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A uniform F-algebra with locally compact spectrum and with nonglobal peak points

by

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Abstract. We exhibit a uniform Fréchet algebra A with locally compact spectrum $M(A)$ which has local peak points in $M(A)$ but has no global peak points.

1. Introduction. For the basic concept of uniform F-algebras we refer the reader to the book of Kramm [2].

By a *uniform F-algebra A* we mean a point-separating subalgebra of the algebra of all continuous functions on a hemicompact Hausdorff space which is complete with respect to the compact-open topology and which contains the constants.

A point $x \in M(A)$ is said to be a *local peak point* of A if there exists a neighbourhood U of x in $M(A)$ and a function $f \in \hat{A}$ (the algebra of Gelfand transforms) such that $f(x) = 1$ and $|f(y)| < 1$ if $y \in U \setminus \{x\}$. If A is a uniform Banach algebra then, by a well-known theorem of Rossi, every local peak point is a *global peak point*, i.e., U can be taken to be $M(A)$. This result is not valid for uniform F-algebras. Meyers [3] constructed an example of a uniform F-algebra which has local but no global peak points. In this example $M(A)$ is not locally compact. Later Carpenter [1] proved that locally accessible points are accessible; however, the question remained open whether an analogue of Rossi's result is valid for uniform F-algebras with locally compact spectrum (cf. [4]).

It is the purpose of this note to answer this question in the negative. Moreover, our example—in the opinion of the author—seems to be very simple.

2. The example. We first define the following sets in C^2:

$$X_0 = \{(z_1, z_2) \in C^2\colon |z_1| \leqslant 1,\ z_2 = 1\},$$

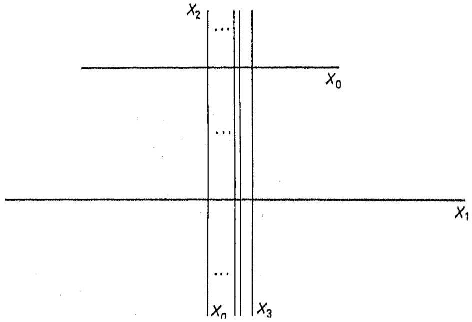
$$X_1 = C \times \{0\},$$

$$X_2 = \{0\} \times C,$$

$$X_n = \{1/n\} \times C \quad \text{for } n > 2 \quad \text{and}$$

$$X = \bigcup X_n.$$

X may be illustrated in the following way.



It is clear that X is locally compact. If we denote by K_n the set $X \cap \{(z_1, z_2) \in C^2\colon |z_1| \leqslant n, |z_2| \leqslant n\}$ for $n \in N$, then $K_n \subset K_{n+1} \subset \ldots$ is an admissible exhaustion of X, i.e. $\bigcup K_n = X$ and K_n is contained in the interior of K_{n+1}.

Moreover, each K_n is polynomially convex. To prove this let $x = (x_1, x_2) \notin K_n$. If $|x_1| > n$ or $|x_2| > n$ then $|z_1(x)| > \|z_1\|_{K_n} = \sup\{|z_1(y)|\colon y \in K_n\}$ or $|z_2(x)| > \|z_2\|_{K_n}$ where z_1, z_2 denote the coordinate functions. Now let $|x_1| \leqslant n$ and $|x_2| \leqslant n$ and assume that $x_2 \neq 1$. Since $x \notin K_n$ there is $m \in N$ such that $|x_1| > 1/m$. For sufficiently large $l \in N$ we have $|p(x)| > \|p\|_{K_n}$ for the polynomial

$$p(z_1, z_2) = \prod_{i=3}^{m} (z_1 - 1/i) \cdot (z_2 - 1) \cdot z_2 \cdot (z_1/|x_1|)^l.$$

The case $x_2 = 1$ can be treated similarly.

Since K_n is polynomially convex we can identify K_n with the spectrum of $P(K_n)$, the algebra of all continuous functions on K_n which can be approximated uniformly on K_n by polynomials. We consider the algebra $A = \varprojlim P(K_n)$, the projective limit with respect to the natural restriction mappings, and obtain a uniform F-algebra. The spectrum of A can be identified as a set with $\bigcup M(P(K_n)) = X$. Since each $f \in A$ is a continuous function on every K_l and $K_n \subset K_{n+1} \subset \ldots$ is an admissible exhaustion of X, the Gelfand topology and the induced Euclidean topology are equivalent on X (note that sets of the form $\{x \in X\colon |f_i(x) - f_i(y)| < \varepsilon, i = 1, \ldots, r\}$, $f_1, \ldots, f_r \in A$, $\varepsilon > 0$, are a basis of open neighbourhoods of $y \in X$ for the Gelfand topology).

Now let $f \in A = \hat{A}$ be a function with $|f(x)| \leqslant 1$ for all $x \in X$. Since f is the uniform limit of polynomials, $f|_{X_i}$ $(i \geqslant 1)$ can be regarded as holomorphic functions on C and $f|_{X_0}$ can be regarded as an element of the disc algebra. Since X is connected it follows from Liouville's theorem and the identity theorem that f is constant. Hence A has no global peak points. On the other hand, the polynomials $(z_1 + x_1)/2x_1 \in A$ peak locally at each point $(x_1, 1)$ of the set

$$\{(z_1, z_2)\colon |z_1| = 1, z_2 = 1\} \subset X.$$

References

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[4] —, *Local peak sets of certain algebras of functions on non-compact spaces*, in: Papers from the Summer Gathering on Function Algebras at Aarhus, Matematisk Inst. Aarhus Univ., 1969, 67–71.

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