STUDIA MATHEMATICA, T. LXXXVII. (1987)

A generalized skew product

by

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Abstract. We define a skew product of a family of non-measure preserving transformations $\{T_x\}_{x\in X}$ to obtain the transformation which preserves the product measure. We apply such skew products to perturbations of measure preserving transformations and to factors of endomorphisms.

Let h be an endomorphism of a probability space $(X, \mathscr{B}, m)$ and let $\{T_x\}_{x\in X}$ be a measurable family of transformations of a probability space $(Y, \mathscr{D}, p)$. By the *generalized skew product* of the endomorphism h and the family $\{T_x\}_{x\in X}$ we mean the transformation T of $X\times Y$ given by

$$T(x, y) = \big(h(x),\ T_x(y)\big) \tag{1}$$

and satisfying the following condition:

$$\forall A\in\mathscr{B}\ \forall D\in\mathscr{D}:\quad m(A)\,p(D) = \int_{h^{-1}(A)} p\big(T_x^{-1}(D)\big)\,dm(x). \tag{2}$$

LEMMA 1. *The transformation T defined by* (1) *preserves the product measure $m\times p$ iff it satisfies* (2).

Proof. We show that the equality

$$(m\times p)\big(T^{-1}(A\times D)\big) = (m\times p)(A\times D)$$

for all $A\in\mathscr{B}$ and $D\in\mathscr{D}$ is equivalent to (2). Using the Fubini theorem we get

$$(m\times p)\big(T^{-1}(A\times D)\big) = \int_{h^{-1}A} dm(x)\int_Y 1_D\big(T_x(y)\big)\,dp(y)$$

$$= \int_{h^{-1}A} p\big(T_x^{-1}(D)\big)\,dm(x).\ \blacksquare$$

From (2) we get

COROLLARY 1.

$$\forall D\in\mathscr{D}:\quad p(D) = \int_X p\big(T_x^{-1}(D)\big)\,dm(x). \tag{3}$$

We note that (3) is not equivalent to (2). Obviously the condition (2) is equivalent to the following:

$$\forall D \in \mathscr{D},\ p(D) > 0: \quad E\left(\frac{p(T_x^{-1}(D))}{p(D)}\middle| h^{-1}(\mathscr{B})\right) = 1. \tag{4}$$

For the rest of the paper we shall assume that the spaces considered are Lebesgue spaces.

Remark 1. If h is an automorphism then $h^{-1}(\mathscr{B}) = \mathscr{B}$ and hence the transformations T_x are endomorphisms. This is part of Theorem 1 in [1].

Let f be a real measurable function defined on X.

Definition. We say that f is *independent of a σ-field* $\mathscr{A} \subseteq \mathscr{B}$ if $f^{-1}(\mathscr{B}_{\mathbf{R}})$ is independent of $\mathscr{A}$ $(f^{-1}(\mathscr{B}_{\mathbf{R}}) \perp \mathscr{A})$.

Here $\mathscr{B}_{\mathbf{R}}$ denotes the σ-field of measurable subsets of $\mathbf{R}$.

Corollary 2. *If for every* $D \in \mathscr{D}$, $p(D) > 0$, *the function*

$$f(x) = \frac{p(T_x^{-1}(D))}{p(D)}$$

is independent of $h^{-1}(\mathscr{B})$ *and* $\int_X f\,dm = 1$ *then* (2) *holds.*

Another condition equivalent to (2) uses the Frobenius–Perron operator P_h for h:

$$P_h g(x) = \int_{h^{-1}(x)} g\,dm_{h^{-1}(x)}$$

where $g \in L_1(m)$ and $\{m_{h^{-1}(x)}\}_{x\in X}$ is the canonical system of measures for the partition $h^{-1}\varepsilon$. Here $\varepsilon = \{\{x\}: x \in X\}$.

By the equivalence of (4) and (2) we get

Lemma 2. *The condition* (2) *is equivalent to the following*:

$$\forall D \in \mathscr{D},\ p(D) > 0: \quad P_h f(x) = 1 \quad \text{a.e.}$$

where $f(x) = p(T_x^{-1}(D))/p(D)$.

Remark 2. Assume that the transformations $\{T_x\}_{x\in X}$ are negative nonsingular for a.e. x. Let

$$J_{T_x^{-1}}(y) = \frac{dp\,T_x^{-1}}{dp}(y).$$

If $J_{T_x^{-1}}(y)$ is a measurable function of x for a.e. y then the condition (2) is equivalent to $P_h J_{T_x^{-1}} = 1$ m-a.e. for a.e. y.

Now, we are in a position to give an example of a generalized skew product where the function $f(x)$ is not independent of $h^{-1}(\mathscr{B})$.

Example 1. Let $h(x) = 2x \bmod 1$ and let

$$f_1(x) = \tfrac{1}{2}1_{A_1}(x) + 1_{A_2\cup A_4}(x) + \tfrac{3}{2}1_{A_3}(x)$$

where $A_1 = [0, \frac{1}{4}]$, $A_2 = (\frac{1}{4}, \frac{1}{2}]$, $A_3 = (\frac{1}{2}, \frac{3}{4}]$, $A_4 = (\frac{3}{4}, 1]$. We also put

$$f_2(x) = \tfrac{3}{2}1_{A_1}(x) + 1_{A_2\cup A_4}(x) + \tfrac{1}{2}1_{A_3}(x).$$

A family of transformations $\{T_x\}_{x\in[0,1]}$ of the interval $[0, 1]$ into itself with the measure $p = m_0$ (the Lebesgue measure) is defined as follows: for $x \in A_2 \cup A_4$ we put $T_x(y) = y$, for $x \in A_1$

$$T_x(y) = \begin{cases} 2y & \text{for } y \in [0, \frac{1}{4}], \\ \frac{2}{3}y + \frac{1}{3} & \text{for } y \in [\frac{1}{4}, 1], \end{cases}$$

and for $x \in A_3$

$$T_x(y) = \begin{cases} \frac{2}{3}y & \text{for } y \in [0, \frac{3}{4}], \\ 2y - 1 & \text{for } y \in [\frac{3}{4}, 1]. \end{cases}$$

For any Borel set $D \subset [0, 1]$ with $m(D) > 0$ we define

$$f_D(x) = \frac{m(D \cap [0, \frac{1}{2}])}{m(D)} f_1(x) + \frac{m(D \cap (\frac{1}{2}, 1])}{m(D)} f_2(x).$$

Obviously $P_h f_D = 1$ and f_D is not independent of $h^{-1}(\mathscr{B})$ because $A_2 \cup A_4 \in h^{-1}(\mathscr{B})$.

In the above example the partition $\xi = f^{-1}(\mathbf{R})$ consists of three elements.

Observation 1. *If* $\operatorname{card}\xi = 2$ *where* $\xi = f^{-1}(\mathbf{R})$ *then* ξ *is independent of* $h^{-1}(\mathscr{B})$.

Proof. Here $f = a1_B + b1_{B^c}$ where $a \neq b$. By (4) we get

$$\begin{aligned} m(A) &= \int_{h^{-1}A} (a1_B + b1_{B^c})\,dm \\ &= am(h^{-1}(A) \cap B) + bm(h^{-1}(A) \cap B^c) \end{aligned}$$

for every $A \in \mathscr{B}$. Hence we obtain the following equalities:

$$a\frac{m(h^{-1}(A) \cap B)}{m(A)} + b\frac{m(h^{-1}(A) \cap B^c)}{m(A)} = 1,$$

$$\frac{1-b}{a-b} = \frac{m(h^{-1}(A) \cap B)}{m(A)}.$$

For $A = X$ we get $\dfrac{1-b}{a-b} = m(B)$. This implies

$$m(h^{-1}(A) \cap B) = m(h^{-1}(A))\,m(B). \ \blacksquare$$

As an application of the generalized skew product we shall describe a class of functions $\varphi(x, y)$ satisfying

$$P_h \varphi(x, y) = 1. \tag{5}$$

Let $\varphi(x)$ be an integrable function satisfying $P_h \varphi(x) = 0$ and $f(y)$ a measurable function on Y. $\varphi(x, y)$ defined by

$$\varphi(x, y) = 1 + \varphi(x) f(y) \tag{6}$$

satisfies (5). If $\varphi(x, y) = J_{T_x^{-1}}(y)$ for a family of transformations $\{T_x\}_{x\in X}$ then by $\varphi(x, y) \geqslant 0$ and

$$\int J_{T_x^{-1}}\, dp = p\big(T_x^{-1}(Y)\big) = 1$$

we get $\int f(y)\, dp = 0$.

It is well known that $\ker P_h \neq \{0\}$ iff h is a noninvertible endomorphism; in that case there exists a function $g_1 \in L^\infty$, $g_1 \neq 0$, $g_1 \in \ker P_h$.

Let us fix an $\varepsilon > 0$ and let $\varphi(x) = \varepsilon g_1(x)/\|g_1\|_\infty$. Assume that $Y = [0, 1]$ and $p = m_0$ is the Lebesgue measure. Then for any function $g \in C^1[0, 1]$ such that $g(0) = 0$, $g(1) = 1$, satisfying the inequalities

$$1 - \varepsilon^{-1} \leqslant g'(y) \leqslant \varepsilon^{-1} + 1, \qquad |g(y)| \leqslant 1,$$
$$0 \leqslant \big(1 - \varphi(x)\big) y + \varphi(x) g(y) \leqslant 1$$

we can define a family of transformations $\{T_x\}_{x\in X}$ such that

$$J_{T_x^{-1}}(y) = 1 + \varphi(x)\big(g'(y) - 1\big) = \varphi(x, y).$$

Namely, we obtain

$$[T_x^{-1}(z)]' = 1 + \varphi(x)\big(g'(z) - 1\big)$$

if we put

$$T_x^{-1}(z) = \big(1 - \varphi(x)\big) z + \varphi(x) g(z).$$

$\{T_x\}$ is a family of ε-perturbations of the identity transformation. Here $T_x(0) = 0$ and $T_x(1) = 1$.

Let T be any transformation of $[0, 1]$ into itself preserving the measure m_0. Define a family of stationary ε-perturbations of T by putting $\tilde{T}_x = T_x \circ T$. Here

$$J_{\tilde{T}_x^{-1}}(y) = J_{T^{-1}} J_{T_x^{-1}}(y) = \varphi(x, y).$$

The family $\{\tilde{T}_x\}$ and $h(x)$ determine the generalized skew product $T_1(x, y) = \big(h(x), \tilde{T}_x(y)\big)$. Using Birkhoff's ergodic theorem we get

Observation 2. *For every noninvertible endomorphism h and a family of stationary ε-perturbations of T the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f\big(\tilde{T}_{h^k(x)} \circ \ldots \circ \tilde{T}_x(y)\big)$$

exists a.e. and in L_1 for every $f \in L_1(m_0)$.

The second application of generalized skew products is connected with factors of endomorphisms of Lebesgue spaces. It is well known that if T is an endomorphism of the space $(X, \mathscr{B}, m)$ then any factor of T is a T_ξ transformation where ξ is a measurable partition such that $T^{-1}\xi \leqslant \xi$. Let us assume that ξ has an independent complement, i.e. a partition η such that $\sigma(\xi) \perp \sigma(\eta)$ and $\xi \vee \eta = \varepsilon$. Then $X = X_\xi \times X_\eta$ and $m = m_\xi \times m_\eta$. In this case we may define the family of transformations $\{T_C\}_{C\in\xi}$, T_C: $X_\eta \to X_\eta$ by the equality $T_C(D) = PT(C, D)$ where P: $X \to X_\eta$, $P(C, D) = D$ for $D \in \eta$. $\{T_C\}_{C\in\xi}$ is a measurable family of transformations. From Lemma 1 we conclude that $T(C, D) = \big(T_\xi(C), T_C(D)\big)$ is the generalized skew product of $\{T_C\}_{C\in\xi}$ and T_ξ. The family $\{T_C\}_{C\in\xi}$ is uniquely determined by ξ in the sense that if $\eta' = \eta$ modulo a set of measure zero then $T_C' = T_C$ a.e. for every $C \in \xi$. It is not difficult to find an endomorphism T and a partition ξ, $T^{-1}\xi \leqslant \xi$, such that ξ has no independent complement. A partial solution of the problem of existence of an independent complement is given by the following theorem:

Theorem 1. *Let ξ be a partition such that $T^{-1}\xi \leqslant \xi$ and $\xi \vee T^{-1}\varepsilon = \varepsilon$ for an ergodic endomorphism T. The following conditions are equivalent:*

(a) *m_B has an atom for a.e. $B \in \xi$.*

(b) *There exists a set X', $m(X') = 1$, and a finite number k such that* $\operatorname{card}(B \cap X') = k$ *for a.e.* $B \in \xi$.

Proof. The implication (b) $\Rightarrow$ (a) is obvious. Now, we show the converse. Using the reasoning of Rokhlin (see [2], p. 41) we get a measurable set C such that for a.e. $B \in \xi$, $C \cap B$ consists of an atom of the greatest measure. Let us observe that

$$B \cap T^{-1}(C) = \{y\colon\ \exists D \in \xi\colon\ y \in T^{-1}(D \cap C) \cap B\}$$
$$= \{y\colon\ \exists D \in \xi\colon\ y \in T^{-1}(D \cap C) \cap B \text{ and } T(B) \cap D \neq \emptyset\}.$$

By the definition of ξ, $T(B) \cap D \neq \emptyset$ implies $T(B) \subset D$ and there exists exactly one such D. Denote it by D_B. Hence

$$B \cap T^{-1}(C) = \{y\colon\ y \in T^{-1}(D_B \cap C) \cap B\}.$$

Since $T \restriction B$ is 1-1, there exists exactly one $y_B \in T^{-1}(D_B \cap C) \cap B$. Therefore $\operatorname{card}\big(B \cap T^{-1}(C)\big) = 1$ for a.e. B. By induction we get this property for the sets $T^{-i}C$ for $i = 1, 2, \ldots$ The ergodicity of T implies

$$m\Big(\bigcup_{i=0}^{\infty} T^{-i}(C)\Big) = 1.$$

From the equality

$$m(C) = m(T^{-i}(C)) = \int_{X_\xi} \Big(\int_B 1_{T^{-i}(C)}(x)\, dm_B\Big) dm_\xi$$

it follows that $T^{-i}C \cap B$ is an atom of maximum measure. Hence the sum of measures of atoms of maximum measure in B is equal to 1. Therefore there exists a set X' of measure 1 such that $\operatorname{card}(B \cap X') < \aleph_0$ for a.e. $B \in \xi$ and $B \cap X'$ consists of atoms of the same measure. Hence (b) follows easily.

COROLLARY 2. *If a partition ξ satisfies the assumptions of Theorem 1 then it has an independent complement. In this case T_C is 1-1.*

We now show that as a matter of fact if an endomorphism $T(x, y) = (g(x), T_x(y))$ has a 1-sided generator of finite entropy and T_x is 1-1 for a.e. x then T is a skew product. Namely, the following theorem holds:

THEOREM 2. *If T has a 1-sided generator of finite entropy and the transformations $\{T_x\}_{x\in X}$ are 1-1 for a.e. x then T_x is an automorphism for a.e. x.*

The proof is based on the following lemma:

LEMMA 3. *If T satisfies the hypotheses of Theorem 2 then*

$$J_T(x, y) = J_g(x) J_{T_x}(y) \quad \text{a.e.}$$

Proof of Lemma 3. Let $B \in \mathscr{B}$ be a set such that $g \restriction B$ is 1-1. Then for any $D \in \mathscr{D}$, $T \restriction B \times D$ is 1-1 a.e. and

$$\int_B J_g(x) \int_D J_{T_x}(y)\, dp\, dm = \int_B J_g(x)\, p\big(T_x(D)\big)\, dm = \int_{g(B)} p\big(T_{(g\restriction B)^{-1}(x)}(D)\big)\, dm$$
$$= (m\times p)\big(T(B\times D)\big) = \int_{B\times D} J_T(x, y)\, d(m\times p).$$

Now, let $C \in \mathscr{B}$. The endomorphism g is countable-to-one because T is countable-to-one as a transformation with a countable generator. Therefore C is a disjoint sum of sets C_i where $g\restriction C_i$ is 1-1 for $i = 0, 1, \ldots$ and

$$\int_C J_g \int_D J_{T_x}\, dp\, dm = \sum_{i=0}^{\infty} \int_{C_i} J_g \int_D J_{T_x}\, dp\, dm$$
$$= \sum_{i=0}^{\infty} \int_{C_i\times D} J_T\, d(m\times p) = \int_{C\times D} J_T\, d(m\times p).$$

The above equalities hold for any $C \in \mathscr{B}$ and $D \in \mathscr{D}$. Hence by the separability of the Lebesgue space we get $J_T(x, y) = J_g(x) J_{T_x}(y)$ a.e. ∎

Proof of Theorem 2. By the hypotheses of the theorem and by Lemma 3

$$h(T) = \int \log J_T\, d(m\times p) = \int \log(J_g J_{T_x})\, d(m\times p)$$
$$= \int \log J_g\, dm + \int \log J_{T_x}\, d(m\times p).$$

Here $\int \log J_g\, dm \leqslant h(g) \leqslant h(T)$. Hence $0 \leqslant \int \log J_{T_x}\, d(m\times p)$. By the concavity of $\log x$ we get

$$0 \leqslant \int \log J_{T_x}\, d(m\times p) \leqslant \log \int J_{T_x}(y)\, d(m\times p)$$

and by the T-invariance of the product measure $m\times p$ we have

$$\int J_{T_x}(y)\, dp = p\big(T_x(Y)\big) = 1 \quad \text{for a.e. } x.$$

Therefore

$$\int \log J_{T_x}\, d(m\times p) = \log \int J_{T_x}\, d(m\times p) = 0.$$

The above equality holds only in the case $\log J_{T_x}(y) = 0$ a.e. and hence $J_x(y) = 1$ a.e. This implies that T_x is an automorphism for a.e. x. ∎

In general, the assumption that T has a 1-sided generator and the transformations $\{T_x\}_{x\in X}$ are 1-1 for a.e. x is necessary.

EXAMPLE 2. Let $X = \prod_{-\infty}^{+\infty}\{1, \ldots, k\}$, let T be the 2-sided Bernoulli shift and m a product measure on X. We represent X as the product $Y\times Z$ where $y \in Y$ and $z \in Z$ are sequences $y = (x_{-1}, x_{-2}, \ldots)$, $z = (x_0, x_1, \ldots)$, and

$$(y, z) \sim (\ldots, x_{-2}, x_{-1}, x_0, \ldots).$$

Here $m = m_1 \times m_2$ and $T(y, z) = (\sigma(y), T_y(z))$ where

$$\sigma(y) = (x_{-2}, x_{-3}, \ldots), \qquad T_y(z) = (x_{-1}, x_0, x_1, \ldots).$$

Then T_y is 1-1 and does not preserve the measure m_2.

The above example was suggested by M. Misiurewicz.

EXAMPLE 3. Let $h(x) = 2x \bmod 1$,

$$\varphi(x) = \begin{cases} \tfrac12, & x \in [0, \tfrac12], \\ -\tfrac12, & x \in (\tfrac12, 1], \end{cases} \qquad f(y) = \begin{cases} 1, & y \in [0, \tfrac12], \\ -1, & y \in (\tfrac12, 1]. \end{cases}$$

We define a family of transformations $\{T_x\}_{x\in[0,1]}$ such that $J_{T_x^{-1}}(y) = 1 + \varphi(x) f(y)$ by

$$T_x(y) = \begin{cases} \tfrac43 y, & y \in [0, \tfrac38], \\ \tfrac43 y - \tfrac12, & y \in (\tfrac38, \tfrac34], \\ 4y - \tfrac52, & y \in (\tfrac34, \tfrac78], \\ 4y - 3, & y \in (\tfrac78, 1] \end{cases} \quad \text{for } x \in [0, \tfrac12],$$

$$T_x(y) = \begin{cases} 4y, & y \in [0, \tfrac18], \\ 4y - \tfrac12, & y \in (\tfrac18, \tfrac14], \\ \tfrac43 y + \tfrac16, & y \in (\tfrac14, \tfrac58], \\ \tfrac43 y - \tfrac13, & y \in [\tfrac58, 1] \end{cases} \quad \text{for } x \in (\tfrac12, 1].$$

The generalized skew product $T(x, y) = (h(x), T_x(y))$ has the 1-sided generator

$$\alpha = \{[0, \tfrac{1}{8}], (\tfrac{1}{8}, \tfrac{1}{4}], (\tfrac{1}{4}, \tfrac{3}{8}], (\tfrac{3}{8}, \tfrac{5}{8}], (\tfrac{5}{8}, \tfrac{3}{4}], (\tfrac{3}{4}, \tfrac{7}{8}], (\tfrac{7}{8}, 1]\} \times \{[0, \tfrac{1}{2}], (\tfrac{1}{2}, 1]\}$$

because $T \restriction C$ is expanding for any $C \in \alpha$. By $J_{T_x^{-1}} \neq 1$, the transformations T_x do not preserve the measure m_0.

References

[1] L. M. Abramov and V. A. Rokhlin, *Entropy of the skew product of measure preserving transformations*, Vestnik Leningrad. Univ. 1962, no. 7, 5–13.

[2] V. A. Rokhlin, *On the fundamental ideas of measure theory*, Amer. Math. Soc. Transl. Ser. 1 10 (1962), 1–54; Mat. Sb. 25 (1949), 107–150.

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Received October 12, 1985
Revised version August 1, 1986 (2110)

STUDIA MATHEMATICA, T. LXXXVII. (1987)

On the space of Bloch harmonic functions and interpolation of spaces of harmonic and holomorphic functions

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Abstract. We prove that the orthogonal projection P from $L^2(D)$ onto $L^2\operatorname{Harm}(D)$, the space of square-integrable harmonic functions, maps $L^\infty(D)$ onto the space $\operatorname{Bl}\operatorname{Harm}(D)$ of Bloch harmonic functions on D if D is a smooth bounded domain in $\boldsymbol{R}^n$. We prove an interpolation theorem which permits us to interpolate between Sobolev or Hölder spaces of harmonic functions and the space $L^p\operatorname{Harm}(D, |\varrho|^r)$ of harmonic functions from $L^p(D, |\varrho|^r)$, where ϱ is a defining function for D. We prove analogous results for spaces of holomorphic functions on strictly pseudoconvex domains.

I. Introduction and the statement of results. The present paper is the direct continuation of [14] and [15]. First, let us recall some notation from those papers.

For a bounded domain D in $\boldsymbol{R}^n$ we denote by P the orthogonal projection from $L^2(D)$ onto the space $L^2\operatorname{Harm}(D)$ of square-integrable harmonic functions. If D is a domain in $\boldsymbol{C}^n$, we denote by B the orthogonal projection from $L^2(D)$ onto the space $L^2\operatorname{Hol}(D)$ of square-integrable holomorphic functions (the Bergman projection). $\operatorname{Harm}_p^s(D)$ is the space of harmonic functions from the Sobolev space $W_p^s(D)$, $-\infty < s < +\infty$, $1 < p < \infty$, and $\Lambda_s\operatorname{Harm}(D)$ the space of harmonic functions from the Hölder space $\Lambda_s(D)$; analogously, $\operatorname{Hol}_p^s(D)$ denotes the space of holomorphic functions from $W_p^s(D)$ and $\Lambda_s\operatorname{Hol}(D)$ the space of holomorphic functions from $\Lambda_s(D)$. If D is a C^∞-smooth domain in $\boldsymbol{R}^n$ then a function $\varrho \in C^\infty(\boldsymbol{R}^n)$ is a *defining function* for D iff $D = \{x \in \boldsymbol{R}^n\colon \varrho(x) < 0\}$ and $\operatorname{grad}\varrho \neq 0$ on ∂D.

The space of *Bloch harmonic functions* on D consists of functions h harmonic on D such that

$$\|h\|_{\mathrm{Bl}} = \sup_{x \in D}\big(|\varrho(x)\,h(x)| + |\varrho(x)\operatorname{grad} h(x)|\big) < \infty$$

for a defining function ϱ. We denote it by $\operatorname{Bl}\operatorname{Harm}(D)$. If $D \subset \boldsymbol{C}^n$ then $\operatorname{Bl}\operatorname{Hol}(D)$ denotes the subspace of $\operatorname{Bl}\operatorname{Harm}(D)$ consisting of holomorphic functions.