \boldsymbol{K})$ *onto* $M_{\mathrm{HP}}(\mathscr{C}(\Omega, E), \boldsymbol{K})$.

(5.4) Theorem. *Let $T \in M_{\mathrm{HP}}(\mathscr{C}(\Omega, E), F)$ be regular and μ the representation content of T. Then T is s-bounded if and only if μ is of s-bounded semivariation on A, i.e. for every sequence A_n of disjoint sets in A the sequence $\|\chi_{A_n}\|_{\mu,\alpha}$ converges to zero for each $\alpha > 0$.*

Proof. Assume that μ is of s-bounded semivariation on A and let $\varphi_n \in \mathscr{C}(\Omega)$, $0 \leqslant \varphi_n \leqslant 1$, satisfy $\varphi_n \wedge \varphi_m = 0$ for $m \neq n$. Set $A_n := \{\varphi_n \neq 0\} \in \boldsymbol{P} \subset A$. Then, by (4.4)(a),

$$\|\varphi_n\|_{T,\alpha} \leqslant \|\chi_{A_n}\|_{T,\alpha} \leqslant \|\chi_{A_n}\|_{\mu,\alpha} \to 0 \quad \text{as } n \to \infty.$$

Hence T is s-bounded. Conversely, assume that T is s-bounded. As we remarked at the end of the second section, $\| \ \|_\alpha$ is "schwach halbadditiv" on $\{\psi \in \mathscr{C}(\Omega)\colon \psi \geqslant 0\}$ in the sense of [S, Section 2.5]. Therefore we can apply [S, (2.5.2)]. It follows that, for every sequence $\varphi_n \in \mathfrak{J}(\Omega)$, $0 \leqslant \varphi_n \leqslant 1$, such that $\varphi_n \wedge \varphi_m = 0$ for $m \neq n$, the sequence $\|\varphi_n\|_\alpha$ converges to zero. Now let A_n be a sequence of disjoint sets in A. Then $\chi_{A_n} \in \mathfrak{J}(\Omega)$, hence $\|\chi_{A_n}\|_\alpha$ converges to zero for each $\alpha > 0$. By (4.4), (2.2)(b),

$$\|\chi_{A_n}\|_{\mu,\alpha} \leqslant \|\chi_{A_n}\|_{T,\alpha} \leqslant \|\chi_{A_n}\|_\alpha.$$

Hence $\|\chi_{A_n}\|_{\mu,\alpha}$ converges to zero. This completes the proof.

(5.5) Theorem. *Let $T \in M_{\mathrm{HP}}(\mathscr{C}(\Omega, E), F)$ be regular and μ the representation content of T. Then the following statements are equivalent:*

(i) *T is (weakly) compact.*

(ii) *$f \mapsto \int f \, d\mu$ is (weakly) compact on $\mathfrak{M}(A, E)$.*

(iii) *$f \mapsto \int f \, d\mu$ is (weakly) compact on $\mathfrak{E}(A, E)$, i.e. the set*

$$\Big\{\sum_{k=1}^{n} \mu(A_k) y_k\colon n \in N;\ y_k \in E,\ |y_k| \leqslant \alpha,\ A_k \in A \text{ disjoint } (k = 1, \ldots, n)\Big\}$$

is relatively (weakly) compact for each $\alpha > 0$.

The proof is similar to that in the linear case [VW2, Theorem 1.5.5] and is omitted.

6. Smooth transformations. A transformation $T\colon \mathscr{C}(\Omega, E) \to F$ is called *τ-smooth* (*σ-smooth*) if, for every decreasingly directed (countable) subset Ψ of $\mathscr{C}(\Omega)$,

$$\inf \Psi = 0 \quad \text{implies} \quad \inf_{\psi \in \Psi} \|\psi\|_{T,\alpha} = 0 \quad \text{for each } \alpha > 0.$$

It follows from Dini's theorem that every transformation $T \in M(\mathscr{C}(\Omega, E), F)$ is σ-smooth (τ-smooth) if Ω is pseudocompact (compact) (see [VW2, (1.6.7)]).

By the next theorem the smoothness of a regular transformation is expressed in terms of its representation content.

(6.1) Theorem. *Let $T \in M_{\mathrm{HP}}(\mathscr{C}(\Omega, E), F)$ be regular and μ the representation content of T. Then*

(a) *T is τ-smooth if and only if μ is of τ-smooth semivariation on $\boldsymbol{Z}$, i.e. for every decreasingly directed subset $\boldsymbol{Z}_1$ of $\boldsymbol{Z}$,*

$$\bigcap \boldsymbol{Z}_1 = \emptyset \quad \textit{implies} \quad \inf_{Z \in \boldsymbol{Z}_1} \|\chi_Z\|_{\mu,\alpha} = 0 \quad \textit{for each } \alpha > 0.$$

(b) *The following statements are equivalent:*

(i) *T is σ-smooth.*

(ii) *μ is of σ-smooth semivariation on $\boldsymbol{Z}$.*

(iii) *μ is of s-bounded and σ-smooth semivariation on $\boldsymbol{Z}$.*

The proof of this theorem is similar to that of [VW2, (1.6.4)] and is omitted. We remark that [VW2, (1.6.2)] remains valid for Riesz pseudonorms.

In the next theorem we show that a σ-smooth regular transformation $T \in M_{\mathrm{HP}}(\mathscr{C}(\Omega, E), F)$ is representable in the form

$$Tf = \int f \, d\omega, \quad f \in \mathscr{C}(\Omega, E),$$

with a Baire measure ω (with respect to the topology of uniform convergence on bounded sets in $M(E, F)$) of regular semivariation. By a measure we mean a σ-additive content and by the system of Baire sets the σ-algebra $\sigma(\boldsymbol{P})$ generated by $\boldsymbol{P}$. Observe that a content with values in $M(E, F)$ is a measure if it is of σ-smooth semivariation on its domain.

(6.2) THEOREM. *Let $T \in M_{\mathrm{HP}}(\mathscr{C}(\Omega, E), F)$ be a σ-smooth regular transformation and μ the representation content of T. Then there exists a (uniquely determined) extension $\omega \in \mathrm{rba}(\sigma(\boldsymbol{P}), E, F)$ of μ and ω is of σ-smooth semivariation on $\sigma(\boldsymbol{P})$.*

The proof is similar to that of [VW2, [1.6.5)].

If $T \in M(\mathscr{C}(\Omega, E), F)$ is σ-smooth and s-bounded, then an analogue of Lebesgue's dominated convergence theorem holds for $\bar{T}\colon \mathfrak{I}(\Omega, E) \to F$ defined in Section 2.

(6.3) THEOREM. *Let $T \in M(\mathscr{C}(\Omega, E), F)$ be σ-smooth and s-bounded. Let f_n be a sequence in $\mathfrak{I}(\Omega, E)$ and let $f \in \mathfrak{B}(\Omega, E)$ be such that $f_n(x) \to f(x)$ for every $x \in \Omega$. Assume that there is a real α such that $\|f_n\|_\infty \leqslant \alpha$ for every n.*

Then f_n converges to f in $(\mathfrak{B}(\Omega, E), \tau)$. In particular, $f \in \mathfrak{I}(\Omega, E)$ and $\bar{T}f_n \to \bar{T}f$.

Proof. This follows easily from [VW1, Section 1.2]. The "upper norms" which are needed for the integration theory of [VW1] can be defined as in Section 2 or as in [VW1, (2.1.1)].

(6.4) COROLLARY. *The following statements are equivalent for every $T \in M_{\mathrm{HP}}(\mathscr{C}(\Omega, E), F)$:*

(i) *T is s-bounded and σ-smooth.*

(ii) *T is regular and σ-smooth.*

(iii) *For every $\alpha > 0$ and all sequences $f_n, g_n \in \mathscr{C}(\Omega, E)_\alpha$,*

$$f_n(x) - g_n(x) \to 0 \quad (n \to \infty;\ x \in \Omega) \quad \textit{implies} \quad Tf_n - Tg_n \to 0.$$

Proof. (i) $\Rightarrow$ (ii) follows from (5.2)(c) and (ii) $\Rightarrow$ (i) from (6.1)(b) and (5.4).

(iii) $\Rightarrow$ (i). Assume that T is not s-bounded. Then there are $\alpha > 0$, $\varepsilon > 0$ and sequences f_n, g_n in $\mathscr{C}(\Omega, E)_\alpha$ such that $|f_n - g_n| \wedge |f_m - g_m| = 0$ for $m \neq n$. and $|Tf_n - Tg_n| > \varepsilon$ for every n. Therefore (iii) is not satisfied.

Similarly, if T is not σ-smooth then it follows immediately that (iii) is not satisfied.

(i) $\Rightarrow$ (iii). Let α, f_n, g_n be as in (iii). Set $\varphi_n := |f_n - g_n| \in \mathscr{C}(\Omega)$ and choose $y \in E$, $|y| = 1$. Then $\varphi_n(x)y$ converges to zero as $n \to \infty$ for every x and $\|\varphi_n y\|_\infty \leqslant 2\alpha$. Hence, by Theorem (6.3),

$$|Tf_n - Tg_n| \leqslant |||f_n - g_n|||_\alpha = \|\varphi_n y\|_\alpha \to 0.$$

(6.5) COROLLARY. *If Ω is pseudocompact then every weakly compact transformation $T \in M_{\mathrm{HP}}(\mathscr{C}(\Omega), F)$ takes a weak Cauchy sequence into a strong Cauchy sequence.*

Proof. Let $T \in M_{\mathrm{HP}}(\mathscr{C}(\Omega), F)$ be weakly compact and f_n a weak Cauchy sequence in $\mathscr{C}(\Omega)$. Then the limits $f(x) := \lim_{n\to\infty} f_n(x)$ exist for every $x \in \Omega$ and there exists a real α with $\|f_n\|_\infty \leqslant \alpha$. By Theorem (5.2), T is s-bounded. Hence, by Theorem (6.3), we have $f \in \mathfrak{I}(\Omega, E)$ and Tf_n converges to $\bar{T}f$. This completes the proof.

Batt [B2, Theorem 9] has shown the same result by different methods.

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UNIVERSITÄT ESSEN – GHS, FB MATHEMATIK
D-4300 Essen 1, West Germany
and
UNIVERSITÀ DEGLI STUDI DELLA BASILICATA FACOLTÀ
DI SCIENZE MATEMATICHE, FISICHE E NATURALI
I-85100 Potenza, Italy

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