Hence $\Phi \in L^{(p+\delta)',1}(0, \infty)$.

To prove (ii) and (iii) we observe that it is enough to show $\Phi(2^N) \leqslant C 2^{-N/p'}/|N|$ and $\Phi(2^N) \leqslant C 2^{-N/p'}/|N|^2$ respectively. For $N > 0$, we use Lemma 4 to derive the estimates. For $N < 0$ we use Lemma 3.

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On Δ-uniform convexity and drop property

by

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Abstract. Let $(X, \|\ \|)$ be a real Banach space. The norm $\|\ \|$ is called Δ-uniformly convex if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for each convex set E contained in the unit ball B with measure of noncompactness greater than ε, $\inf\{\|x\|: x \in E\} < 1-\delta$. It is shown that the norm $\|\ \|$ is Δ-uniformly convex if and only if it satisfies uniformly a certain condition (α) equivalent to the drop property. The paper contains an example of a reflexive space in which there is no Δ-uniformly convex norm equivalent to the given one.

Let $(X, \|\ \|)$ be a real Banach space. The norm $\|\ \|$ is called *uniformly convex* [2] if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for $x, y \in X$ such that $\|x\| = \|y\| = 1$ and

$$\|x-y\| > \varepsilon, \tag{1}$$

we have

$$\|\tfrac{1}{2}(x+y)\| < 1-\delta. \tag{2}$$

Of course, in this definition we can replace condition (2) by

$$\inf\{\|z\|: z \in \operatorname{conv}(\{x, y\})\} < 1-\delta \tag{3}$$

where $\operatorname{conv}(A)$ denotes the convex hull of a set A.

Indeed, (2) trivially implies (3). On the other hand, if (3) holds then there is $z \in \operatorname{conv}(\{x, y\})$ such that

$$\|z\| < 1-\delta. \tag{4}$$

We have two possibilities: either

$$\tfrac{1}{2}(x+y) = (1-t)x+tz \quad \text{for some } t,\ 0 \leqslant t \leqslant 1,$$

or

$$\tfrac{1}{2}(x+y) = (1-t)y+tz \quad \text{for some } t,\ 0 \leqslant t \leqslant 1.$$

In both cases $t > \frac{1}{2}$ and the norm of $\frac{1}{2}(x+y)$ can be estimated as follows:

$$\|\tfrac{1}{2}(x+y)\| \leqslant (1-t)+t(1-\delta) = 1-t\delta < 1-\tfrac{1}{2}\delta, \tag{5}$$

and we obtain (2) with δ replaced by $\frac{1}{2}\delta$.

Goebel and Sękowski [8] extend the definition of uniform convexity replacing condition (1) by a condition involving the Kuratowski measure of noncompactness.

Let A be a set in a Banach space X. The *Kuratowski measure of noncompactness* of A is the infimum $\alpha(A)$ of those $\varepsilon > 0$ for which there is a covering of A by a finite number of sets A_i such that $\operatorname{diam}(A_i) = \sup\{\|x-y\|: x, y \in A_i\} < \varepsilon$. It has the following properties (see for example [1]):

(a) $\alpha(A) = 0$ if and only if the closure $\bar{A}$ of A is compact.

(b) $\alpha(A) = \alpha(\bar{A})$.

(c) $\alpha(\operatorname{conv}(A)) = \alpha(A)$.

(d) $\alpha(A+B) = \alpha(A)+\alpha(B)$.

(e) $\alpha(\lambda A) = |\lambda|\,\alpha(A)$.

A norm $\|\ \|$ in a Banach space X is *Δ-uniformly convex* [8] if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for each convex set E contained in the closed unit ball $B = \{x \in X: \|x\| \leqslant 1\}$ such that

(6) $$\alpha(E) > \varepsilon,$$

we have

(7) $$\inf\{\|x\|: x \in E\} < 1-\delta.$$

Goebel and Sękowski [8] have shown that if $\|\ \|$ is Δ-uniformly convex, then each nonexpansive mapping T of a closed convex set $C \subset X$ into itself has a fixed point.

We say that a Banach space X is *superreflexive* (*Δ-uniformly convexifiable*) if there is a norm $\|\ \|$ which is equivalent to the given one and uniformly convex (Δ-uniformly convex).

Let $(X, \|\ \|)$ be a Banach space. We say that the norm has the *drop property* [10] if for any closed set C disjoint with the unit ball there is a point $a \in C$ such that

(8) $$D(a, B) \cap C = \{a\}$$

where for brevity we have put

(9) $$D(a, B) = \operatorname{conv}(\{a\} \cup B);$$

we call $D(a, B)$ a *drop* [3].

It was shown by Rolewicz [10] and Montesinos [9] that a Banach space is reflexive if and only if there is a norm $\|\ \|$ equivalent to the given one such that $\|\ \|$ has the drop property.

In the present paper we shall discuss the relations between the drop property, the Δ-uniform convexity and uniform convexity of norms, as well as the relations between reflexivity, Δ-uniform convexifiability and superreflexivity.

Let $(X, \|\ \|)$ be a Banach space. We say that the norm $\|\ \|$ satisfies *condition* (α) if for each continuous linear functional f of norm one

(10) $$\lim_{\varepsilon\to 0}\alpha(S(f, \varepsilon)) = 0,$$

where $S(f, \varepsilon)$ denotes the "slice"

(11) $$S(f, \varepsilon) = \{x \in X: \|x\| \leqslant 1, f(x) \geqslant 1-\varepsilon\}.$$

THEOREM 1 ([9], [10]). *The norm $\|\ \|$ has the drop property if and only if it satisfies condition* (α).

COROLLARY 1 ([9]). *Let $(X, \|\ \|)$ be a Banach space. Let $(Y, \|\ \|)$ be a subspace of X. If the norm $\|\ \|$ has the drop property then*

(i) *The norm $\|\ \|$ restricted to Y has the drop property.*

(ii) *The norm $\|[x]\|_Q = \inf\{\|x+y\|: y \in Y\}$ in the quotient space X/Y also has the drop property.*

Proof. (i) Let f be an arbitrary functional of norm 1 defined on Y. Let $\tilde{f}$ be a norm one extension of f to X. Then $S(f, \varepsilon) \subset Y \cap S(\tilde{f}, \varepsilon)$ and $\alpha(S(f, \varepsilon)) \leqslant \alpha(S(\tilde{f}, \varepsilon))$ which tends to zero, because the norm $\|\ \|$ on X has the drop property.

(ii) Let f be a functional of norm one defined on the quotient space X/Y. It induces a functional $\tilde{f}$ of norm one on X by the formula $\tilde{f}(x) = f([x])$. Observe that for each ε

$$S(f, \varepsilon) = \{[x]: x \in S(\tilde{f}, \varepsilon)\}.$$

Since $\operatorname{diam}\{[x]: x \in A\} \leqslant \operatorname{diam} A$, we have

$$\alpha(S(f, \varepsilon)) \leqslant \alpha(S(\tilde{f}, \varepsilon))$$

and the drop property of the norm $\|\ \|$ implies the drop property of the quotient norm.

THEOREM 2. *Let $(X, \|\ \|)$ be a Banach space. Let x_0 be a point of norm greater than 1. Let*

(12) $$B_0 = \operatorname{conv}(\{x_0, -x_0\} \cup B).$$

The set B_0 induces a new norm $\|\ \|_0$ equivalent to the given one.

If the norm $\|\ \|$ has the drop property, then so does the norm $\|\ \|_0$.

The proof is based on some propositions.

Let f be a continuous linear functional on X of norm 1. We write

$$g^f(\varepsilon) = \alpha(S(f, \varepsilon)).$$

PROPOSITION 1. *For $0 < \lambda < 1$ and $0 < \varepsilon < 1$,*

(13) $$g^f(\lambda\varepsilon) \geqslant \lambda g^f(\varepsilon).$$

Proof. Let δ be such that $0 < \delta < \varepsilon$. Let x_δ^f be an element of norm 1 such that

(14) $$f(x_\delta^f) = 1-\delta.$$

By the convexity of the unit ball

$$x_\delta^f+\lambda\frac{\varepsilon-\delta}{\varepsilon}(S(f,\varepsilon)-x_\delta^f)\subset S(f,\lambda\varepsilon). \tag{15}$$

Thus

$$\lambda\frac{\varepsilon-\delta}{\varepsilon}\alpha(S(f,\varepsilon))\leqslant\alpha(S(f,\lambda\varepsilon)).$$

Letting δ tend to 0, we get (13).

We do not know whether the function g^f is always concave.

PROPOSITION 2. *Let $(X, \|\ \|)$ be a Banach space. Let $X_1=X\times\mathbf{R}$, where the norm $\|\ \|_1$ in X_1 is defined by*

$$\|(x,t)\|_1=\|x\|+|t|. \tag{16}$$

If the norm $\|\ \|$ has the drop property, then so does the norm $\|\ \|_1$.

Proof. Let f be an arbitrary linear functional of norm one in X_1, $\|f\|_1=1$. Let f_0 denote the restriction of f to X, $f_0=f|_X$. Of course, $\|f_0\|\leqslant 1$. We write

$$S(f_0,\varepsilon)=\{x\in X:\ \|x\|\leqslant 1,\ f_0(x)\geqslant 1-\varepsilon\}. \tag{17}$$

Of course, if $\|f_0\|<1$ the set $S(f_0,\varepsilon)$ is void for sufficiently small ε. Now we have two possibilities:

(i) Neither $(0, 1)$ nor $(0, -1)$ is a point of support of the functional f.

(ii) Either $(0, 1)$ or $(0, -1)$ is a point of support of f.

In case (i) it is easy to observe that for sufficiently small ε

$$S(f,\varepsilon)\subset\operatorname{conv}(\{(0,1),(0,-1)\}\cup S(f_0,\varepsilon)) \tag{18}$$

and by property (b) of the measure of noncompactness

$$\alpha(S(f,\varepsilon))\leqslant\alpha(S(f_0,\varepsilon)). \tag{19}$$

Now we consider case (ii). Without loss of generality we may assume that $f(0,1)=1$. Let t be an arbitrary number, $-1\leqslant t\leqslant 1$. Let $A_t=\{x\in X:\ (x,t)\in S(f,\varepsilon)\}$. Of course

$$S(f,\varepsilon)\subset\bigcup_{-1\leqslant t\leqslant 1}A_t\times\{t\}. \tag{20}$$

Using a compactness argument we can easily show that (20) implies

$$\alpha(S(f,\varepsilon))=\max_{-1\leqslant t\leqslant 1}\alpha(A_t). \tag{21}$$

Now we shall estimate $\alpha(A_t)$. We divide the interval $[-1, 1]$ into three sections: $[-1, 0]$, $[0, 1-\varepsilon]$, $[1-\varepsilon, 1]$. In $[-1, 0]$

$$A_t\times\{t\}\subset\operatorname{conv}(\{(0,-1)\}\cup(S(f_0,\varepsilon)\times\{0\})) \tag{22}$$

and so

$$\alpha(A_t)\leqslant\alpha(S(f_0,\varepsilon)). \tag{23}$$

In $[1-\varepsilon, 1]$, $A_t=(1-t)B$, where B denotes the closed unit ball in X. Thus

$$\alpha(A_t)\leqslant 2\varepsilon. \tag{24}$$

The most complicated case is the interval $[0, 1-\varepsilon]$. In this section we obtain A_t by cutting off a piece of the ball $(1-t)B$ by a hyperplane with distance from the center not smaller than ε. In other words,

$$A_t\subset(1-t)S\left(f_0,\frac{\varepsilon}{1-t}\right) \tag{25}$$

and by (b)

$$\alpha(A_t)\leqslant(1-t)g^{f_0}\left(\frac{\varepsilon}{1-t}\right). \tag{26}$$

By Proposition 1

$$\sup_{0\leqslant t\leqslant 1-\varepsilon}\alpha(A_t)\leqslant g^{f_0}(\varepsilon). \tag{27}$$

Therefore by (23), (24), (27) and (21)

$$\alpha(S(f,\varepsilon))\leqslant\max(2\varepsilon, g^{f_0}(\varepsilon)). \tag{28}$$

Thus the norm $\|\ \|_1$ satisfies condition (α), which finishes the proof of Proposition 2.

Proof of Theorem 2. We embed X into the space X_1 described in Proposition 2. Let T be a projection of X_1 onto X such that $T(0,1)=x_0$. It is easy to see that

$$TB_1=\operatorname{conv}(\{x_0,-x_0\}\cup B),\qquad B_1=\{(x,t):\ \|(x,t)\|_1\leqslant 1\},$$

and by Corollary 1 the norm $\|\ \|_0$ in X has the drop property.

If the convergence in formula (10) is uniform with respect to all f, $\|f\|=1$, then we say the norm $\|\ \|$ satisfies the *uniform condition* (α).

More precisely, we say that a norm $\|\ \|$ in a Banach space $(X, \|\ \|)$ satisfies the uniform condition (α) if for each $\varepsilon>0$ there is a $\delta>0$ such that for each continuous linear functional f of norm one

$$\alpha(S(f,\delta))\leqslant\varepsilon. \tag{29}$$

THEOREM 3. *Let $(X, \|\ \|)$ be a Banach space. The norm $\|\ \|$ is Δ-uniformly convex if and only if it satisfies the uniform condition (α).*

Proof. Observe that the norm is Δ-uniformly convex if and only if for each $\varepsilon>0$ there is a $\delta>0$ such that for each convex subset E of the unit ball B

(30) $$\inf\{\|x\|:\ x\in E\}\geqslant 1-\delta$$

implies

(31) $$\alpha(E)<\varepsilon.$$

Let E be an arbitrary convex subset of the unit ball satisfying (30). Then by the separation theorem there is a continuous linear functional f of norm one such that

(32) $$E\subset S(f,\delta).$$

Thus the uniform condition (α) implies (31). On the other hand, $S(f,\delta)$ is a convex subset of the unit ball satisfying (30). Thus the Δ-uniform convexity implies the uniform condition (α).

In a similar way as in Corollary 1 we obtain

PROPOSITION 3. *Let* $(X,\|\ \|)$ *be a Banach space. Let* $(Y,\|\ \|)$ *be a subspace of* X. *If the norm* $\|\ \|$ *is* Δ*-uniformly convex then*:

(i) *The norm* $\|\ \|$ *restricted to* Y *is also* Δ*-uniformly convex.*

(ii) *The quotient norm* $\|[x]\|_Q=\inf\{\|x+y\|:\ y\in Y\}$ *is* Δ*-uniformly convex in the quotient space* X/Y.

PROPOSITION 4. *Let* $(X,\|\ \|)$ *be a Banach space. Let* $X_1=X\times\boldsymbol{R}$ *with the norm* $\|(x,t)\|_1=\|x\|+|t|$. *If the norm* $\|\ \|$ *is* Δ*-uniformly convex, then so is the norm* $\|\ \|_1$.

Proof. The proof is a slight modification of the proof of Proposition 2. Let ε be a small positive number. Let f be an arbitrary continuous linear functional of norm one defined on X_1.

Without loss of generality we may assume that $f(0,1)\geqslant 0$. We shall consider two cases:

(i) $f(0,1)\leqslant 1-\varepsilon$.

(ii) $f(0,1)>1-\varepsilon$.

As previously, we denote by f_0 the restriction of f to X.

In case (i)

(33) $$S(f,\delta)\subset\operatorname{conv}(\{(0,1),(0,-1)\}\cup S(f_0,\delta))$$

for $\delta<\varepsilon$ and by property (b) of the Kuratowski measure of noncompactness

(34) $$g^f(\delta)\leqslant g^{f_0}(\delta)\quad\text{ for } \delta<\varepsilon.$$

In the second case we also introduce the sets

$$A_t=\{x\in X:\ (x,t)\in S(f,\delta)\}.$$

We divide the interval $[-1,1]$ into three sections $[-1,0]$, $[0,1-\varepsilon-\delta)$, $[1-\varepsilon-\delta,1]$.

In the first section the estimation is the same as in the proof of Proposition 2. The proofs for $[0,1-\varepsilon-\delta)$, $[1-\varepsilon-\delta,1]$ are also more or less the same, except that we replace ε by $\varepsilon+\delta$. Hence we have finally

$$\alpha(S(f,\varepsilon))\leqslant\alpha(S(f_0,2\varepsilon)),$$

which finishes the proof.

By Propositions 3 and 4 in the same way as in Theorem 2 we get

THEOREM 4. *Let* $(X,\|\ \|)$ *be a Banach space. Let* B *denote the unit ball in* X. *Let* $x_0\notin B$ *and let*

$$B_1=\operatorname{conv}(\{x_0,-x_0\}\cup B).$$

The norm induced by B_1 *will be denoted by* $\|\ \|_1$. *If the norm* $\|\ \|$ *is* Δ*-uniformly convex, then so is* $\|\ \|_1$.

Of course, each superreflexive space is Δ-uniformly convexifiable, and each Δ-uniformly convexifiable space is reflexive. The converse implications are not true in general as follows from the following theorems.

THEOREM 5. *Let* $(X_n,\|\ \|_n)$ *be a sequence of finite-dimensional Banach spaces. Let* $X=(X_1\times X_2\times\ldots)_{l^p}$, $1<p<+\infty$, *be the space of all sequences* $x=\{x_n\}$, $x_n\in X_n$, *such that*

(35) $$\|x\|=\Big(\sum_{n=1}^{\infty}\|x_n\|^p\Big)^{1/p}<+\infty.$$

Then the norm $\|\ \|$ *is* Δ*-uniformly convex.*

Proof. Let f be an arbitrary continuous linear functional on X of norm one and let ε be an arbitrary number such that $0<\varepsilon<\frac{1}{2}$.

By the construction of the space we can find an index N such that the restriction of f to the space

$$Z=X_{N+1}\times X_{N+2}\times\ldots$$

has norm smaller than ε:

(36) $$\|f|_Z\|<\varepsilon.$$

Let

$$Y=X_1\times\ldots\times X_N.$$

In this way we obtain a decomposition of the space X into the direct sum of two spaces Y and Z, $X=Y+Z$, such that (36) holds, Y is finite-dimensional and for $y\in Y$, $z\in Z$ and $x=y+z$

(37) $$\|x\|^p=\|y+z\|^p=\|y\|^p+\|z\|^p.$$

Let as before $S(f,\varepsilon)=\{x\in X:\ \|x\|\leqslant 1, f(x)\geqslant 1-\varepsilon\}$. Let $x\in S(f,\varepsilon)$. We represent x as the sum $x=y+z$, $y\in Y$, $z\in Z$. By (36) and (37), $|f(z)|<\varepsilon$ and

(38) $$f(y)=f(x-z)\geqslant f(x)-|f(z)|>1-\varepsilon-\varepsilon=1-2\varepsilon.$$

The functional f has norm one. Thus (38) implies

(39) $$\|y\| \geqslant 1-2\varepsilon.$$

By (37) and (39),

(40) $$\|z\|^p = \|x\|^p - \|y\|^p \leqslant 1-(1-2\varepsilon)^p < 2p\varepsilon,$$

and so

(41) $$\|z\| < \sqrt[p]{2p\varepsilon}.$$

Thus by (37), (38), (41),

$$S(f, \varepsilon) \subset \big(S(f, 2\varepsilon) \cap Y\big) + \{z \in Z: \|z\| \leqslant \sqrt[p]{2p\varepsilon}\}.$$

The set $S(f, 2\varepsilon) \cap Y$ is compact since Y is finite-dimensional. Thus by properties (a) and (d) of the Kuratowski measure of noncompactness

$$\alpha\big(S(f, \varepsilon)\big) = \alpha\big(\{z \in Z: \|z\| \leqslant \sqrt[p]{2p\varepsilon}\}\big) = 2\sqrt[p]{2p\varepsilon}.$$

COROLLARY 2. *There is a Δ-uniformly convexifiable space which is not superreflexive.*

Proof. Taking for X_n n-dimensional spaces either with the c_0 norm, i.e.

$$\|x\|_n = \sup_{1 \leqslant i \leqslant n} |\xi_i|, \quad x = (\xi_1, \ldots, \xi_n),$$

or with the l^{p_n} norm, i.e.

$$\|x\|_n = \Big(\sum_{i=1}^{n} |\xi_i|^{p_n}\Big)^{1/p_n},$$

with $p_n \to \infty$, we obtain the classical examples of Day [5] of nonsuperreflexive spaces. By Theorem 5, those spaces are Δ-uniformly convexifiable.

THEOREM 6. *Let $X_n = l^{p_n}$ with the standard norm. Assume that $p_n \to \infty$. Then the space*

$$X = (X_1 \times X_2 \times \ldots)_{l^p}, \quad 1 < p < +\infty,$$

is not Δ-uniformly convexifiable.

Proof. We shall denote the standard norm by $\|\ \|$. Suppose that there is a Δ-uniformly convex norm $|\ |$ in X equivalent to $\|\ \|$. This means that there are two positive numbers m, M such that $\|x\| \leqslant m|x| \leqslant M\|x\|$. Replacing $\|\ \|$ by $m\|\ \|$ we may assume without loss of generality that

(43) $$\|x\| \leqslant |x| \leqslant M\|x\|.$$

This means that the unit ball in the standard norm contains the unit ball in the new norm $|\ |$ and that the unit ball in the new norm contains the ball of radius $\alpha = 1/M$ in the standard norm.

We shall denote by the same symbols $\|\ \|$ and $|\ |$ the restrictions of the norms $\|\ \|$ and $|\ |$ to each component $X_n = l^{p_n}$. The calculation will be done in one space $X_n = l^{p_n}$ with p_n sufficiently large. The choice of p_n will follow from the construction. For brevity we put

$$p_n = \bar{p}, \quad X_n = \bar{X}.$$

We decompose $\bar{X}$ into two infinite-dimensional subspaces by decomposing the set of natural numbers into two disjoint infinite sets N_1, N_2 and putting

$$Y = \{x \in X: x_i = 0,\ i \in N_2\}, \quad Z = \{x \in X: x_i = 0,\ i \in N_1\}.$$

Of course, $\bar{X} = Y+Z$ and for $y \in Y$, $z \in Z$

$$\|y+z\|^{\bar{p}} = \|y\|^{\bar{p}} + \|z\|^{\bar{p}}.$$

Now let $\varepsilon = \frac{1}{2}\alpha$. Since we have assumed that the norm $|\ |$ is Δ-uniformly convex there is $\delta > 0$ such that for each convex set $E \subset \{x: |x| \leqslant 1\}$ such that $\alpha(E) < \varepsilon$ we have

(44) $$\inf\{|x|: x \in E\} < 1-\delta.$$

Let y be an arbitrary element of Y such that

(45) $$\|y\| \leqslant \alpha(1-1/2^{\bar{p}})^{1/\bar{p}}.$$

Then, of course, for an arbitrary $z \in Z$ such that $\|z\| \leqslant \frac{1}{2}\alpha$

(46) $$\|y+z\| \leqslant \alpha.$$

Thus

$$y + \tfrac{1}{2}\alpha\{z \in Z: \|z\| \leqslant 1\} \subset \{x \in \bar{X}: \|x\| \leqslant \alpha\} \subset \{x \in \bar{X}: |x| \leqslant 1\}.$$

The set $y + \frac{1}{2}\alpha\{z \in Z: \|z\| \leqslant 1\}$ has the Kuratowski measure of noncompactness in the standard norm not smaller than $\varepsilon = \frac{1}{2}\alpha$. By (43) the same is true in the new norm. Thus there is $z \in Z$, $\|z\| < \frac{1}{2}\alpha$, such that $|y+z| < 1-\delta$. Of course $|y-z| \leqslant 1$ and finally

$$|y| \leqslant \tfrac{1}{2}(|y+z| + |y-z|) < 1-\tfrac{1}{2}\delta.$$

Thus $|y/(1-\frac{1}{2}\delta)| < 1$ and we have shown that for each $y \in Y$ such that

$$\|y\| \leqslant \alpha\frac{(1-1/2^{\bar{p}})^{1/\bar{p}}}{1-\frac{1}{2}\delta}$$

we have $|y| \leqslant 1$.

Now, $\bar{p}$ ought to be chosen so that

(47) $$\frac{(1-1/2^{\bar{p}})^{1/\bar{p}}}{1-\frac{1}{2}\delta} > 1.$$

This is possible since $p_n \to \infty$. Now repeating the decomposition procedure n times we deduce that there is an infinite-dimensional space Y_n such that for all $y \in Y_n$ such that

$$\|y\| \leqslant \alpha \left[\frac{(1-1/2^{\bar{p}})^{1/\bar{p}}}{1-\frac{1}{2}\delta}\right]^n \tag{48}$$

we have

$$\|y\| \leqslant 1. \tag{49}$$

By (47) this contradicts (43).

COROLLARY 3. *There are reflexive spaces which are not Δ-uniformly convexifiable.*

Proof. The space X described in Theorem 6 is reflexive [5].

Now we shall distinguish a property lying between uniform convexity and Δ-uniform convexity. The starting point is the following.

PROPOSITION 5 [10]. *Let $(X, \|\ \|)$ be a Banach space. Let x not to belong to the unit ball. Let*

$$R(x) = D(x, B)\backslash B. \tag{50}$$

The norm $\|\ \|$ is uniformly convex if and only if for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\|x\| < 1+\delta \tag{51}$$

implies

$$\operatorname{diam}(R(x)) < \varepsilon. \tag{52}$$

Proposition 5 suggests the investigation of the following condition on the norm:

(β) For each $\varepsilon > 0$ there is a $\delta > 0$ such that $\|x\| < 1+\delta$ implies

$$\alpha(R(x)) < \varepsilon. \tag{53}$$

PROPOSITION 6. *If a norm $\|\ \|$ satisfies condition (β) then it is Δ-uniformly convex.*

Proof. Suppose that the norm is not Δ-uniformly convex. Then by Theorem 3 it does not satisfy condition (α). This means that there is an $\varepsilon_0 > 0$ and sequences of continuous linear functionals of norm one $\{f_n\}$ and positive numbers $\delta_n > 0$ such that

$$\alpha(S(f_n, \delta_n)) \geqslant \varepsilon_0. \tag{54}$$

Let x_n be an element such that $1+2\delta_n \leqslant \|x_n\| \leqslant 1+3\delta_n$ and

$$f_n(x_n) = 1+2\delta_n. \tag{55}$$

By (55) for each element of the form $\frac{1}{2}(x_n+y)$, $y \in S(f_n, \delta_n)$, we have

$$f\left(\tfrac{1}{2}(x_n+y)\right) > 1$$

and $\frac{1}{2}(x_n+y) \notin B$. On the other hand, $\frac{1}{2}(x_n+y) \in D(x_n, B)$. Thus $\frac{1}{2}(x_n+y) \in R(x_n)$. Observe that the set $\{\frac{1}{2}(x_n+y)\colon y \in S(f_n, \delta_n)\}$ is homothetic to the set $S(f_n, \delta_n)$ with coefficient 1/2. Thus, by (54) and property (e) of the Kuratowski measure of noncompactness

$$\alpha(R(x_n)) \geqslant \varepsilon_0/2,$$

which completes the proof.

Observe that Δ-uniform convexity does not imply condition (β). Indeed, in Proposition 4 we have constructed a Δ-uniformly convex space which is the l^1-product of a Δ-uniformly convex space $(X, \|\ \|)$ by $\boldsymbol{R}$. It is easy to see that for $x_\delta = (0, 1+\delta)$ the closure of $R(x_\delta) = D(x_\delta, B)\backslash B$ contains the unit sphere in X, thus $\alpha(R(x_\delta)) \geqslant 1$ independently of $\delta > 0$.

We shall say that a Banach space $(X, \|\ \|)$ is a *(β)-space* if there is a norm $\|\ \|_1$ equivalent to $\|\ \|$ such that $\|\ \|_1$ satisfies condition (β).

We have shown that every superreflexive space is a (β)-space and every (β)-space is Δ-uniformly convexifiable. We do not know anything about the converse implications.

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