STUDIA MATHEMATICA, T. LXXXVII. (1987)

Two-weight mixed norm inequalities for maximal operators and extrapolation results for the fractional maximal operator

by

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Abstract. Two-weight mixed norm inequalities are studied for a generalized maximal operator through the use of a rearrangement inequality. Necessary and sufficient conditions are obtained for the restricted weak type norm inequality. One application of the methods presented in this paper is to study the problem of lowering a two-weight norm inequality for the fractional maximal operator. Necessary and sufficient conditions are established for this problem which is the two-weight fractional maximal operator analogue of the A_p implies $A_{p-\varepsilon}$ result for the Hardy–Littlewood maximal operator.

1. Let μ and ν be Borel measures on $\boldsymbol{R}^n$ and $f\colon \boldsymbol{R}^n \to \boldsymbol{R}$ a Borel measurable function. For every cube Q in $\boldsymbol{R}^n$ let there be associated a Borel measurable function φ_Q supported in Q. We define the general maximal operator as

$$Mf(x) = \sup \int \varphi_Q f \, d\nu,$$

where the supremum is taken over all φ_Q where the center y of Q satisfies $|x-y| < \frac{1}{2}\operatorname{diam} Q$. For $0 < p \leqslant q < \infty$, we first establish the following rearrangement inequality.

THEOREM 1. $(Mf)^*_\mu(\xi^{q/p}) \leqslant A \int_0^\infty \Phi(t) f^*_\nu(t\xi)\, dt, \quad 0 < \xi < \infty.$

The rearrangement of a function g with respect to a measure ω is $g^*_\omega(t) = \inf\{s\colon \omega\{|g| > s\} \leqslant t\}$. The function Φ will depend upon μ, ν, p and q. If $p \geqslant 1$ we may apply Hölder's inequality on the right and obtain

$$\lambda^{1/q}(Mf)^*_\mu(\lambda) \leqslant C\, \|\Phi\|_{p',\infty} \|f\|_{p,1,\nu}.$$

Thus M maps $L^{p,1}_\nu$ boundedly into $L^{q,\infty}_\mu$ if Φ belongs to $L^{p',\infty}(0, \infty)$. Other variations can easily be obtained and we use some of them in our later application to the fractional maximal operator. However, we are able to reverse the above which we state as Theorem 2. That is, if the functions φ_Q are compatible (Definition 1), which includes most reasonable examples, or if μ satisfies a doubling condition, then Φ belongs to $L^{p',\infty}(0, \infty)$ if M maps $L^{p,1}_\nu$ boundedly into $L^{q,\infty}_\mu$. Thus we can establish if and only if conditions for two-weight, mixed, restricted type norm inequalities for many general maximal operators. We should note that by [4, Theorem 4] this is the best possible result in this direction.

The advantage of using Φ to study weighted norm inequalities is that Φ can be decomposed into simple pieces (Lemma 1). This gives us a simple picture which we can use to study all weighted norm inequalities for M. In particular, we shall study $M_\alpha f(x) = \sup \int_Q f\,dx/|Q|^\alpha$, $0 < \alpha \leqslant 1$, in Sections 3 and 4. The case $\alpha = 1$ is the Hardy–Littlewood maximal operator which has been studied extensively for $p = q$ (see for instance [4]–[8]). Given $\|M_\alpha f\|_{q,\infty,\mu} \leqslant A\|f\|_{p,1,\nu}$, $1 < p \leqslant q < \infty$, we establish (Theorem 3) necessary and sufficient conditions to have

$$\|M_\alpha f\|_{q_\varepsilon,\mu} \leqslant A_\varepsilon \|f\|_{p-\varepsilon,\nu}, \qquad q_\varepsilon = \left(\frac{p-\varepsilon}{p}\right) q.$$

And we establish (Theorem 4) sufficient conditions to have

$$\|M_\alpha f\|_{q_\delta,\mu} \leqslant A_\delta \|f\|_{p+\delta,\nu}, \qquad q_\delta = \left(\frac{p+\delta}{p}\right) q.$$

This is the two-weight fractional maximal operator generalization of the single weight A_p implies $A_{p-\varepsilon}$ result of [6]. However, key ideas for obtaining the above results are derived in [4] and [5] where the case $\alpha = 1$, $p = q$ is handled. We define pseudo-iterated operators as

$$M_{\alpha,j} f(x) = \sup \frac{1}{|Q|^\alpha} \int_0^{|Q|} (f\chi_Q)^*(t) \frac{\log^j(|Q|^\alpha/t + e)}{j!}\,dt,$$

$$M_\alpha^j f(x) = \sup \frac{1}{|Q|^\alpha} \int_0^{|Q|} (f\chi_Q)^*(t) \frac{\log^j(t/|Q|^\alpha + e)}{j!}\,dt$$

where $j = 1, 2, \ldots$

We study and estimate the functions Φ they generate. We are thus able to show that if the above operators are bounded from $L_\nu^{p,1}$ to $L_\mu^{q,\infty}$ by a geometric constant A^j, then we may push the norm inequality for M_α down and up respectively, keeping the ratio p/q fixed.

We note that when $\alpha = 1$, $M_{\alpha,j}$ is the $(j+1)$-iterated Hardy–Littlewood maximal operator. M_1^j is the Hardy–Littlewood maximal operator divided by $j!$ for which pushing up the norm inequality is trivial.

2. Let μ and ν be Borel measures on $\boldsymbol{R}^n$; to avoid technicalities we will assume Borel sets always have nonnegative measure. The *nonincreasing rearrangement* of a function g with respect to a measure ω is defined as $g_\omega^*(t) = \inf\{s\colon \omega\{|g| > s\} \leqslant t\}$. We define the space $L_\omega^{p,q}$ [3] as the collection of all

g with $\|g\|_{p,q,\omega} < \infty$, where

$$\|g\|_{p,q,\omega} = \begin{cases} \left(\dfrac{q}{p}\displaystyle\int_0^\infty (t^{1/p} g_\omega^*(t))^q \frac{dt}{t}\right)^{1/q}, & 1 \leqslant p < \infty,\ 1 \leqslant q < \infty, \\ \sup\limits_{t>0} t^{1/p} g_\omega^*(t), & 1 \leqslant p \leqslant \infty,\ q = \infty. \end{cases}$$

For each cube $Q \subset \boldsymbol{R}^n$ let $\varphi_Q\colon \boldsymbol{R}^n \to (0, \infty)$ be Borel measurable and supported in Q. We consider the maximal operator $Mf(x) = \sup \int \varphi_Q f\,d\nu$, where the supremum is taken over all φ_Q where the center y of Q satisfies $|x-y| < \frac{1}{2}\operatorname{diam} Q$. Given $0 < p \leqslant q < \infty$ let

$$\Phi(t) = \sup \mu^{p/q}(Q)(\varphi_Q)_\nu^*(\mu^{p/q}(Q)\,t)$$

where the supremum is over all φ_Q. We note that p and p' will always be related by $1/p + 1/p' = 1$ and A, B, C will denote constants depending only upon the dimension n and possibly μ and ν, with a subscript denoting further dependence.

THEOREM 1. *Let* $0 < p \leqslant q \leqslant \infty$. *We have*

$$(Mf)_\mu^*(\xi^{q/p}) \leqslant A \int_0^\infty \Phi(t) f_\nu^*(t\xi)\,dt$$

for $0 < \xi < \infty$ *where A depends only upon the dimension n.*

Proof. We let $M_r f(x) = \sup \int \varphi_Q f\,d\nu$, where the supremum is restricted to cubes Q with center y and $|x-y| < \frac{1}{2}\operatorname{diam} Q$, $|Q| < r$. It suffices to prove the theorem for $M_r f$ and then let $r \uparrow \infty$.

Let $E_\tau = \{x\colon M_r f(x) > \tau\}$ and $E_{\tau,R} = E_\tau \cap \{|x| \leqslant R\}$. For every $x \in E_{\tau,R}$ we have a Q_x with center y, $|x-y| < \frac{1}{2}\operatorname{diam} Q_x$ and $\tau \leqslant \int \varphi_{Q_x} f\,d\nu$. We may apply the Besicovitch covering lemma [2] and select $\{Q_j\} \subset \{Q_x\colon x \in E_{\tau,R}\}$ such that $E_{\tau,R} \subset \bigcup Q_j$ and $\sum \chi_{Q_i} \leqslant C$, where C depends only upon dimension n. Set

$$H_N = \sum_{j=1}^N \mu^{p/q}(Q_j), \qquad \Phi_N = \sum_{j=1}^N \mu^{p/q}(Q_j)\varphi_{Q_j}.$$

Then since $p \leqslant q$ we have

$$\text{(I)} \qquad \tau\Big(\sum_{j=1}^N \mu(Q_j)\Big)^{p/q} \leqslant \tau H_N \leqslant \int \Phi_N(y) f(y)\,d\nu \leqslant \int_0^\infty (\Phi_N)_\nu^*(t) f_\nu^*(t)\,dt.$$

We claim that $(\Phi_N)_\nu^*(t) \leqslant C\Phi(t/H_N)$ where C is the Besicovitch constant. To see this consider $\alpha > 0$. If $\Phi_N(x) > \alpha$ then $x \in \bigcup Q_j$ and the number of Q_j's containing x is at most C. Thus $\mu^{p/q}(Q_j)\varphi_{Q_j}(x) > \alpha/C$ for some j. We have

$$\{x\colon \Phi_N(x) > \alpha\} \subset \bigcup \{x\colon \mu^{p/q}(Q_j)\varphi_{Q_j}(x) > \alpha/C\},$$

$$\nu\{x\colon \mu^{p/q}(Q_j)\varphi_{Q_j}(x) > \alpha/C\} \leqslant \mu^{p/q}(Q_j)\,|\{t\colon \Phi(t) > \alpha/C\}|.$$

The two statements imply

$$|\{t\colon (\Phi_N)_\nu^*(t) > \alpha\}| \leqslant H_N |\{t\colon \Phi(t) > \alpha/C\}|.$$

Hence $(\Phi_N)_\nu^*(t) = \inf\{\alpha\colon |\{(\Phi_N)_\nu^* > \alpha\}| \leqslant t\} \leqslant C\Phi(t/H_N)$.

Inequality (I) now becomes

$$\tau H_N \leqslant C\int_0^\infty \Phi(t/H_N) f_\nu^*(t)\,dt \leqslant C\int_0^\infty \Phi(t/H) f_\nu^*(t)\,dt$$

where $H = \sum \mu^{p/q}(Q_j) \geqslant H_N$. Since $H_N \uparrow H$ and $\mu^{p/q}(E_{\tau,R}) \leqslant H$ we infer

$$\tau \leqslant \frac{C}{H}\int_0^\infty \Phi(t/H) f_\nu^*(t)\,dt = C\int_0^\infty \Phi(t) f_\nu^*(tH)\,dt \leqslant C\int_0^\infty \Phi(t) f_\nu^*(\mu^{p/q}(E_{\tau,R})t)\,dt.$$

Let $\tau_0 = (M_r f)_\mu^*(\xi) = \inf\{\tau\colon \mu(E_\tau) \leqslant \xi\}$. Then for $\tau < \tau_0$, $\mu(E_\tau) > \xi$, and for some $R > 0$, $\mu(E_{\tau,R}) > \xi$. From this we have

$$\tau \leqslant C\int_0^\infty \Phi(t) f_\nu^*(t\xi^{p/q})\,dt,$$

and letting $\tau \uparrow \tau_0$ completes the proof.

By applying Hölder's inequality and Minkowski's inequality to the rearrangement inequality of Theorem 1 we derive, respectively,

$$\|Mf\|_{q,\infty,\mu} \leqslant A\|\Phi\|_{p',r'}\|f\|_{p,r,\nu} \quad \text{and} \quad \|Mf\|_{q,s,\mu} \leqslant A\|\Phi\|_{p',1}\|f\|_{p,s,\nu}.$$

Thus we have norm inequalities if we know that Φ lies in $L^{p',r'}(0,\infty)$.

Next we show that if μ is doubling, $\mu \in D_\infty$, i.e. if $\mu(2Q) \leqslant C\mu(Q)$, then $\|Mf\|_{q,\infty,\mu} \leqslant B\|f\|_{p,1,\nu}$ if and only if $\Phi \in L^{p',\infty}(0,\infty)$. If μ is not doubling then we must require that the collection $\{\varphi_Q\}$ satisfy a compatibility condition which we define below in Definition 1. We note that the Hardy–Littlewood and fractional maximal operators satisfy this condition.

Definition 1. The set $\{\varphi_Q\} \in C_\infty$ if given φ_Q and f there exists a $\varphi_{\bar{Q}}$ such that $2Q \subset \bar{Q}$, the centers x and $\bar{x}$ of Q and $\bar{Q}$ satisfy $|x - \bar{x}| < \frac{1}{2}\operatorname{diam}(\bar{Q})$ and

$$C\int f\varphi_{\bar{Q}}\,d\nu \geqslant \int f\varphi_Q\,d\nu,$$

where C is independent of the choice of f, φ_Q and $\varphi_{\bar{Q}}$.

Theorem 2. *Suppose* $\{\varphi_Q\}$ *belongs to* C_∞ *or the measure* μ *belongs to* D_∞. *Then* (i) *and* (ii) *below are equivalent*:

(i) $\|Mf\|_{q,\infty,\mu} \leqslant B\|f\|_{p,1,\nu}$, $\quad f \in L_\nu^{p,1}$, $1 \leqslant p \leqslant q < \infty$.

(ii) $\Phi \in L^{p',\infty}(0,\infty)$, $\quad 1/p' + 1/p = 1$, $\quad 1 \leqslant p < \infty$.

Proof. We first prove (ii) implies (i). Applying Hölder's inequality to the left side of the rearrangement inequality of Theorem 1, we obtain

$$(Mf)_\mu^*(\xi^{q/p}) \leqslant A\|\Phi\|_{p',\infty}\|f\|_{p,1,\nu}/\xi^{1/p}, \qquad 0 < \xi < \infty.$$

Equivalently, $\sup_\lambda \lambda^{1/q}(Mf)_\mu^*(\lambda) \leqslant A\|\Phi\|_{p',\infty}\|f\|_{p,1,\nu}$.

We now assume (i). Given φ_Q choose f such that $\operatorname{supp} f \subset Q$, $\|f\|_{p,1,\nu} = 1$ and $\int \varphi_Q f\,d\nu \geqslant C\|\varphi_Q\|_{p',\infty,\nu}$. If $\mu \in D_\infty$ then for $x \in Q/4$ we have $Mf(x) \geqslant C\|\varphi_Q\|_{p',\infty,\nu}$. If not, we require the compatibility condition to hold for $\{\varphi_Q\}$ to obtain $Mf(x) \geqslant C_2\|\varphi_Q\|_{p',\infty,\nu}$ for $x \in Q$. In either case our assumption implies $\mu^{1/q}(Q) \leqslant C/\|\varphi_Q\|_{p',\infty,\nu}$. Thus it follows that

$$\mu^{p/q}(Q)(\varphi_Q)_\nu^*(\mu^{p/q}(Q)t) \leqslant \frac{\mu^{p/q}(Q)}{t^{1/p'}[\mu(Q)]^{p/(qp')}}\sup \tau^{1/p'}(\varphi_Q)_\nu^*(\tau) \leqslant Ct^{-1/p'},$$

where $1/p + 1/p' = 1$.

Hence $\Phi(t) \in L^{p',\infty}(0,\infty)$ since our choice of φ_Q was arbitrary. This completes the proof.

By Theorem 2, Φ is the correct quantity for establishing if and only if conditions to obtain two-weight restricted weak type mixed norm inequalities. By [4, Theorem 4], we can infer that this result cannot be strengthened to include weak type or strong type norm inequalities. We note that weak type norm inequalities are classified by a variant of Muckenhoupt's A_p condition, i.e. $\mu^{p/q}(Q)\left(\int \varphi_Q^{p'}\,d\nu\right)^{p-1} \leqslant C$.

As for strong type norm inequalities, we have Sawyer's condition [8]

$$\int_Q (M_\alpha(\chi_Q \nu^{-p'/p}))^q\,d\mu \leqslant C\left(\int_Q \nu^{-p'/p}\right)^{q/p},$$

for the case $\varphi_Q = \chi_Q/(\nu|Q|^\alpha)$, $0 < \alpha \leqslant 1$. Presumably a variant of Sawyer's condition might classify strong type norm inequalities for our general maximal operators.

Our next result is crucial in that it makes Φ a useful object of study. In Lemma 1 we decompose Φ and obtain a simple picture that applies to all two-weight problems involving our general maximal operators. In the applications of the following sections, Lemma 1 is decisive in computing needed estimates.

We require $\mu(E) = \int_E \mu\,dx$ and $\nu(E) = \int_E \nu\,dx$ for measurable functions μ and ν. Since Φ is basically a rearrangement, we assume we have perturbed φ_Q slightly so as to have $(\varphi_Q)_\nu^*(t)$ strictly decreasing for $t \in (0, \nu(\operatorname{supp}\varphi_Q))$. This can be done so as not to appreciably affect the size of Φ.

Lemma 1. *Let* $N \in \mathbf{Z}$. *There exist* φ_{Q_N}, α_N *and parallel rectangles* R_N *and* R'_N *with* $R_N \cap R'_N = \emptyset$, $R_N, R'_N \subset Q_N$ *and each of* R_N, R'_N *having two parallel* *sides as large as a side of Q_N. Moreover, there is a set $S_N \subset R_N$ such that the following estimates are true for $1 \leqslant p \leqslant q < \infty$:*

(i) $\Phi(2^N) \leqslant C_1 \mu^{p/q}(R'_N)\alpha_N$.

(ii) $\alpha_N \leqslant \varphi_{Q_N}(x) \leqslant 5\alpha_N, \quad x \in S_N$.

(iii) $\frac{1}{5} 4^{p/q-1} \mu^{p/q}(R'_N) 2^N \leqslant \nu(S_N) \leqslant 4^{p/q-1} \mu^{p/q}(R'_N) 2^N$.

(iv) $\alpha_N \leqslant (\varphi_{Q_N} \chi_{S_N})^*_\nu (4^{p/q-1} \mu^{p/q}(R'_N) 2^N) \leqslant 5\alpha_N$.

If $\Phi \in L^{p',\infty}(0, \infty)$, and $\mu \in D_\infty$ or $\{\varphi_Q\} \in C_\infty$, then we also have

(v) $\mu^{p/q}(R'_N)\alpha_N \leqslant C_2 2^{-N/p'}$.

Proof. We begin by choosing a $\bar{Q}_N$ and $\varphi_{\bar{Q}_N}$ for which

$$\Phi(2^N) \leqslant 2\mu^{p/q}(\bar{Q}_N)(\varphi_{\bar{Q}_N})^*_\nu(\mu^{p/q}(\bar{Q}_N) 2^N).$$

Partition $\bar{Q}_N$ into four rectangles $\{R_i\}$ using three parallel $(n-1)$-planes such that $\mu(Q_N) = \frac{1}{4}\mu(R_i)$, $i = 1, 2, 3, 4$. Then from the inequality

$$(\varphi_{\bar{Q}_N})^*_\nu(\tau) \leqslant \sum_{i=1}^{4} (\varphi_{\bar{Q}_N} \chi_{R_i})^*_\nu(\tau/4),$$

we have an i such that for each $j = 1, 2, 3, 4$,

$$\Phi(2^N) \leqslant C 4^{p/q} \mu^{p/q}(R_j)(\varphi_{\bar{Q}_N} \chi_{R_i})^*_\nu(4^{p/q-1} \mu^{p/q}(R_j) 2^N).$$

Select a j so that $R_j \cap R_i = \emptyset$. Let $R_N = R_i$ and $R_N^* = R_j$. Let us denote by R' a rectangle in R_N^* which has one side equal to the side of R_N^* furthest from R_N. Set

$$\bar{\Phi}(2^N) = \sup_{R'} \mu^{p/q}(R')(\varphi_{\bar{Q}_N} \chi_{R_N})^*_\nu(4^{p/q-1} \mu^{p/q}(R') 2^N).$$

We observe $\bar{\Phi}(2^N) \geqslant \Phi(2^N)/(C 4^{p/q})$. Select an R' for which the sup is nearly attained. Let

$$\alpha_N = (\varphi_{\bar{Q}_N} \chi_{R_N})^*_\nu(4^{p/q-1} \mu^{p/q}(R') 2^N),$$

$$S_N = \{x \in R_N \colon 5\alpha_N \geqslant \varphi_{\bar{Q}_N}(x) \geqslant \alpha_N\}, \quad \text{and}$$

$$S'_N = \{x \in R_N \colon \varphi_{\bar{Q}_N}(x) \geqslant \alpha_N\}.$$

Since $\nu\{\varphi_{\bar{Q}_N}^{-1}(t)\} = 0$, $t > 0$, we see that

$$\nu(S'_N) = \nu\{\varphi_{\bar{Q}_N} \chi_{R_N} > \alpha_N\} = 4^{p/q-1} \mu^{p/q}(R') 2^N.$$

We claim that $\nu(S_N) \geqslant \frac{1}{5}\nu(S'_N)$. To prove this we assume $\nu(S'_N) > \nu(S_N)$. If $\nu(S_N) < \frac{1}{5}\nu(S'_N)$, then

$$4^{p/q-1} \mu^{p/q}(R') 2^N \geqslant \nu(S'_N \setminus S_N) > \tfrac{4}{5}\nu(S'_N) = \tfrac{4}{5} 4^{p/q-1} \mu^{p/q}(R') 2^N.$$

We choose $R'' \subset R_N^*$ for which

$$4^{p/q-1} \mu^{p/q}(R'') 2^N \leqslant \nu(S'_N \setminus S_N) \leqslant 4^{p/q-1} \mu^{p/q}(R'') 2^{N+1}$$

and R'' is a candidate for the sup of $\bar{\Phi}$. Then $\mu^{p/q}(R'') > \frac{2}{5}\mu^{p/q}(R')$ and since

$$(\varphi_{\bar{Q}_N} \chi_{S'_N \setminus S_N})^*_\nu(4^{p/q-1} \mu^{p/q}(R'') 2^N) \geqslant 5\alpha_N,$$

we get

$$\bar{\Phi}(2^N) \geqslant \mu^{p/q}(R'')(\varphi_{\bar{Q}_N} \chi_{S'_N \setminus S_N})^*_\nu(4^{p/q-1} \mu^{p/q}(R'') 2^N)$$
$$> 2\mu^{p/q}(R')\alpha_N \geqslant \bar{\Phi}(2^N).$$

Hence our claim is established and

$$\tfrac{1}{5} 4^{p/q-1} \mu^{p/q}(R') 2^N \leqslant \nu(S_N) \leqslant 4^{p/q-1} \mu^{p/q}(R') 2^N.$$

If we now let R'_N be R' properties (i), (ii), and (iv) follow.

To establish (v) we must work a bit harder than expected since it is not obvious that $\bar{\Phi}(2^N) \leqslant C\Phi(2^N)$. Assuming $\Phi \in L^{p',\infty}(0, \infty)$ by Theorem 2(i) we have $\|Mf\|_{q,\infty,\mu} \leqslant B\|f\|_{p,1,\nu}$ for $f \in L^{p,1}_\nu$. In the proof of Theorem 2 we have seen that this implies

$$\mu^{1/q}(Q) \sup_\tau (\varphi_Q)^*_\nu(\tau)\tau^{1/p'} \leqslant C.$$

Thus

$$\bar{\Phi}(2^N) \leqslant \sup_{R'} \mu^{p/q}(R')(\varphi_{Q_N} \chi_{R_N})^*_\nu(4^{p/q-1} \mu^{p/q}(R') 2^N)$$
$$\leqslant \sup_R \frac{\mu^{1/q}(R')}{2^{N/p'}} \sup_\tau \tau^{1/p'}(\varphi_{R_N})^*_\nu(\tau) \leqslant C 2^{-N/p'},$$

since $R'_N, R_N \subset Q_N$. Thus we have (v) and our proof is complete.

3. We shall investigate the problem of extrapolating a two-weight mixed norm inequality for the fractional maximal operator defined as

$$M_\alpha f(x) = \sup \frac{1}{|Q|^\alpha} \int_Q f(x)\,dx,$$

where $0 < \alpha \leqslant 1$ and the supremum is taken over all cubes with center y such that $|x-y| < \frac{1}{2}\operatorname{diam} Q$. We require $\mu(E) = \int_E \mu\,dx$ and $\nu(E) = \int_E \nu\,dx$ for measurable functions μ and ν.

Remark. To avoid the trivial cases of $\mu = 0$ almost everywhere and $\nu = \infty$ almost everywhere, it will be assumed that $\alpha \leqslant 1/q + 1/p'$. This additional restriction makes no difference in the following computations. This fact can be interpreted as follows: If we bound the size of the cubes Q used in defining M_α away from zero, then the above trivial cases do not arise necessarily for $\alpha > 1/q + 1/p'$ and we still have a theory.

Assuming $\|M_\alpha f\|_{q,\infty,\mu} \leqslant A\|f\|_{p,1,\nu}$, $1 < p \leqslant q < \infty$, we deduce by Theorem 2 that $\Phi(t) \leqslant C/t^{1/p'}$, $0 < t < \infty$. We observe that $t^{-1/p'}$ is not in $L^{s,1}(0,\infty)$ for any s. Thus the problem of extrapolating down is equivalent to showing $\Phi(t) \leqslant C/t^{1/(p-\varepsilon)'}$ for $0 < t \leqslant 1$ and some $\varepsilon > 0$. And similarly the problem of extrapolating up is equivalent to showing $\Phi(t) \leqslant C/t^{1/(p+\delta)'}$ for $1 \leqslant t < \infty$ and some $\delta > 0$.

In [4],[5] the problem of extrapolating down was solved for the case $\alpha = 1$ and $p = q$ by considering iterations of the Hardy–Littlewood maximal operator which has the following equivalence:

$$\underbrace{M \ldots M}_{j+1 \text{ times}} f(x) \sim \sup_{x\in Q} \frac{1}{|Q|} \int_0^{|Q|} (f\chi_Q)^*(t) \frac{\log^j(|Q|/t)}{j!}\, dt.$$

It is unreasonable to iterate M_α for $0 < \alpha < 1$ so we will use the following pseudo-iterations to solve the problem of extrapolation.

DEFINITION 2. For $j = 1, 2, \ldots$ we define $M_{\alpha,j}$ and M_α^j respectively as

$$M_{\alpha,j} f(x) = \sup \frac{1}{|Q|^\alpha} \int_0^{|Q|} (f\chi_Q)^*(t) \frac{\log^j(|Q|^\alpha/t + e)}{j!}\, dt,$$

$$M_\alpha^j f(x) = \sup \frac{1}{|Q|^\alpha} \int_0^{|Q|} (f\chi_Q)^*(t) \frac{\log^j(t/|Q|^\alpha + e)}{j!}\, dt,$$

where the supremum is taken over cubes Q whose center y satisfies $|x-y| < \frac{1}{2}\operatorname{diam} Q$. We may realize the above integrals as integrals over Q for functions f satisfying $|f^{-1}(t)| = 0$ by replacing $|Q|^\alpha/t$ by $|Q|^\alpha/\varrho_Q(x)$, where $\varrho_Q(x) = \inf\{t\colon x \in \{z\colon |f(z)| \geqslant (f\chi_Q)^*(t)\}\}$.

The maximal operators $M_{\alpha,j}$ and M_α^j have an associated Φ function which we will denote by Φ_j and Φ^j respectively. For the remainder of this section we shall just investigate $M_{\alpha,j}$ and Φ_j together with the problem of extrapolating down. The problem of extrapolating up and the role of M_α^j and Φ^j is discussed in Section 4.

The next two lemmas provide the needed estimates on Φ and Φ_j respectively. Lemma 2 shows that the boundedness of $M_{\alpha,j}$ provides bounds on $\Phi(t)$ for the critical range $0 < t \leqslant 1$. Lemma 3 shows that Φ_j is weakly controlled by Φ. We list the implications of these estimates as Theorem 3, thus showing that the problem of extrapolation down is entirely controlled by $M_{\alpha,j}$.

LEMMA 2. *Suppose* $\|M_{\alpha,j} f\|_{q,\infty,\mu} \leqslant A_j \|f\|_{p,1,\nu}$, $f \in L_\nu^{p,1}$, *for* $j = 0, 1, 2, \ldots$, $1 < p \leqslant q < \infty$. *Then there is a constant* $B > 0$ *such that for every* j *and positive* N,

$$\Phi(2^{-N}) \leqslant Bp^j A_j \frac{j!}{N^j} 2^{N/p'}.$$

Proof. We may assume without loss of generality that $|\nu^{-1}(t)| = 0$ for $t > 0$. Let $N > 0$. By Lemma 1 we may choose Q_N and rectangles $R_N, R_N' \subset Q_N$, $R_N \cap R_N' = \emptyset$, and $S_N \subset R_N$ such that

(i) $\Phi(2^{-N}) \leqslant C_1 \mu^{p/q}(R_N')\alpha_N$.

(ii) $\alpha_N \leqslant \dfrac{\chi_{Q_N}}{\nu(x)|Q_N|^\alpha} \leqslant 5\alpha_N, \quad x \in S_N$.

(iii) $\dfrac{C_2}{5} \mu^{p/q}(R_N') 2^{-N} \leqslant \nu(S_N) \leqslant C_2 \mu^{p/q}(R_N') 2^{-N}$.

(iv) $\alpha_N \leqslant \left(\dfrac{\chi_{S_N}}{\nu(x)|Q_N|^\alpha}\right)_\nu^* \left(C_2 \mu^{p/q}(R_N') 2^{-N}\right) \leqslant 5\alpha_N$.

(v) $\mu^{p/q}(R_N')\alpha_N \leqslant C_3 2^{N/p'}$.

We note that for $y \in R_N'$ and $\varrho_Q(x) = \inf\{t\colon x \in \{z\colon |\nu^{1-p'}(z)| \geqslant (\nu^{1-p'}\chi_Q)^*(t)\}\}$,

$$M_{\alpha,j}(\nu^{1-p'}\chi_{S_N})(y) \geqslant \frac{C}{|Q_N|^\alpha} \int_{Q_N} \nu^{1-p'}(x)\chi_{S_N}(x) \frac{\log^j}{j!}\left(\frac{|Q_N|^\alpha}{\varrho_{Q_N}(x)} + e\right) dx.$$

We begin with

$$y^p \mu^{p/q}\{x\colon M_{\alpha,j}(\nu^{1-p'}\chi_{S_N})(x) > y\} \leqslant A_j^p \|\nu^{1-p'}\chi_{S_N}\|_{p,1,\nu}^p,$$

and observe, using $\nu^{1-p'}(x) \sim (\alpha_N |Q_N|^\alpha)^{p'-1}$ for $x \in S_N$, that

$$\mu^{p/q}(R_N')\left[\frac{1}{|Q_N|^\alpha} \int_{Q_N} \nu^{1-p'}(x)\chi_{S_N}(x) \frac{\log^j}{j!}\left(\frac{|Q_N|^\alpha}{\varrho_{Q_N}(x)} + e\right) dx\right]^p$$
$$\leqslant A_j^p (5\alpha_N)^{p'-1} |Q_N|^{\alpha(p'-1)} |S_N|$$

which reduces further to

$$\mu^{p/q}(R_N')\left[\alpha_N^{p'-1} \frac{|S_N|}{|Q_N|^{\alpha(2-p')}} \frac{\log^j}{j!}\left(\frac{|Q_N|^\alpha}{|S_N|} + e\right)\right]^p$$
$$\leqslant A_j^p (5\alpha_N)^{p'-1} |Q_N|^{\alpha(p'-1)} |S_N|.$$

From this we derive

$$\alpha_N \mu^{p/q}(R_N')\left(\frac{|S_N|}{|Q_N|^\alpha}\right)^{p-1} \leqslant C\left[\frac{A_j j!}{\log^j(|Q_N|^\alpha/S_N| + e)}\right]^p.$$

We use $\frac{1}{5} C_2 \alpha_N \mu^{p/q}(R_N') 2^{-N} \leqslant \alpha_N \nu(S_N) \leqslant |S_N|/|Q_N|^\alpha$ and

$$|S_N|/|Q_N|^\alpha \leqslant 5\alpha_N \nu(S_N) \leqslant 5C_2 \alpha_N \mu^{p/q}(R_N') 2^{-N} \leqslant 2^{-N/p}$$

to estimate the left and right sides respectively to obtain

$$[\alpha_N \mu^{p/q}(R_N')]^p 2^{-(p-1)N} \leqslant C[p^j A_j j!/N^j]^p.$$

We take the power $1/p$ to complete the proof.

LEMMA 3. *We have the following estimate for* $N\in \mathbf{Z}, j=1, 2, \ldots$.

$$\Phi_j(2^N) \leqslant C\left(\frac{3^j|N|^j}{j!}\Phi_0(2^{N-1})+3^j\left[\frac{|N|^j}{j!}+1\right]2^{-2|N|}\right).$$

Proof. For $N\neq 0$ consider a $Q\subset \mathbf{R}^n$ and let

$$L_{Q,j} = \mu^{p/q}(Q)\left(\frac{\chi_Q \log^j(|Q|^\alpha/\varrho_Q+e)}{v|Q|^\alpha j!}\right)^*_v(\mu^{p/q}(Q)2^N),$$

where ϱ_Q is any function supported in Q with $|\{x:\ 0<\varrho_Q(x)\leqslant t\}|=t$, for $0<t<|Q|$. Let

$$Q_N = \{x\in Q:\ \log(|Q|^\alpha/\varrho_Q(x)+e) > 3|N|\log(2)\}$$

and observe that $|Q_N| = |Q|^\alpha(2^{3|N|}-e)^{-1}$. Hence

$$L_{Q,j} \leqslant \mu^{p/q}(Q)\left(\frac{3^j|N|^j\chi_{Q\setminus Q_N}}{v|Q|^\alpha j!}\right)^*_v(\mu^{p/q}(Q)2^{N-1})$$
$$+\mu^{p/q}(Q)\left(\frac{\chi_{Q_N}\log^j(|Q|^\alpha/\varrho_Q+e)}{v|Q|^\alpha j!}\right)^*_v(\mu^{p/q}(Q)2^{N-1}).$$

The first term on the right of the inequality is at most $(3^j|N|^j/j!)\,\Phi_0(2^{N-1})$. To estimate the second term we assume $v(Q_N)\geqslant \mu^{p/q}2^{N-1}$. Construct a set $S_N\subset Q_N$ such that if we let

$$\alpha_N = \left(\frac{\chi_{Q_N}\log^j(|Q|^\alpha/\varrho_Q+e)}{v|Q|^\alpha j!}\right)^*_v(\mu^{p/q}(Q)2^{N-1}),$$

then

$$\tfrac{1}{2}\mu^{p/q}(Q)2^{N-1}\leqslant v(S_N)\leqslant \mu^{p/q}(Q)2^{N-1},$$

$$\alpha_N\leqslant \frac{\log^j(|Q|^\alpha/\varrho_Q(x)+e)}{v(x)|Q|^\alpha j!}\quad \text{for } x\in S_N.$$

We compute

$$\alpha_N v(S_N)\leqslant \frac{1}{|Q|^\alpha}\int_0^{|Q_N|}\frac{\log^j(|Q|^\alpha/t+e)}{j!}dt$$
$$\leqslant \frac{1}{|Q|^\alpha}\int_0^{|Q_N|}\frac{\log^j(2|Q|^\alpha/t)}{j!}dt \leqslant \frac{|Q_N|}{|Q|^\alpha}\left[\frac{\log^j}{j!}\left(\frac{2|Q|^\alpha}{|Q_N|}\right)+1\right]2^j$$

Thus

$$\alpha_N\mu^{p/q}(Q)\leqslant C2^{-N}\alpha_N v(S_N)\leqslant C2^{-N}\frac{|Q_N|}{|Q|^\alpha}\left[\frac{\log^j}{j!}\left(\frac{2|Q|^\alpha}{|Q_N|}\right)+1\right]2^j$$
$$\leqslant C2^{-N}2^{-3|N|}3^j\left[\frac{|N|^j}{j!}+1\right]$$

and the proof is complete.

THEOREM 3. (i) *If*

$$\sup_{\|f\|_{p,1,v}=1}\|M_{\alpha,j}f\|_{q,\infty,\mu} = o(A^j),\qquad 1<p\leqslant q<\infty,$$

then there exists $\varepsilon>0$ *such that* $\Phi\in L^{(p-\varepsilon)',1}(0,\infty)$ *and thus* $\|M_\alpha f\|_{q_\varepsilon,\mu}\leqslant C\|f\|_{p-\varepsilon,v}$, *where* $q_\varepsilon = ((p-\varepsilon)/p)q$.

(ii) *If* $\|Mf\|_{q_i,\infty,\mu}\leqslant A\|f\|_{p_i,1,v}$ *for* $i=1, 2$ *and* $p_1/q_1 = p_2/q_2$, $p_2>p_1\geqslant 1$, *then for* $p_1<p_0<p_2$ *and* $q_0 = p_0(q_1/p_1)$ *we have*

$$\|M_{\alpha,j}f\|_{q_0,\mu}\leqslant (A_{p_0})^j\|f\|_{p_0,v}.$$

Proof. To prove (i) we use Lemma 2 for $N<0$ to get

$$\Phi(2^{-N})\leqslant C(A/N)^j j!\,2^{N/p'}$$

for some constants A and C. We use Stirling's formula $j!\sim\sqrt{2\pi}\,e^{-j}j^{j+1/2}$ to get

$$\Phi(2^{-N})\leqslant C\left(\frac{A_j}{eN}\right)^j j^{1/2}2^{N/p'}.$$

Let $\gamma = e/(2A)$ and choose $j=[\gamma N]$. Then

$$\Phi(2^{-N})\leqslant \frac{CN^{1/2}}{2^{\gamma N}}2^{N/p'}\leqslant C_\varepsilon\frac{2^{N/(p-\varepsilon)'}}{N^2}$$

for some $\varepsilon>0$. This implies $\sum_{N>0}\Phi(2^{-N})2^{-N/(p-\varepsilon)'}<\infty$. Since $\Phi\in L^{p',\infty}(0,\infty)$ we have $\Phi(2^N)\leqslant C2^{-N/p'}$ and we infer

$$\sum_{N>0}\Phi(2^N)2^{+N/(p-\varepsilon)'}<\infty.$$

Thus $\Phi\in L^{(p-\varepsilon)',1}(0,\infty)$.

To prove (ii) we use Lemma 3 to derive

$$\Phi_j(2^N)\leqslant \frac{C3^j|N|^j}{j!}[\Phi(2^{N-1})+2^{-2|N|}]+3^j2^{-2|N|},\qquad N\in\mathbf{Z}.$$

Since Φ is in $L^{p_1',\infty}(0,\infty)$ and $L^{p_2',\infty}(0,\infty)$ we have

$$\Phi(t)\leqslant\begin{cases}C_1/t^{1/p_1'}, & 0<t\leqslant 1,\\ C_2/t^{1/p_2'}, & 1\leqslant t<\infty.\end{cases}$$

Thus for p_0 with $p_1<p_0<p_2$, we estimate

$$\|\Phi_j\|_{p_0',1}\leqslant CA^j\left[\int_0^1\frac{\log^j(1/t)}{j!}\left(\frac{1}{t^{1/p_1'}}\right)\frac{dt}{t^{1/p_0}}+\int_1^\infty\frac{\log^j(t)}{j!}\left(\frac{1}{t^{1/p_2'}}\right)\frac{dt}{t^{1/p_0}}\right].$$

We compute the right hand side to be less than CB^j, where B depends upon p_0, p_1 and p_2.

4. The behavior of M_α, $0<\alpha<1$, differs markedly from M_1, the Hardy–Littlewood maximal operator, in the problem of extrapolating up. If we do not make any restrictions on the size of cubes used in defining M_1 (see Remark) then extrapolating up is trivial. In the case $p=q$, if M_1 maps $L_\nu^{p,1}$ boundedly into $L_\mu^{p,\infty}$ then $\mu(x)\le C\nu(x)$ a.e. and extrapolating up is simply interpolation between L^p and L^∞ spaces.

To investigate the problem of extrapolating up a mixed norm inequality of M_α, we use the behavior of the operator M_α^j (Definition 2) to obtain estimates on $\Phi(t)$ for $1\le t<\infty$. We list these estimates as Lemma 4. Finally, the last theorem (Theorem 4) contains our result on this problem as part (i) and two results (ii) and (iii) using both $M_{\alpha,j}$ and M_α^j to obtain weak type sufficiency conditions to have M_α map L_ν^p to $L_\mu^{q,\infty}$ and L_ν^p to L_μ^q boundedly.

LEMMA 4. *Suppose* $\|M_\alpha^j f\|_{q,\infty,\mu}\le A_j\|f\|_{p,1,\nu}$ *for* $f\in L_\nu^{p,1}$ *and* $j=0,1,2,\ldots$, $1\le p\le q<\infty$. *Then there is a constant* $B_\varepsilon>0$ *depending upon* $0<\varepsilon<1/p$ *such that for every* j *and* $N>0$,

$$\Phi(2^N)\le\max\left\{2^{-N(1-\varepsilon)},\ B_\varepsilon\frac{A_j j!}{\varepsilon^j N^j}2^{-N/p'}\right\}.$$

Proof. Given ε with $0<\varepsilon<1/p$ we assume $\Phi(2^N)>2^{-N(1-\varepsilon)}$. We begin the proof, just as in Lemma 2, by fixing N and then applying Lemma 1 to obtain Q_N, R_N, R'_N and S_N satisfying properties (i) through (v). We apply the hypothesis to $M_\alpha^j(\nu^{1-p'}\chi_{S_N})$ to derive

$$\mu^{p/q}(R'_N)\left[\alpha_N^{p'-1}\frac{1}{|Q_N|^{\alpha(2-p')}}\int\chi_{S_N}\frac{\log^j}{j!}\left(\frac{\varrho_Q(x)}{|Q_N|^\alpha}+e\right)dx\right]^p \le A_j^p(5\alpha_N)^{p'-1}|Q_N|^{\alpha(p'-1)}|S_N|.$$

We use

$$\frac{|S_N|}{|Q_N|^\alpha}\cdot\frac{1}{2^N}\ge C\frac{\mu^{p/q}(R'_N)}{\mu^{p/q}(R'_N)\,2^N}\int_0^{\mu^{p/q}(R'_N)2^N}\left(\frac{\chi_{S_N}}{\nu|Q|^\alpha}\right)_\nu^*(s)\,ds \ge C\Phi(2^N)\ge C2^{-N(1-\varepsilon)}$$

to estimate $|S_N|/|Q_N|^\alpha\ge C\,2^{N\varepsilon}$.

We reduce the left side of the first inequality to obtain

$$\mu^{p/q}(R'_N)\left[\frac{\alpha_N^{p'-1}|S_N|}{|Q_N|^{\alpha(2-p')}}\cdot\frac{\log^j(|S_N|/(2|Q_N|^\alpha)+e)}{j!}\right]^p\le CA_j^p(\alpha_N^{p'-1})|Q_N|^{\alpha(p'-1)}|S_N|$$

or

$$\alpha_N\mu^{p/q}(R'_N)\left(\frac{|S_N|}{|Q_N|^\alpha}\right)^{p-1}\le C\left[\frac{A_j j!}{\log^j(|S_N|/(2|Q_N|^\alpha)+e)}\right]^p.$$

We now use $\frac{1}{5}C_2 2^N\alpha_N\mu^{p/q}(R'_N)\le\alpha_N\nu(S_N)\le|S_N|/|Q_N|^\alpha$ for the left and $|S_N|/|Q_N|^\alpha\ge C2^{+N\varepsilon}$ for the right side. Thus the above becomes

$$[\alpha_N\mu^{p/q}(R'_N)]^p 2^{(p-1)N}\le C_\varepsilon\left[\frac{A_j j!}{\varepsilon^j N^j}\right]^p.$$

The proof is completed by taking the power $1/p$ of both sides.

THEOREM 4. (i) *If*

$$\sup_{\|f\|_{p,1,\nu}=1}\|M_\alpha^j f\|_{q,\infty,\mu}\le o(A^j),\qquad 1\le p\le q<\infty,$$

then there exists $\delta>0$ *such that* $\Phi\in L^{(p+\delta)',1}(0,\infty)$ *and thus*

$$\|M_\alpha f\|_{q_\delta,\mu}\le C\|f\|_{p+\delta,\nu},\quad \textit{where}\quad q_\delta=\left(\frac{p+\delta}{p}\right)q.$$

(ii) *If* $1<p\le q<\infty$, $\|M_{\alpha,1}f\|_{q,\infty,\mu}\le A\|f\|_{p,1,\nu}$ *and* $\|M_\alpha^1 f\|_{q,\infty,\mu}\le A\|f\|_{p,1,\nu}$, *then* $\Phi\in L^{p',p'}(0,\infty)$ *and thus*

$$\|M_\alpha f\|_{q,\infty,\mu}\le C\|f\|_{p,\nu}.$$

(iii) *If* $1<p\le q<\infty$, $\|M_{\alpha,2}f\|_{q,\infty,\mu}\le A\|f\|_{p,1,\nu}$ *and* $\|M_\alpha^2 f\|_{q,\infty,\mu}\le A\|f\|_{p,1,\nu}$, *then* $\Phi\in L^{p',1}(0,\infty)$ *and thus*

$$\|M_\alpha f\|_{q,s,\mu}\le C\|f\|_{p,s,\nu},\qquad 1\le s\le\infty.$$

Proof. To prove (i) we assume there exists $C>0$ and $\varepsilon>0$ such that $\Phi(2^N)>C2^{-N(1-\varepsilon)}$ for all $N>0$. Otherwise we are done since $\Phi(2^{-N})<2^{N/p'}$ and $\Phi(2^N)\le C2^{-N}$ for $N>0$. We use Lemma 4 to derive

$$\Phi(2^N)\le C\left(\frac{A}{eN}\right)^j j!\,2^{-N/p'},\qquad N>0.$$

We use Stirling's formula $j!\sim\sqrt{2\pi}\,e^{-j}j^{j+1/2}$ to get

$$\Phi(2^N)\le C\left(\frac{A_j}{eN}\right)^j j^{1/2}2^{-N/p'}.$$

Let $\gamma=e/(2A)$ and choose $j=[\gamma N]$. Then

$$\Phi(2^N)\le\frac{CN^{1/2}}{2^{\gamma N}}2^{-N/p'}\le C_\delta\frac{2^{-N/(p+\delta)'}}{N^2}$$

for some $\delta>0$. Thus $\sum_{N>0}\Phi(2^N)2^{-N/(p+\delta)'}<\infty$. Since $\Phi\in L^{p',\infty}(0,\infty)$, we have $\Phi(2^{-N})\le C2^{N/p'}$ and

$$\sum_{N>0}\Phi(2^{-N})2^{N/(p+\delta)'}<\infty.$$

Hence $\Phi \in L^{(p+\delta)',1}(0, \infty)$.

To prove (ii) and (iii) we observe that it is enough to show $\Phi(2^N) \leqslant C 2^{-N/p'}/|N|$ and $\Phi(2^N) \leqslant C 2^{-N/p'}/|N|^2$ respectively. For $N > 0$, we use Lemma 4 to derive the estimates. For $N < 0$ we use Lemma 3.

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Received May 5, 1986 (2166)

STUDIA MATHEMATICA, T. LXXXVII. (1987)

On Δ-uniform convexity and drop property

by

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Abstract. Let $(X, \|\ \|)$ be a real Banach space. The norm $\|\ \|$ is called Δ-uniformly convex if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for each convex set E contained in the unit ball B with measure of noncompactness greater than ε, $\inf\{\|x\|: x \in E\} < 1-\delta$. It is shown that the norm $\|\ \|$ is Δ-uniformly convex if and only if it satisfies uniformly a certain condition (α) equivalent to the drop property. The paper contains an example of a reflexive space in which there is no Δ-uniformly convex norm equivalent to the given one.

Let $(X, \|\ \|)$ be a real Banach space. The norm $\|\ \|$ is called *uniformly convex* [2] if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for $x, y \in X$ such that $\|x\| = \|y\| = 1$ and

$$\|x-y\| > \varepsilon, \tag{1}$$

we have

$$\|\tfrac{1}{2}(x+y)\| < 1-\delta. \tag{2}$$

Of course, in this definition we can replace condition (2) by

$$\inf\{\|z\|: z \in \operatorname{conv}(\{x, y\})\} < 1-\delta \tag{3}$$

where $\operatorname{conv}(A)$ denotes the convex hull of a set A.

Indeed, (2) trivially implies (3). On the other hand, if (3) holds then there is $z \in \operatorname{conv}(\{x, y\})$ such that

$$\|z\| < 1-\delta. \tag{4}$$

We have two possibilities: either

$$\tfrac{1}{2}(x+y) = (1-t)x+tz \quad \text{for some } t,\ 0 \leqslant t \leqslant 1,$$

or

$$\tfrac{1}{2}(x+y) = (1-t)y+tz \quad \text{for some } t,\ 0 \leqslant t \leqslant 1.$$

In both cases $t > \frac{1}{2}$ and the norm of $\frac{1}{2}(x+y)$ can be estimated as follows:

$$\|\tfrac{1}{2}(x+y)\| \leqslant (1-t)+t(1-\delta) = 1-t\delta < 1-\tfrac{1}{2}\delta, \tag{5}$$

and we obtain (2) with δ replaced by $\frac{1}{2}\delta$.

Goebel and Sękowski [8] extend the definition of uniform convexity replacing condition (1) by a condition involving the Kuratowski measure of noncompactness.