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Estimates in Sobolev norms $\|\cdot\|_p^s$ for harmonic and holomorphic functions and interpolation between Sobolev and Hölder spaces of harmonic functions

by

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Abstract. In the paper the duality theory is extended to the case of Sobolev spaces of harmonic functions whose derivatives are in $L^p(D)$. The behaviour of Bell's operators L_s^t on each space is studied. These operators together with the orthogonal projection P on harmonic functions are used to the study of interpolation between L^p, Sobolev and Hölder spaces of harmonic functions. It turns out that all these spaces form a double interpolation scale. If P maps $L^\infty(D)$ onto the space of Bloch harmonic functions, as in the case of the unit ball, then this last space is the vertex of this scale. No assumptions on the existence of traces on the boundary are needed in this approach. The possible use of above approach in the study of the regularity of the Bergman projection and of solutions of the $\bar\partial$-Neumann problem is discussed. The duality and interpolation theorems are also proved for the spaces of holomorphic functions on strictly pseudoconvex domains.

I. Introduction and the statement of results. The present paper is a continuation of [16]–[19]. We extend the duality theory for spaces of harmonic functions, originated by S. Bell [3], [4] and developed in [5], [10], [16]–[18], to the spaces $\mathrm{Harm}_p^s(D)$ of harmonic functions belonging to the Sobolev space $W_p^s(D)$, $1 < p < \infty$. In [16] we gave the detailed description of this duality for $p = 2$. Let us define the "negative Sobolev spaces" $W_p^{-s}(D)$, $1 < p < \infty$, s an integer, $s > 0$, as the spaces of distributions g on the domain D such that $g = \sum_{|\zeta|=s} D^\zeta g_\zeta + g_0$, where $g_0, g_\zeta \in L^p(D)$. The space $W_p^{-s}(D)$ is the adjoint space to $\mathring{W}_q^s(D)$ which is the closure of $C_0^\infty(D)$ in $W_q^s(D)$, $q = p/(p-1)$.

In fact, $W_p^{-s}(D)$ and $\mathring{W}_q^s(D)$ are mutually dual with respect to the L^2 scalar product $\langle\ ,\ \rangle$. We equip $W_p^{-s}(D)$ with the dual norm of $(\mathring{W}_q^s(D))^*$.

If s is not an integer, we define $W_p^s(D)$ as the value of the complex interpolation functor $[W_p^{[s]}(D), W_p^{[s]+1}(D)]_{[\theta]}$ for $\theta = s-[s]$, where $[s]$ is the integer part of s. If $s > 0$ then the "negative Sobolev space" $W_p^{-s}(D)$ represents the dual space to $\mathring{W}_q^s(D) = [\mathring{W}_q^{[s]}(D), \mathring{W}_q^{[s]+1}(D)]_{[\theta]}$, $\theta = s-[s]$, $q = p/(p-1)$. The space $\mathring{W}_q^s(D)$ is equal to the closure of $C_0^\infty(D)$ in $W_q^s(D)$ for $s \neq k+1/q$, $k = 0, 1, 2, \ldots$

In the sequel we shall need the following important property which follows from the W_p^s estimates for the solution of the Dirichlet problem $\Delta^m v = f$, v vanishes on ∂D up to order $m-1$ (see [1] and [21], Theorem 5.4):

(*) If D is a bounded domain in $\boldsymbol{R}^n$ with boundary of class C^∞, then the operator Δ^s is an isomorphism between $\mathring{W}_p^s(D)$ and $W_p^{-s}(D)$ for each integer $s > 0$.

If $p = 2$ this can be proved elementarily (see [16]).

Let D be a bounded domain with C^∞ boundary. We shall call a function $\varrho \in C^\infty(\boldsymbol{R}^n)$ a *defining function* for D if $D = \{x \in \boldsymbol{R}^n\colon \varrho(x) < 0\}$ and $\operatorname{grad}\varrho \neq 0$ on ∂D.

Let us recall a well-known property of the spaces $\mathring{W}_p^s(D)$. If $f \in \mathring{W}_p^s(D)$ then $f/|\varrho|^s \in L^p(D)$ and $\|f/|\varrho|^s\|_{L^p} \lesssim \|f\|_p^s$ (this follows e.g. from Theorem 2, 1.3.1 of [20], or [22]). This means that $\mathring{W}_p^s \subset L^p(D, 1/|\varrho|^{ps})$ and the inclusion is continuous.

Let us define *Bell's operators* [3]:

$$L^r u = u - \Delta\Big(\sum_{k=0}^{r-1} \theta_k \varrho^{k+2}\Big),$$

$$\theta_t = \frac{\varphi}{(t+2)!}|\nabla\varrho|^{-2}\Big(\frac{\partial}{\partial\eta}\Big)^t L^t u, \qquad \frac{\partial}{\partial\eta} = \frac{\sum_{i=1}^{n}\frac{\partial\varrho}{\partial x_i}\frac{\partial}{\partial x_i}}{|\nabla\varrho|^2},$$

$$L^1 u = u - \Delta(\theta_0\varrho^2), \qquad \theta_0 = \frac{1}{2}\frac{\varphi u}{|\nabla\varrho|^2},$$

where φ is an arbitrarily chosen C^∞ function equal to 1 in a neighbourhood of ∂D and equal to 0 in a neighbourhood of the set $\{\nabla\varrho = 0\}$.

We should mention here the work of E. Straube [26], [27] who constructed an operator T from $C^\infty(\bar{D})$ into the space $C_0^\infty(\bar{D})$ of functions vanishing on ∂D up to infinite order, which extends to a continuous operator from $W_2^s(D)$ into $\mathring{W}_2^s(D)$ for each $s \geqslant 0$ and has the following property: $P(Tv) = P(v)$ for each v, where P denotes as usual the orthogonal projection from $L^2(D)$ onto the space of square-integrable harmonic functions.

We will prove

THEOREM 1. *Let D be a bounded domain in $\boldsymbol{R}^n$ with C^∞-smooth boundary and let ϱ be a defining function for D. Then*

(a) *The operator $T_k f = \varrho^k f$ maps continuously* $\mathrm{Harm}_p^s(D)$ *into* $W_p^{s+k}(D)$ *for every $-\infty < s < +\infty$, s real, k a nonnegative integer.*

(b) *The operator L^r maps continuously* $\mathrm{Harm}_p^s(D)$ *into* $W_p^s(D) \cap L^p(D, 1/|\varrho|^{ps})$, *$s \geqslant 0$, $r \geqslant s$, r a nonnegative integer.*

THEOREM 2. *Let D and ϱ be as above.*

(a) $\mathrm{Harm}_p^s(D)$ *and* $\mathrm{Harm}_q^{-s}(D)$, *$q = p/(p-1)$, are mutually dual with respect to the pairing $\langle\ ,\ \rangle_r$ for $r \geqslant s$, r an integer, where $\langle u, v\rangle_r = \langle u, L^r v\rangle$ for $u \in \mathrm{Harm}_q^{-s}(D)$, $v \in \mathrm{Harm}_p^s(D)$.*

(b) $\mathrm{Harm}_q^{-s}(D)$ *is equal to* $L^q\,\mathrm{Harm}(D, |\varrho|^{qs})$ *with equivalent norms.*

It should be mentioned here that if both u and v are in $L^2\,\mathrm{Harm}(D)$ then $\langle\ ,\ \rangle_r = \langle\ ,\ \rangle$ ($\langle\ ,\ \rangle$ is as above the usual $L^2(D)$ scalar product). Theorems 1 and 2 for $p = 2$ and integer s were proved in [16].

The proof of Theorems 1 and 2 is based on the following two facts.

PROPOSITION 1. *The projector P maps continuously $L^p(D, 1/|\varrho|^{ps})$ onto* $\mathrm{Harm}_p^s(D)$.

In [17] it was proved that P maps $L^\infty(D, 1/|\varrho|^\alpha)$ onto the space $\Lambda_\alpha\,\mathrm{Harm}(D)$ of harmonic functions belonging to the space $\Lambda_\alpha(D)$ of Hölder functions on D. $L^\infty(D, 1/|\varrho|^\alpha)$ denotes here the space of functions f such that $|f|/|\varrho|^\alpha$ is bounded on D with norm $\|f\|_\infty^\alpha = \sup(|f|/|\varrho|^\alpha)$.

PROPOSITION 2. *If $1 < p < \infty$ and $1 < q \leqslant \infty$ then*

$$[L^p(D, 1/|\varrho|^s), L^q(D, 1/|\varrho|^r)]_{[\theta]} = L^m(D, 1/|\varrho|^t)$$

where $0 < \theta < 1$ and

$$\frac{1}{m} = \frac{1-\theta}{p} + \frac{\theta}{q} \quad \text{and} \quad t = \frac{s(1-\theta)q + r\theta p}{q(1-\theta) + p\theta} \quad \text{if } q \neq \infty,$$

$$m = \frac{p}{1-\theta} \quad \text{and} \quad t = s + \frac{p\theta r}{1-\theta} \quad \text{if } q = \infty.$$

The statement of Proposition 2 needs some explanation. If (A_1, A_2) is an admissible pair of Banach spaces then $[A_1, A_2]_{[\theta]}$ denotes the value of an interpolation functor at θ, $0 < \theta < 1$. We shall always use the complex interpolation method. For the definition and the properties of the interpolation functor see [6] and [11]. We shall also denote by $[A_1, A_2]_{[\theta]}^{\sim}$ the completion of $[A_1, A_2]_{[\theta]}$ with respect to the space $A_1 + A_2$ (see [11]).

We can now state our interpolation theorem.

THEOREM 3 (interpolation theorem). *Let D be a bounded domain with C^∞-smooth boundary in $\boldsymbol{R}^n$. Then*

(a) $$[\mathrm{Harm}_{p_1}^{s_1}(D), \mathrm{Harm}_{p_2}^{s_2}(D)]_{[\theta]} = \mathrm{Harm}_q^t(D),$$

$$\frac{1}{q} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \qquad t = (1-\theta)s_1 + \theta s_2, \qquad s_1, s_2 \geqslant 0,\ 1 < p_1, p_2 < \infty.$$

(b) $$[\mathrm{Harm}_p^s(D), \Lambda_\alpha\,\mathrm{Harm}(D)]_{[\theta]} = \mathrm{Harm}_q^t(D),$$

$$q = \frac{p}{1-\theta}, \qquad t = (1-\theta)s + \theta\alpha, \qquad s \geqslant 0, \alpha > 0, 1 < p < \infty.$$

(c) $$[\Lambda_\alpha\,\mathrm{Harm}(D), \Lambda_\beta\,\mathrm{Harm}(D)]_{[\theta]}^{\sim} = \Lambda_t\,\mathrm{Harm}(D),$$

$$t = (1-\theta)\alpha + \theta\beta, \qquad \alpha, \beta > 0.$$

This means that if we write

$$A_p^s = \operatorname{Harm}_p^s(D), \quad 1 < p < \infty, \quad A_\infty^s = \Lambda_s \operatorname{Harm}(D), \quad s > 0,$$

we get a double interpolation scale of spaces.

The bottom row of this scale is formed by the spaces $A_p^0 = \operatorname{Harm}_p(D) = L^p \operatorname{Harm}(D)$, and the right border column by the Hölder spaces of harmonic functions $A_\infty^s = \Lambda_s \operatorname{Harm}(D)$, $s > 0$. Then the question arises: What could stand at the "vertex" of this scale, that means, what is A_∞^0?

We conjecture that it should always be the space of *Bloch harmonic functions*, i.e. the space of harmonic u such that

$$\|u\|_{\mathrm{Bl}} = \sup_{z\in D} |\varrho(z) u(z)| + \sup_{z\in D} |\varrho(z) \operatorname{grad} u(z)| < \infty.$$

At present we are only able to prove the following.

PROPOSITION 3. *Let D be as above. If the projector P maps continuously $L^\infty(D)$ onto* $\operatorname{Bl}\operatorname{Harm}(D)$ *then $A_\infty^0(D) = \operatorname{Bl}\operatorname{Harm}(D)$ and*

(a) $[A_p^s, A_\infty^0]_{[\theta]} = A_r^t, \quad t = (1-\theta)s, \quad r = \dfrac{p}{1-\theta}, \quad 1 < p < \infty, \; s \geqslant 0.$

(b) $[A_\infty^s, A_\infty^0]_{[\theta]}^{\sim} = A_\infty^{(1-\theta)s}, \quad s \geqslant 0.$

Unfortunately, we can verify the assumptions of Proposition 3 only in the case when D is the unit ball in $\boldsymbol{R}^n$. This can be done by exhibiting an explicit formula for the projector P for the unit ball. The details will be given in the paper *The reproducing kernel for harmonic functions and the space of Bloch harmonic functions on the unit ball in $\boldsymbol{R}^n$* (in preparation).

We think that the assumptions of Proposition 1 are valid for every smooth domain D and that it can be proved via the estimates similar to the Hölder estimates in [1]; but this is not done yet.

Theorem 1 can be used to prove the following.

THEOREM 4. *Let D be a smooth bounded domain in $\boldsymbol{C}^n$. If the Bergman projection B maps continuously $\Lambda_\alpha(D)$ into $\Lambda_\alpha(D)$, $\alpha > 0$, and $L^\infty(D)$ into* $\operatorname{Bl}\operatorname{Harm}(D)$ *then it maps continuously $W_p^s(D)$ into $W_p^s(D)$ for $s \geqslant 0$ and $1 < p < \infty$.*

Recall that the *Bergman projector* is the orthogonal projection of $L^2(D)$ onto the space $L^2 H(D)$ of holomorphic square-integrable functions on D.

We shall denote by $\Lambda_\alpha H(D)$ the space of Hölder holomorphic functions, by $H_p^s(D)$ the space of holomorphic functions from $W_p^s(D)$, by $L^p H(D, |\varrho|^r)$ the space of holomorphic functions from $L^p(D, |\varrho|^r)$ and by $\operatorname{Bl} H(D)$ the space of Bloch holomorphic functions.

In [2] and [23] the Hölder estimates for the projector B for a strictly pseudoconvex domain D with C^∞ boundary were proved. In [15] a more elementary proof of the Hölder estimates for a strictly pseudoconvex domain with C^{k+4}-smooth boundary with $\alpha < k$ was given (see Remark 1 at the end of the present paper). In [19] it was proved that if D is a strictly pseudoconvex domain with boundary of class C^4 then B maps $L^\infty(D)$ onto $\operatorname{Bl} H(D)$. In view of the above results Theorem 4 yields the following.

THEOREM 5. *If D is a strictly pseudoconvex domain with C^∞ boundary then B maps continuously $W_p^s(D)$ onto $H_p^s(D)$ and $L^p(D, 1/|\varrho|^{ps})$ onto $H_p^s(D)$, $s \geqslant 0$, $1 < p < \infty$.*

It turns out that the most elementary and easy way to get the estimates in Sobolev norms for the projector B of a C^∞-smooth strictly pseudoconvex domain is to prove Hölder estimates as in [15], Bloch norm estimates as in [19] and use our interpolation theorem. Note that the estimates in [15] can be essentially simplified by proving that B maps $L^\infty(D, 1/|\varrho|^\alpha)$ into $\Lambda_\alpha(D)$, which is equivalent to the fact that $B\colon \Lambda_\alpha(D) \to \Lambda_\alpha(D)$ and needs simpler gradient estimates (see [15], [17]). Theorem 5 for $s = 0$ was proved in [23].

Theorem 5 implies the following facts.

THEOREM 6. *Let D be a bounded strictly pseudoconvex domain with C^∞-smooth boundary. Then*

(a) *$H_p^s(D)$ and $H_q^{-s}(D)$, $q = p/(p-1)$, are mutually dual with respect to the pairing $\langle\ ,\ \rangle_r$, $r \geqslant s$.*

(b) *$H_q^{-s}(D)$ is equal to $L^q H(D, |\varrho|^{qs})$ with an equivalent norm.*

THEOREM 7. *The projector B extends to a continuous projection from $L^q \operatorname{Harm}(D, |\varrho|^t)$ onto $L^q H(D, |\varrho|^t)$ for all $1 < q < \infty$, $t \geqslant 0$, and D as above.*

THEOREM 8 (Interpolation theorem). *Let D be as above. Then*

(a) $[H_{p_1}^{s_1}(D), H_{p_2}^{s_2}(D)]_{[\theta]} = H_q^t(D),$

$$\frac{1}{q} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad t = (1-\theta)s_1 + \theta s_2, \quad s_1, s_2 \geqslant 0, \; 1 < p_1, p_2 < \infty.$$

(b) $[H_p^s(D), \Lambda_\alpha H(D)]_{[\theta]} = H_q^t(D),$

$$q = \frac{p}{1-\theta}, \quad t = (1-\theta)s + \theta\alpha, \quad s \geqslant 0, \; \alpha > 0, \; 1 < p < \infty.$$

(c) $[\Lambda_\alpha H(D), \Lambda_\beta H(D)]_{[\theta]}^{\sim} = \Lambda_t H(D), \quad t = (1-\theta)\alpha + \theta\beta, \quad \alpha, \beta > 0.$

(d) $[H_p^s(D), \operatorname{Bl} H(D)]_{[\theta]} = H_q^t(D),$

$$t = (1-\theta)s, \quad q = \frac{p}{1-\theta}, \quad 1 < p < \infty, \; s \geqslant 0.$$

(e) $[\Lambda_s H(D), \operatorname{Bl} H(D)]_{[\theta]}^{\sim} = \Lambda_{(1-\theta)s} H(D), \quad s \geqslant 0.$

Theorem 8 implies that the spaces $F_p^s = H_p^s(D)$, $s \geqslant 0$, $1 < p < \infty$,

$F^s_\infty(D) = \Lambda_s H(D)$, $s > 0$, and $F^0_\infty = \mathrm{Bl}\, H(D)$ form a double interpolation scale and the space of Bloch holomorphic functions is the vertex of this scale.

The fact analogous to Theorem 6 also holds for the Szegö projection (see Remark 2 at the end of the paper).

The interpolation theorem (Th. 3) could also be used in the theory of the $\bar\partial$ Neumann problem (see [8]). We state the following

PROPOSITION 4. *Let D be a smooth pseudoconvex domain in C^n. Let N denote the operator solving the $\bar\partial$-Neumann problem on D. Then if N maps $L^2_{(p,q)}(D)$ into $W^s_{2,(p,q)}(D)$ and $\Lambda_{\alpha,(p,q)}(D)$ into $\Lambda_{\alpha+s,(p,q)}(D)$ for some s, $0 \leqslant s < 1$, then N maps $W^k_{r,(p,q)}(D)$ into $W^{k+s}_{r,(p,q)}(D)$ for all $k > 0$, $2 < r < \infty$.*

$L^2_{(p,q)}(D)$, $\Lambda_{\alpha,(p,q)}(D)$, $W^k_{r,(p,q)}(D)$ are the spaces of differential forms of type (p, q) with coefficients from $L^2(D)$, $\Lambda_\alpha(D)$, $W^k_r(D)$ respectively. However, we again have the same situation as with Bloch harmonic functions since the only domain for which the Hölder estimates for the $\bar\partial$-Neumann problem are known is the unit ball in C^n. M. Range [24] proved that if D is the unit ball in C^n then N maps $\Lambda_{\alpha,(p,q)}(D) \to \Lambda_{\alpha+1,(p,q)}(D)$ and $\partial^* N\colon \Lambda_{\alpha,(p,q)}(D) \to \Lambda_{\alpha+1/2,(p,q)}(D)$. Thus by Proposition 4, N maps $W^s_{r,(p,q)}(D) \to W^{s+1}_{r,(p,q)}(D)$, $s \geqslant 0$, $2 < r < \infty$, and $\partial^* N\colon W^s_{r,(p,q)}(D) \to W^{s+1/2}_{r,(p,q)}(D)$.

It should be mentioned here that I. Lieb and M. Range [12]–[14] proved the same estimates (they proved them for $\partial^* N$, but this also implies the estimates for N) for the $\bar\partial$-Neumann problem connected with the Levi metric, i.e. with the Kähler metric whose potential is a strictly plurisubharmonic function ϱ, which is also a defining function for D. However, the complex Laplacian for such a metric is not equal to $\frac{1}{2}\Delta$ as in the case of the Euclidean metric.

We hope that our method can be extended to fit also this case. Our optimism is based on the following observation: Let T be a strongly elliptic selfadjoint differential operator with C^∞ coefficients (on R^n), and let D be a smooth bounded domain. Let P_T denote the orthogonal projection from $L^2(D)$ onto the space $L^2\,\mathrm{Harm}_T(D)$ of functions from $L^2(D)$ on which $T = 0$. Then we can construct the family of Bell's operators L^r_T in the following manner:

$$L^r_T u = u - T\Big(\sum_{k=0}^{r-1} \theta_k \varrho^{k+2m}\Big),$$

$$\theta_t = \frac{\varphi}{(t+2m)!}(\sigma(\nabla\varrho))^{-1}\left(\frac{\partial}{\partial\eta}\right)^t L^t_T u, \qquad \frac{\partial}{\partial\eta} = \frac{\sum \frac{\partial\varrho}{\partial x_l}\frac{\partial}{\partial x_l}}{|\nabla\varrho|^2},$$

$$L^1_T u = u - T(\theta_0 \varrho^{2m}), \qquad \theta_0 = \frac{1}{(2m)!}\frac{\varphi u}{\sigma(\nabla\varrho)},$$

where $\sigma(\nabla\varrho)$ denotes the value of the principal symbol of T at the vector $\nabla\varrho$, $2m$ is the order of T and φ is an arbitrarily chosen C^∞ function equal to 1 in a neighbourhood of ∂D and to 0 in a neighbourhood of the set $\{\nabla\varrho = 0\}$.

In this general case, $P_T f = f - TG_{T^2}Tf$, where G_{T^2} denotes the operator solving the Dirichlet problem $T^2 u = v$, u vanishes on ∂D up to order $2m-1$.

The most difficult problem is to prove Proposition 2 and the fact that P_T maps $L^\infty(D, 1/|\varrho|^\alpha)$ into $\Lambda_\alpha(D)$. We hope that this can be solved in the affirmative.

At present we are able to prove that all results of this paper and of [17] remain valid when $T = \Delta^m$, i.e. for the spaces $\mathrm{Harm}^s_{(m),p}(D)$ and $\Lambda_\alpha \mathrm{Harm}_{(m)}(D)$ of m-polyharmonic functions. The details will be given in the paper *On duality and interpolation for spaces of polyharmonic functions* (in preparation).

We end this introduction with a comparison of our interpolation theorem (Theorem 3) and the results which can be achieved by applying the more standard method of identifying harmonic functions with their traces on ∂D and interpolating between spaces of traces.

In fact, if $s > 1/p$ then the trace on ∂D of the space $\mathrm{Harm}^s_p(D)$ is equal to the Besov space $B^{s-1/p}_{p,p}(\partial D)$ (for definition of Besov spaces see [6], [25], [23]). Moreover, the Besov space $B^\alpha_{\infty,\infty}(\partial D)$ is equal to the Hölder space $\Lambda_\alpha(\partial D)$. In [6], Theorem 6.4.5, it is proved that

$$[B^{s-1/p}_{p,p}, B^\alpha_{\infty,\infty}]_{[\theta]} = B^t_{q,q}, \qquad t = (1-\theta)s + \theta\alpha - \frac{1-\theta}{p}, \qquad q = \frac{p}{1-\theta}.$$

(This is proved there for the Besov spaces on R^{n-1}, but remains true also in our case.) The space $B^t_{q,q}(\partial D)$ is the space of traces of functions from $\mathrm{Harm}^r_q(D)$, $r = (1-\theta)s + \theta\alpha$. Thus we can obtain the statement of Theorem 3 by this classical method but only for $s > 1/p$. If $s < 1/p$ then the trace may not exist at all. In particular, we get no information about interpolation between $L^p\,\mathrm{Harm}(D)$ and $\Lambda_\alpha \mathrm{Harm}(D)$ which seems to be most useful. Thus our method gives essentially new information.

II. Proofs.

1. Proof of Proposition 2. For $q < \infty$ our proposition is a direct consequence of the Stein–Weiss interpolation theorem (see [6], Theorem 5.5.3). Let $q = \infty$. The space $L^\infty(D, 1/|\varrho|^r)$ is the dual space to $L^1(D, |\varrho|^r)$, and $L^p(D, 1/|\varrho|^s)$ is dual to $L^{p/(p-1)}(D, |\varrho|^{s/(p-1)})$. By the duality theorem ([6], Th.4.5.1) we have

$$[L^p(D, 1/|\varrho|^s), L^\infty(D, 1/|\varrho|^r)]_{[\theta]} = ([L^{p/(p-1)}(D, |\varrho|^{s/(p-1)}), L^1(D, |\varrho|^r)]_{[\theta]})^*.$$

Thus again by the Stein–Weiss theorem

$$[L^p(D, 1/|\varrho|^s), L^\infty(D, 1/|\varrho|^r)]_{[\theta]} = (L^a(D, |\varrho|^b))^*,$$

$$a = \frac{p}{p-1+\theta}, \qquad b = \frac{s(1-\theta)+rp\theta}{p-1+\theta}.$$

We have

$$(L^a(D, |\varrho|^b))^* = L^m(D, 1/|\varrho|^t), \qquad m = \frac{p}{1-\theta}, \qquad t = s + \frac{r\theta p}{1-\theta}.$$

This ends the proof of Proposition 2.

Before proving our Proposition 1 we observe that the projector P can be written as $Pu = u - \Delta G_2 \Delta u$, where G_2 is the operator solving the Dirichlet problem $\Delta^2 f = g$, f vanishes on ∂D up to the first order. Thus by (*) and [1], [21], [28], P is a continuous projection from $W_p^s(D)$ onto $\mathrm{Harm}_p^s(D)$, $s \geqslant 0$, $1 < p < \infty$. Thus

$$[\mathrm{Harm}_{p_1}^{s_1}(D), \mathrm{Harm}_{p_2}^{s_2}(D)]_{[\theta]} = \mathrm{Harm}_q^t(D),$$
$$t = (1-\theta)s_1 + \theta s_2, \quad \frac{1}{q} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}.$$

This follows immediately from the well-known results on interpolation of Sobolev spaces $W_p^s(D)$ (see [6]). This also means that the first part of Theorem 3 is a direct consequence of the continuity of P.

2. Proof of Proposition 1. The above considerations imply that P maps $L^p(D)$ into $L^p(D)$. Let s be a positive integer. Just as in the proof of Proposition 2 in [17] we have for $f \in L^p(D)$

$$Pf = \Delta(v - G_2 \Delta^2 v),$$

where v is the solution of the Dirichlet problem $\Delta v = f$, $v = 0$ on ∂D. The properties of Green functions imply that

$$cPf(x) = \Delta\left(\int_D \frac{f(y)\,dV_y}{|x-y|^{n-2}} - G_2 \Delta^2\left(\int_D \frac{f(y)\,dV_y}{|x-y|^{n-2}}\right)\right).$$

G_2 denotes here the same operator as above, c is a constant depending only on n.

If $f \in L^p(D, 1/|\varrho|^{ps})$ then $f = u|\varrho|^s$, where $u \in L^p(D)$ and

$$\|f\|_{L^p(D,1/|\varrho|^{ps})} \lesssim \|u\|_{L^p(D)}.$$

We have

$$cPf(x) = \Delta\left[\int_D \frac{|\varrho(y)|^s u(y)\,dV_y}{|x-y|^{n-2}} - G_2 \Delta^2\left(\int_D \frac{|\varrho(y)|^s u(y)\,dV_y}{|x-y|^{n-2}}\right)\right].$$

The expression in square brackets is the biharmonic extension of the function

$$g(x) = \int_D \frac{|\varrho(y)|^s u(y)\,dV_y}{|x-y|^{n-2}}.$$

The estimates from [1] and [21], Theorem 5.3, yield that in order to prove

our proposition it suffices to show that the trace $\mathrm{Tr}\, g$ on ∂D belongs to $\mathrm{Tr}\, W_p^{s+2}(D)$ and $\mathrm{Tr}\, \partial g/\partial n$ on ∂D belongs to $\mathrm{Tr}\, W_p^{s+1}(D)$. Now the boundary values of g are the same as those of the function

$$w(x) = \int_D \frac{(\varrho(x)-\varrho(y))^s u(y)}{|x-y|^{n-2}}\,dV_y$$

and the boundary values of $\partial g/\partial n$ are equal to those of

$$w_1(x) = \int_D (\varrho(x)-\varrho(y))^s \frac{\partial}{\partial n}\frac{u(y)}{|x-y|^{n-2}}\,dV_y$$
$$= \frac{\partial}{\partial n} w(x) - s\int_D \frac{(\varrho(x)-\varrho(y))^{s-1} u(y) \frac{\partial}{\partial n}\varrho(y)}{|x-y|^{n-2}}\,dV_y.$$

If we differentiate $w(x)$ s times then we obtain the following expression for $|\alpha| = s$:

$$D^\alpha w(x) = \sum_{\substack{|\beta| = 2m-s-k \\ k,m \leqslant s}} \int_D \frac{(\varrho(x)-\varrho(y))^{s-k}(x-y)^\beta}{|x-y|^{n-2+2m}} u(y)\,\varphi_{k,m}^\beta(y)\,dV_y \tag{1}$$

$$+ \text{less singular terms}$$

where $\varphi_{k,m}^\beta$ are smooth functions on $\boldsymbol{R}^n$. Those less singular terms can be differentiated twice and their second derivatives are integral operators on u with kernels dominated by $c/|x-y|^{n-1}$. Hence these operators are compact from $L^p(D)$ into $L^p(D)$. Now we can always assume that ϱ vanishes identically on some neighbourhood of infinity, disjoint with D. By Taylor's formula we have

$$\varrho(x) - \varrho(y) = \sum_{i=1}^n \frac{\partial\varrho}{\partial x_i}(x_i - y_i) + g(x, y),$$

$g(x, y) = O(|x-y|^2)$ on $\boldsymbol{R}^n \times \boldsymbol{R}^n$. We substitute this formula in (1) and observe that those terms of (1) which contain $g(x, y)$ are also less singular, can be differentiated twice and the resulting operators are compact $L^p(D) \to L^p(D)$. Thus it remains to estimate the operators with kernels of the type

$$\int_D \frac{Q_{2m}(x-y)\,\varphi(y)\,u(y)}{|x-y|^{n+2m-2}}\,dV_y, \tag{2}$$

where Q_{2m} is a monomial of degree $2m$, φ is a smooth function on $\boldsymbol{R}^n$.

It follows from [25], Chap. III, § 3, that Q_{2m} can be written in the form

$$Q_{2m}(t) = |t|^{2m} + |t|^{2m-2} p_2(t) + \ldots + p_{2m}(t),$$

where $p_{2r}(t)$ is a harmonic homogeneous polynomial of order $2r$. Thus (2) consists of terms of the type

$$F(x) = \int_{R^n} \frac{u(y)\varphi(y)}{|x-y|^{n-2}} dV_y \quad \text{and}$$

$$G(x) = \int_{R^n} \frac{p_{2r}(x-y)u(y)\varphi(y)}{|x-y|^{n+2r-2}} dV_y, \quad r > 0.$$

(Putting $u(y) = 0$ outside D we can extend the integration over the whole R^n.)

Let us consider $F(x)$. We have $c\Delta F(x) = u(x)\varphi(x) \in L^p(R^n)$ since u has a bounded support. If R is so large that $D \Subset B(0, R/2)$ then all derivatives $D^\alpha F$ are bounded on the sphere $|x| = R$ by

$$c\frac{2^{n-2+|\alpha|}}{R^{n-2+|\alpha|}}\|u\|_{L^1(D)}.$$

Let $H(x)$ be the solution of the following Dirichlet problem on the ball $B(0, R)$: $c\Delta H = \varphi(y)u(y)$, $H = 0$ on $\partial B(0, R)$. We have $H \in W_p^{s+2}(B(0, R))$ and $F(x) - H(x)$ is the harmonic extension of $F(x)$ from the sphere $\{|x| = R\}$. Therefore

$$F(x) - H(x) \in C^\infty(\overline{B(0, R)}), \quad \|F(x) - H(x)\|_{W_p^{s+2}(B(0,R))} \leqslant c\|u\|_{L^1}.$$

Thus $F(x) \in W_p^{s+2}(D)$ and $\|F(x)\|_{W_p^{s+2}(D)} \leqslant c\|u\|_{L^p(D)}$.

Now let us consider $G(x)$. Since $p_{2r}(t)$ is a harmonic homogeneous polynomial,

$$\Delta\left(\frac{p_{2r}(t)}{|t|^{n+2r-2}}\right) = C\frac{p_{2r}(t)}{|t|^{n+2r}}.$$

Thus we have

$$\Delta G(x) = C\int_{R^n} \frac{p_{2r}(x-y)u(y)\varphi(y)}{|x-y|^{n+2r}} dV_y.$$

The operator on the right is the Riesz transformation of order $2r$. By Theorem 4, Chap. II, § 4 of [25], it maps continuously $L^p(R^n)$ into itself ($1 < p < \infty$). Thus $\|\Delta G(x)\|_{L^p(B(0,R))} \lesssim \|u\|_{L^p(D)}$.

Now we can again solve the Dirichlet problem on the ball $B(0, R)$ and obtain

$$\|G(x)\|_{W_p^{s+2}(D)} \lesssim \|u\|_{L^p(D)}.$$

Thus we have proved that $\|w\|_{W_p^{s+2}(D)} \lesssim \|u\|_{L^p(D)}$.

Now we can repeat the whole procedure for the second term in $w_1(x)$ taking $s_1 = s-1$ and obtain $\|w_1(x)\|_{W_p^{s+1}(D)} \lesssim \|u\|_{L^p(D)}$.

Thus P maps continuously $L^p(D, 1/|\varrho|^{ps})$ onto $\mathrm{Harm}_p^s(D)$. This follows from the estimates from [1] and [21] applied to the biharmonic extension.

Thus we have proved Proposition 1 for integer s. Now we can use Proposition 2 and interpolate between $L^p(D, 1/|\varrho|^{p[s]})$ and $L^p(D, 1/|\varrho|^{p([s]+1)})$ (and between $\mathrm{Harm}_p^{[s]}(D)$ and $\mathrm{Harm}_p^{[s]+1}(D)$). This proves our proposition for every $s \geqslant 0$.

Remark. If $p > 2$ then the above annoying procedure can be avoided by using the results of [16] and [17]. There it was proved that P maps $L^2(D, 1/\varrho^{2s})$ into $\mathrm{Harm}_2^s(D)$ and $L^\infty(D, 1/|\varrho|^\alpha)$ into $\Lambda_\alpha \mathrm{Harm}(D)$. If $s \geqslant 1$ we can interpolate between $\mathrm{Harm}_2^s(D)$ and $\Lambda_\alpha \mathrm{Harm}(D)$ and between $L^2(D, 1/\varrho^{2s})$ and $L^\infty(D, 1/|\varrho|^\alpha)$. This follows from Proposition 2 and the remarks at the end of the introduction. However, this shorter method will not work for $p < 2$.

3. Proof of Theorems 1 and 2. The proof of Theorem 1 for $s > 0$ is the same as that given in [16] for $p = 2$. We shall give a brief account of it. If $s \geqslant 0$ is an integer and $u \in \mathrm{Harm}_p^s(D)$ then $\Delta^k(\varrho^k u) \in W_p^{s-k}(D)$. Thus $\varrho^k u \in W_p^{s+k}(D)$ as the solution of the Dirichlet problem $\Delta^k v = \Delta^k(\varrho^k u)$, v vanishes at ∂D up to order $k-1$. This implies by interpolation that T_k maps $\mathrm{Harm}_p^s(D)$ into $W_p^{s+k}(D)$ for each $s \geqslant 0$. (Interpolation in a similar context was first used by H. Boas in [7].) This also implies that if $u \in \mathrm{Harm}_p^s(D)$ then $D^\alpha u \cdot \varrho^\beta \in W_p^s(D)$ if $\beta \geqslant \alpha$. This yields that $L^r u \in W_p^s(D)$ by the very construction of L^r. Moreover, L^r maps continuously $\mathrm{Harm}_p^s(D)$ into $\mathring{W}_p^s(D) \subset L^p(D, 1/|\varrho|^{ps})$ if $r \geqslant s$. We can again use Proposition 2 and interpolate to get the statement of Theorem 1 for $s \geqslant 0$. Also, $\langle\ ,\ \rangle_s = \langle\ ,\ \rangle_r$ on $\mathrm{Harm}_q^{-s} \times \mathrm{Harm}_p^s$ if $r \geqslant s$, $q = p/(p-1)$.

We can now prove Theorem 2. First we prove that the smooth harmonic functions are dense in $\mathrm{Harm}_q^{-s}(D)$, s an integer. It follows from (*) that if $h \in \mathrm{Harm}_q^{-s}(D)$ then there exists $u \in \mathring{W}_q^s(D)$ such that $\Delta^s u = h$. Thus $\Delta^{s+1} u = 0$ and u is $(s+1)$-polyharmonic. The orthogonal projection onto the space of $(s+1)$-polyharmonic functions $\mathrm{Harm}_{(s+1),q}^s(D)$ is continuous in the $W_q^s(D)$ norm and maps smooth functions to smooth functions. Then there exists a sequence $u_n \to u$ in $\mathrm{Harm}_{(s+1),q}^s(D)$, $u_n \in C^\infty(\bar{D})$. Thus $\Delta^s u_n$ is a sequence of smooth harmonic functions s.t. $\Delta^s u_n \to h$ in $\mathrm{Harm}_q^{-s}(D)$.

Now we can prove Theorem 2 in the same manner as in [16] and [17], following Bell's proof from [3]. Let φ be a continuous functional on $\mathrm{Harm}_q^{-s}(D)$. It can be extended to a continuous functional $\tilde{\varphi}$ on $W_q^{-s}(D)$ and therefore there exists a function $f \in \mathring{W}_p^s(D)$, $p = q/(q-1)$, which represents $\tilde{\varphi}$. Then Pf represents φ on $\mathrm{Harm}_q^{-s}(D)$. Since the smooth functions are dense in $\mathrm{Harm}_q^{-s}(D)$ this representation does not depend on the choice of f and gives us an isomorphism between $(\mathrm{Harm}_q^{-s}(D))^*$ and $\mathrm{Harm}_p^s(D)$.

Now, let ψ be a continuous functional on $\mathrm{Harm}_p^s(D)$. It can be extended to a functional $\tilde{\psi}$ on $W_p^s(D)$. Each such functional can be written in the form

$$\tilde{\psi}(f) = \sum_{|\alpha|=s} \langle D^\alpha f, g_\alpha\rangle + \langle f, g_0\rangle, \quad g_\alpha, g_0 \in L^q(D),\ q = \frac{p}{p-1},$$

$$\|\tilde{\psi}\| = \inf_{g_\alpha, g_0} \left(\sum \|g_\alpha\| + \|g_0\|\right),$$

the infimum being taken over all g_α, g_0 representing $\tilde{\psi}$ (see [20]). If g_α, g_0 and $h \in \mathrm{Harm}_p^s(D)$ are in $C^\infty(\bar{D})$ then

$$\psi(h) = \tilde{\psi}(h) = \sum_{|\alpha|=s} \langle D^\alpha h, L^s g_\alpha\rangle + \langle h, g_0\rangle$$
$$= \langle h, P\big((-1)^s \sum_{|\alpha|=s} D^\alpha L^s g_\alpha + g_0\big)\rangle.$$

It is obvious that

$$S(\psi) = P\big((-1)^s \sum_{|\alpha|=s} D^\alpha L^s g_\alpha + g_0\big)$$

does not depend on the choice of the functions g_α, g_0 representing $\tilde{\psi}$. We must show that this mapping extends to a continuous mapping from $(W_p^s(D))^*$ onto $\mathrm{Harm}_q^{-s}(D)$. We have

$$\|S(\psi)\|_q^{-s} = \sup_{\substack{u \in \mathring{W}_p^s \\ \|u\|_p^s \le 1}} \big|\langle P\big((-1)^s \sum_\alpha D^\alpha g_\alpha + g_0\big), u\rangle\big|$$
$$= \sup_{\substack{u \in \mathring{W}_p^s \\ \|u\|_p^s \le 1}} \big|\langle g_0, Pu\rangle + \sum_\alpha \langle g_\alpha, D^\alpha Pu\rangle\big| \le c\big(\sum_\alpha \|g_\alpha\|_{L^q(D)} + \|g_0\|_{L^q(D)}\big).$$

This means that $P\big((-1)^s \sum D^\alpha L^s g_\alpha + g_0\big)$ extends to a continuous operator from $\prod_{|\alpha|=s} L^q(D) \times L^q(D)$. Hence

$$\|S(\psi)\|_q^{-s} \le c\|\psi\|.$$

Thus $(\mathrm{Harm}_p^s(D))^* = \mathrm{Harm}_q^{-s}(D)$.

To prove (b), we must proceed as in [17] and prove that $\mathrm{Harm}_p^s(D)$ represents the dual space of $L^q\,\mathrm{Harm}(D, |\varrho|^{qs})$, $q = p/(p-1)$. As in [17] we can prove that each $h \in \mathrm{Harm}_p^s(D)$ represents a continuous functional on $\mathring{L}^q\,\mathrm{Harm}(D, |\varrho|^{qs})$, the closure of $L^2\,\mathrm{Harm}(D)$ in $L^q(D, |\varrho|^{qs})$, via the pairing $\langle\ ,\ \rangle_s$.

Since every continuous functional φ on $\mathring{L}^q\,\mathrm{Harm}(D, |\varrho|^{qs})$ can be extended to a continuous functional $\tilde{\varphi}$ on $L^q(D, |\varrho|^{qs})$, it can be represented as $\tilde{\varphi}(f) = \langle f, m\rangle$, $m \in L^p(D, 1/|\varrho|^{ps})$, $p = q/(q-1)$. Since by Proposition 1, P maps $L^p(D, 1/|\varrho|^{ps})$ continuously onto $\mathrm{Harm}_p^s(D)$, it follows that Pm is the desired representation of φ and we have $\varphi(h) = \langle h, L^s Pm\rangle_s$ on $\mathring{L}^q\,\mathrm{Harm}(D, |\varrho|^{qs})$. It can be easily proved that Pm is independent of the choice of extension of φ and thus independent of m. Hence $(\mathring{L}^q\,\mathrm{Harm}(D, |\varrho|^{qs}))^* = \mathrm{Harm}_p^s(D)$.

Now we have

$$\mathring{L}^q\,\mathrm{Harm}(D, |\varrho|^{qs}) \subset \mathrm{Harm}_q^{-s}(D).$$

These two spaces have the same dual space and equivalent norms. Thus $\mathring{L}^q\,\mathrm{Harm}(D, |\varrho|^{qs}) = L^q\,\mathrm{Harm}(D, |\varrho|^{qs}) = \mathrm{Harm}_q^{-s}(D)$ and we have proved Theorem 2 for integer s.

Now we can again use Proposition 2 and interpolate to get our theorem for all s.

Part (b) of Theorem 2 immediately implies Theorem 1 for $s < 0$.

4. Proof of Theorem 3 and Proposition 3. Part (a) of Theorem 3 has been proved above. Part (c) is a direct consequence of the fact that P maps continuously $\Lambda_\alpha(D)$ onto $\Lambda_\alpha\,\mathrm{Harm}(D)$ and $[\Lambda_\alpha(D), \Lambda_\beta(D)]_{[\tilde{\theta}]} = \Lambda_t(D)$, $t = (1-\theta)\alpha + \theta\beta$ (see [6], Th. 6.4.5, $\Lambda_\alpha = B^\alpha_{\infty,\infty}$). Thus only part (b) remains to be proved. Recall that $\mathrm{Harm}_p^s(D)$ is the image of $L^p(D, 1/|\varrho|^{sp})$ under the projection P and $\Lambda_\alpha\,\mathrm{Harm}(D) = P\big(L^\infty(D, 1/|\varrho|^\alpha)\big)$. Recall also that $L^\infty(D, 1/|\varrho|^\alpha) = |\varrho|^\alpha L^\infty(D)$. Thus P maps continuously

$$[L^p(D, 1/|\varrho|^{sp}), L^\infty(D, 1/|\varrho|^\alpha)]_{[\theta]} \to [\mathrm{Harm}_p^s(D), \Lambda_\alpha\,\mathrm{Harm}(D)]_{[\theta]}$$

and L^r ($r \ge \max(s, \alpha)$, r an integer) maps continuously

$$[\mathrm{Harm}_p^s(D), \Lambda_\alpha\,\mathrm{Harm}(D)]_{[\theta]} \to [L^p(D, 1/|\varrho|^{sp}), L^\infty(D, 1/|\varrho|^\alpha)]_{[\theta]}.$$

By Proposition 2

$$[L^p(D, 1/|\varrho|^{sp}), L^\infty(D, 1/|\varrho|^\alpha)]_{[\theta]} = L^q(D, 1/|\varrho|^m),$$
$$q = \frac{p}{1-\theta}, \quad m = \frac{p}{1-\theta}(s(1-\theta) + \theta\alpha).$$

Thus

$$[\mathrm{Harm}_p^s(D), \Lambda_\alpha\,\mathrm{Harm}(D)]_{[\theta]} = P\big(L^q(D, 1/|\varrho|^m)\big) = \mathrm{Harm}_q^t(D),$$
$$t = (1-\theta)s + \theta\alpha.$$

The above used fact that L^r maps $\Lambda_\alpha\,\mathrm{Harm}(D)$ into $L^\infty(D, 1/|\varrho|^\alpha)$ if $r > \alpha$ follows directly from the construction of L^r and the estimate

$$|D^\beta u(x)| \le \frac{c_\beta \|u\|_{\Lambda_\alpha}}{\mathrm{dist}(x, \partial D)^{|\beta|-\alpha}} \quad \text{if } |\beta| > \alpha.$$

The proof is the same as in [17] for $r = [\alpha]+1$.

We now prove Proposition 3. The mean value theorem implies that if $u \in \mathrm{Bl}\,\mathrm{Harm}(D)$ then

$$|D^\beta u(x)| \le \frac{c_\beta \|u\|_{\mathrm{Bl}\,\mathrm{Harm}(D)}}{\mathrm{dist}(x, \partial D)^{|\beta|}}.$$

This and the construction of L^r imply that L^r maps continuously $\mathrm{Bl}\,\mathrm{Harm}(D)$ into $L^\infty(D)$. Thus if P maps continuously $L^\infty(D)$ onto $\mathrm{Bl}\,\mathrm{Harm}(D)$ we can repeat the above procedure and get the proof of part (a) of Proposition 3. To get part (b) we must interpolate between $L^\infty(D)$ and $L^\infty(D, 1/|\varrho|^\alpha)$.

Here we have the following situation: $[L^\infty(D), L^\infty(D, 1/|\varrho|^\alpha)]_{[\theta]}$ is equal to the space $L^\infty_0(D, 1/|\varrho|^{\alpha\theta})$ of functions f from $L^\infty(D, 1/|\varrho|^{\alpha\theta})$ such that

$|f(x)|/|\varrho(x)|^{\alpha\theta} \to 0$ if $\varrho(x) \to 0$. The completion of $L^\infty_0(D, 1/|\varrho|^{\alpha\theta})$ with respect to $L^\infty(D)$ is $L^\infty(D, 1/|\varrho|^{\alpha\theta})$. (Recall that if $E_1 \subset E_2$ then the *completion* of E_1 with respect to E_2 is the space $\tilde{E}_1$ of all elements $a \in E_2$ for which there exists a sequence $a_n \to a$ in E_2, $a_n \in E_1$, $\exists c\ \forall n\ \|a_n\|_{E_1} \leqslant c$. The norm in $\tilde{E}_1$ is the least c for which there exists such a sequence.) Thus we have

$$[L^\infty(D), L^\infty(D, 1/|\varrho|^{\alpha})]_{[\theta]}^{\sim} = L^\infty(D, 1/|\varrho|^{\alpha\theta})$$

(cf. [11], Chap. IV, § 1, items 6 and 9). Hence P maps $L^\infty(D, 1/|\varrho|^{\alpha\theta})$ into $[\mathrm{Bl}\,\mathrm{Harm}(D), \Lambda_\alpha \mathrm{Harm}(D)]_{[\theta]}^{\sim}$ and L^r, $r > 0$, maps the last space into $L^\infty(D, 1/|\varrho|^{\alpha\theta})$. Thus

$$\Lambda_{\alpha\theta}\mathrm{Harm}(D) = P\big(L^\infty(D, 1/|\varrho|^{\alpha\theta})\big) = [\mathrm{Bl}\,\mathrm{Harm}(D), \Lambda_\alpha \mathrm{Harm}(D)]_{[\theta]}^{\sim}.$$

5. Proof of Theorems 4–8. Theorem 4 can be proved in the following way. The projector B is continuous from $L^2(D)$ into $L^2(D)$ from the very definition of B. Since, by the assumptions of Theorem 4, B maps $L^\infty(D)$ into $\mathrm{Bl}\,H(D)$ we find that B maps $[L^2(D), L^\infty(D)]_{[\theta]} = L^{2/(1-\theta)}(D)$ into $[L^2 H(D), \mathrm{Bl}\,H(D)]_{[\theta]}$. The operator L^1 maps $L^2 H(D)$ into $L^2(D)$ and $\mathrm{Bl}\,H(D)$ into $L^\infty(D)$. Thus L^1 maps $[L^2 H(D), \mathrm{Bl}\,H(D)]_{[\theta]}$ into $L^{2/(1-\theta)}(D)$. We have $PL^1 u = u$ if u is holomorphic and hence $[L^2 H(D), \mathrm{Bl}\,H(D)]_{[\theta]} \subset L^{2/(1-\theta)}\mathrm{Harm}(D)$. This means that B maps continuously $L^p(D)$ onto $L^p H(D)$ for each $p \geqslant 2$. Since B is selfadjoint a simple duality argument yields that B maps $L^p(D)$ onto $L^p H(D)$ for every $1 \leqslant p < \infty$. Thus Theorem 3 implies that B maps $\mathrm{Harm}^s_p(D)$ onto $H^s_p(D)$ for all $s \geqslant 0$ and $1 < p < \infty$, since we can interpolate between $L^p\mathrm{Harm}(D)$ and $\Lambda_\alpha\mathrm{Harm}(D)$ for all p and $\alpha > 0$. Since $BP = B$, we find that B is continuous in all Sobolev norms.

Theorem 5 follows immediately from Theorem 4, the Hölder estimates of B, and Bloch norm estimates from [19].

In order to prove Theorem 6 we must show that $L^2 H(D)$ is dense in $L^q H(D, |\varrho|^{qs})$. The rest will follow from Theorem 5 (cf. [3], [16] and [17]).

Let us consider the Forelli–Rudin projections on D. These projections are constructed in the following way. We take a domain $\tilde{D} \subset C^{n+1}$, $\tilde{D} = \{(t, z) \in C^{n+1}\colon |t|^2 + \varrho(z) < 0\}$. $\tilde{D}$ is also a strictly pseudoconvex domain and its Bergman projection maps $L^q(\tilde{D})$ into $L^q(\tilde{D})$ and smooth functions on $\tilde{D}$ into smooth functions.

If f is a function on D we take $B_m(f) = \tilde{B}(t^m f)/t^m$. $\tilde{B}$ denotes here the Bergman projection on $\tilde{D}$. It is easy to check that B_m maps continuously $L^q(D, |\varrho|^{mq/2+1})$ onto $L^q H(D, |\varrho|^{mq/2+1})$. This implies that the smooth functions are dense in $L^q H(D, |\varrho|^{mq/2+1})$.

Thus for each s we can find $s_1 > s$ such that $L^2 H(D)$ is dense in $L^q H(D, |\varrho|^{qs_1})$. Suppose that $L^2 H(D)$ is not dense in $L^q H(D, |\varrho|^{qs})$. Then there exists a functional φ on $L^q H(D, |\varrho|^{qs})$ such that $\varphi \equiv 0$ on $L^2 H(D)$ and $\varphi(h) \neq 0$ for some $h \in L^q H(D, |\varrho|^{qs})$. Note that $h \in L^q H(D, |\varrho|^{qs_1})$. The functional φ can be extended to a continuous functional $\tilde{\varphi}$ on $L^q\,\mathrm{Harm}(D, |\varrho|^{qs})$ and therefore there exists $u \in \mathrm{Harm}^s_p(D)$, $p = q/(q-1)$, which represents $\tilde{\varphi}$. We have $Bu = 0$. Since B is continuous in all Sobolev norms, there exists a sequence of smooth harmonic functions $u_n \to u$ in $\mathrm{Harm}^s_p(D)$, $Bu_n = 0$. Therefore if n is sufficiently large then $\langle h, u_n\rangle_s \neq 0$. Hence $\langle h, u_n\rangle_{s_1} = \langle h, u_n\rangle_s \neq 0$ and $\langle f, u_n\rangle_{s_1} = \langle f, u_n\rangle_0$ for all $f \in L^2 H(D)$. Thus $L^2 H(D)$ cannot be dense in $L^q H(D, |\varrho|^{qs_1})$. Contradiction.

Now, Theorem 7 is a direct consequence of Theorems 6 and 2. Theorem 8 follows immediately from Theorems 3, 5 and 6.

6. Proof of Proposition 4. First we shall prove that it suffices to estimate N on forms with harmonic coefficients. Let

$$\omega(x) = \sum_{I,J} a_{IJ}(x)\, d\bar{z}_I \wedge dz_J$$

be a (p, q)-form on D. Let

$$b_{IJ}(x) = Pa_{IJ}, \qquad c_{IJ}(x) = G_2\,\Delta a_{IJ},$$

where G_2 is the operator defined above. Then we have

$$\omega(x) = \Box\big(N\omega_1(x) + \omega_2(x)\big),$$

$$\omega_1(x) = \sum_{I,J} b_{IJ}(x)\, d\bar{z}_I \wedge dz_J, \qquad \omega_2(x) = 2\sum_{I,J} c_{IJ}(x)\, d\bar{z}_I \wedge dz_J.$$

If $\omega \in W^k_p(D)$ then $\omega_2 \in W^{k+2}_p(D)$ and therefore it suffices to estimate $N\omega_1(x)$.

This also implies that N maps $L^2_{(p,q)}(D)$ into $W^s_{2,(p,q)}(D)$ and $\Lambda_{\alpha,(p,q)}$ into $\Lambda_{\alpha+s,(p,q)}$ iff it maps the space $\Lambda_\alpha\mathrm{Harm}_{(p,q)}(D)$ of forms with harmonic coefficients into the space $\Lambda_\alpha\mathrm{Harm}^{(2)}_{(p,q)}(D)$ of forms with biharmonic coefficients and the space $L^2\mathrm{Harm}_{(p,q)}(D)$ of L^2 forms with harmonic coefficients into the space $\mathrm{Harm}^{(2),s}_2(D)$ of forms with biharmonic coefficients.

It was mentioned in the introduction that all results of this paper remain true if the space of harmonic functions is replaced by the space of m-polyharmonic functions. For biharmonic functions this is really easy to see. It suffices to replace the potential $1/|x-y|^{2n-2}$ in the proof of Proposition 2 by $1/|x-y|^{2n-4}$ and go through the proofs of Theorems 1 and 2 remembering that the Laplacians of biharmonic functions are harmonic. In particular, we can interpolate between $L^2\mathrm{Harm}^{(2)}_{(p,q)}(D)$ and $\Lambda_\alpha\mathrm{Harm}^{(2)}_{(p,q)}(D)$ as in the case of harmonic functions. Thus we can prove that N maps $W^k_{r,(p,q)}(D)$ into $W^{k+s}_{r,(p,q)}(D)$ for all $k \geqslant 0$ and $2 < r < \infty$. The same procedure can be used for the operator $\partial^* N$.

III. Remarks. 1. If the boundary of D is of class C^{k+4} then we can take a special defining function ϱ_0 which is a biharmonic function on D such that

$\varrho_0 = 0$ and $\partial\varrho_0/\partial n = 1$ on ∂D, and using the same methods as in [17] and [18] we can find that Theorems 1–3 remain valid for $s, \alpha < k$. However, the interpolation theorem alone gives worse results than in the smooth case; e.g. we can prove Theorem 4 only for s, p such that $(p-1)k/p > s$. This result is far from the best possible. In [9] Greene and Krantz proved the estimates for the $\bar\partial$-Neumann problem in the $\|\ \|_2^s$ norm if ∂D is of class C^{k+3}. If we use in addition these estimates we get Theorems 5–8 for strictly pseudoconvex domains with C^{k+4} boundary, $s < k$, $p \geq 2$ or $1 < p < 2$, $s < (2p-2)p$.

2. In [19] it was proved that on a smooth strictly pseudoconvex domain the difference between the Szegö projection considered as an operator acting on harmonic functions and the Bergman projection restricted to the space of harmonic functions is a smoothing integral operator. Thus the Szegö projection extends to a continuous mapping from $\mathrm{Harm}_p^s(D)$ to $H_p^s(D)$, $1 \leq p < \infty$, $s \geq 0$. To prove this we use the Hölder estimates for the Szegö and Bergman projections, the fact that the difference between S and B is a compact operator on $L^p\,\mathrm{Harm}(D)$, and Theorems 3 and 4.

3. In [18] and [19] it was proved that the operators: Q of orthogonal projection on pluriharmonic functions, S_r of orthogonal projection on $\mathrm{Re}\, L^2 H(D)$ and $S = S_r \otimes C$ have the same regularity properties as B for any pseudoconvex domain D.

Thus Theorems 5–8 remain valid if B is replaced with one of these projections and the spaces of holomorphic functions with the corresponding spaces of pluriharmonic functions.

4. The duality theorem for the interpolation functor implies that the spaces

$$E_p^s = L^q\,\mathrm{Harm}(D, |\varrho|^{ps}), \quad q = \frac{p}{p-1}, \quad 1 < p < \infty,\ s \geq 0,$$

$$E_\infty^s = \mathring{L}^1\,\mathrm{Harm}(D, |\varrho|^s), \quad s > 0$$

(the closure of $L^2\,\mathrm{Harm}(D)$ in $L^1(D, |\varrho|^s)$)

also form a double interpolation scale. If P maps $L^\infty(D) \to \mathrm{Bl\,Harm}(D)$, then the "vertex" of this scale is $E_\infty^0 = L^1\,\mathrm{Harm}(D)$ since in this case $(L^1\,\mathrm{Harm}(D))^* = \mathrm{Bl\,Harm}(D)$.

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