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On the reproducing kernel for harmonic functions and the space of Bloch harmonic functions on the unit ball in R^n

by

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Abstract. An explicit formula for the orthogonal projection P from the space of square-integrable functions on the unit ball in $\boldsymbol{R}^n$ onto the space of square-integrable harmonic functions is used to the study of the properties of the space of Bloch harmonic functions. We prove that this space is dual to the space $L^1 \operatorname{Harm}(B)$ of integrable harmonic functions on the ball and is the " vertex " of the double interpolation scale formed by the L^p, Sobolev and Hölder spaces of harmonic functions. We use the interpolation property of the space of Bloch harmonic functions to the study of weighted Sobolev spaces of harmonic functions.

1. The reproducing kernel for harmonic functions. Let D be a bounded domain in $\boldsymbol{R}^n$. We shall denote by P the orthogonal projection from $L^2(D)$ onto the space $L^2 \operatorname{Harm}(D)$ of square-integrable harmonic functions. The projection P is an integral operator with kernel $K(x, y)$ equal to the reproducing kernel of the space $L^2 \operatorname{Harm}(D)$. If the boundary of D is sufficiently smooth then P can be written in the form $Pu = u - \Delta G_2 \Delta u$ (see [2]), where G_2 is the operator solving the Dirichlet problem $\Delta^2 g = f$, g vanishes on ∂D up to order 1. Let $G_2(x, y)$ denote the Green function of the above problem. Then $K(x, y) = -\Delta_x \Delta_y G_2(x, y)$. More exactly, $G_2(x, y)$ can be written in the form $G_2(x, y) = v(|x-y|) - \tilde{G}_2(x, y)$, where $v(|x-y|)$ is a fundamental solution of the equation $\Delta^2 g = f$ (this solution is $c|x-y|^{-n+4}$ if $n > 4$ is even and $c|x-y|^{4-n} \ln|x-y|$ if $n \leqslant 4$ or if n is odd) and $\tilde{G}_2(x, y)$ is the symmetric biharmonic function such that for every y, $\tilde{G}_2(x, y) - v(|x-y|)$ vanishes on ∂D up to order 1. We have

$$K(x, y) = \Delta_x \Delta_y \tilde{G}_2(x, y).$$

This formula, however elegant, is impractical, since it is not easy to find an explicit formula for $\tilde{G}_2(x, y)$ even if D is the unit ball B in $\boldsymbol{R}^n$. Fortunately, in that case we can use another approach.

Let G be the operator solving the Dirichlet problem $\Delta g = f$, $g \equiv 0$ on ∂B. We can write

$$Pu = \Delta(Gu - G_2 \Delta^2 Gu).$$

The function in brackets is the biharmonic function h on B such that $h \equiv 0$ on ∂B and $\partial h/\partial n = \partial Gu/\partial n$ on ∂B. Since B is the unit ball, we have $\partial/\partial n = \sum_{i=1}^{n} x_i\, \partial/\partial x_i$. The function h can be written as

$$h = \tfrac{1}{2}(|x|^2-1)\left(\frac{\partial}{\partial n}Gu - G\Delta\frac{\partial}{\partial n}Gu\right),$$

since the function in the right brackets is harmonic, $h=0$ and $\partial h/\partial n = \partial Gu/\partial n$ on ∂D. Hence

$$(*) \qquad Pu = \tfrac{1}{2}\Delta\left[(|x|^2-1)\left(\frac{\partial}{\partial n}Gu - G\Delta\frac{\partial}{\partial n}Gu\right)\right].$$

Now,

$$G\Delta\frac{\partial}{\partial n}Gu = G\frac{\partial u}{\partial n} + 2Gu.$$

We have

$$Gu = \int_B G(x, y)\, u(y)\, dV_y,$$

$$G\frac{\partial u}{\partial n} = -\int_B \frac{\partial}{\partial n_y} G(x, y)\, u(y)\, dV_y - n\int_B G(x, y)\, u(y)\, dV_y$$

where $G(x, y)$ is the Green function of the ball B:

$$G(x, y) = c(n)\left(\frac{1}{|x-y|^{n-2}} - \frac{1}{[|x-y|^2+(1-|x|^2)(1-|y|^2)]^{n/2-1}}\right) \quad \text{if } n>2,$$

$$G(x, y) = c(n)\left(\ln|x-y| - \tfrac{1}{2}\ln(|x-y|^2+(1-|x|^2)(1-|y|^2))\right) \quad \text{if } n=2.$$

Substituting this in $(*)$ we have after elementary calculations

$$Pu = c(n)\,\Delta\left[(|x|^2-1)\int_B \frac{(1-|x|^2|y|^2)\,u(y)}{[|x-y|^2+(1-|x|^2)(1-|y|^2)]^{n/2}}\, dV_y\right]$$

and finally

$$Pu = c(n)\int_B \left(\frac{2n(1-|x|^2|y|^2)(1-\sum_{i=1}^{n} x_i y_i)}{[|x-y|^2+(1-|x|^2)(1-|y|^2)]^{n/2+1}} - \frac{4|x|^2|y|^2}{[|x-y|^2+(1-|x|^2)(1-|y|^2)]^{n/2}}\right)u(y)]\, dV_y.$$

Hence the reproducing kernel for the space $L^2\,\mathrm{Harm}(B)$ is

$$K(x, y) = c(n)\left(\frac{2n(1-|x|^2|y|^2)(1-\sum_{i=1}^{n} x_i y_i)}{[|x-y|^2+(1-|x|^2)(1-|y|^2)]^{n/2+1}} - \frac{4|x|^2|y|^2}{[|x-y|^2+(1-|x|^2)(1-|y|^2)]^{n/2}}\right).$$

2. The space of Bloch harmonic functions. Let D be a bounded domain in $\boldsymbol{R}^n$. A harmonic function u on D is called a *Bloch harmonic function* iff

$$\sup_{z\in D} \operatorname{dist}(z, \partial D)\,|\operatorname{grad} u(z)| < \infty.$$

The Bloch harmonic functions form a Banach space with norm

$$\|u\|_{\mathrm{Bl}} = \sup_{z\in D} \operatorname{dist}(z, \partial D)\left(|u(z)| + |\operatorname{grad} u(z)|\right).$$

Let D be the unit ball in $\boldsymbol{R}^n$. As before, we prefer to denote it by B. Bloch harmonic functions on B are those functions u for which

$$\sup_{z\in B}(1-|z|^2)\,|\operatorname{grad} u(z)| < \infty$$

and the norm

$$\|u\| = \sup_{z\in D}(1-|z|^2)\left(|u(z)| + |\operatorname{grad} u(z)|\right)$$

is equivalent to the above one.

We shall denote the space of Bloch harmonic functions by $\mathrm{Bl\,Harm}(D)$.

We prove the following

THEOREM 1. *The space of Bloch harmonic functions on B is the dual of the space $L^1\,\mathrm{Harm}(B)$ of integrable harmonic functions on B.*

Theorem 1 is a consequence of the following

PROPOSITION 1. *The projector P maps continuously $L^\infty(B)$ onto* $\mathrm{Bl\,Harm}(B)$.

Proof. We shall use the explicit expression for $K(x, y)$ given in Section 1. Let $u \in L^\infty(B)$. We have

$$\frac{\partial}{\partial x_i} Pu = \int_B \frac{\partial}{\partial x_i} K(x, y)\, u(y)\, dV_y.$$

Since

$$|x-y|^2+(1-|x|^2)(1-|y|^2) \geqslant \tfrac{1}{4}[(1-|x|^2)^2+(1-|y|^2)^2],$$
$$1-|x|^2|y|^2 = 1-|x|^2+|x|^2(1-|y|^2),$$
$$1-\sum_{i=1}^{n} x_i y_i = 1-|x|^2+\sum_{i=1}^{n} x_i(x_i-y_i),$$

we have

$$\left|\frac{\partial}{\partial x_i}K(x, y)\right| \leqslant \frac{C_i}{[|x-y|^2+(1-|x|^2)^2+(1-|y|^2)^2]^{n/2+1/2}}$$

and thus, by deleting $(1-|y|^2)^2$, we have

$$\left|\frac{\partial}{\partial x_i}Pu(x)\right| \leqslant C_i \|u\|_{L^\infty(B)} \int_B \frac{dV_y}{[|x-y|^2+(1-|x|^2)^2]^{n/2+1/2}}.$$

The last integral is less than

$$\int_{B(x,2)} \frac{dV_y}{[|x-y|^2+(1-|x|^2)^2]^{n/2+1/2}} \leqslant \frac{c}{1-|x|^2}.$$

Thus P maps $L^\infty(B)$ into $\mathrm{Bl\,Harm}(B)$. The closed graph theorem ensures that P is continuous.

To prove that P is onto we shall use the Bell operator $L^1 u$ (see [2]). If u is a harmonic function on B then

$$L^1 u(x) = u(x)-\tfrac{1}{2}\Delta\left[(|x|^2-1)^2 u(x)\frac{\varphi(x)}{4|x|^2}\right],$$

where $\varphi(x)$ is an arbitrarily chosen C^∞ function equal to 1 in a neighbourhood of $\{|x| = 1\}$ and to 0 in a neighbourhood of zero. It is clear that L^1 maps $\mathrm{Bl\,Harm}(B)$ into $L^\infty(B)$ and $u = P(L^1 u)$.

Proof of Theorem 1. We shall follow the idea of Bell [2] and introduce the pairing

$$\langle v, u\rangle_1 = \int_B v\overline{L^1 u}\,dV.$$

We shall show that $\mathrm{Bl\,Harm}(B)$ represents the dual space to $L^1\,\mathrm{Harm}(B)$ via the pairing $\langle v, u\rangle_1$. It follows from the properties of the operator L^1 that every function $u \in \mathrm{Bl\,Harm}(B)$ determines a continuous linear functional since

$$|\langle v, u\rangle_1| = |\langle v, L^1 u\rangle| \leqslant c\|v\|_{L^1}\|L^1 u\|_{L^\infty} \leqslant c_1\|v\|_{L^1}\|u\|_{\mathrm{Bl\,Harm}(B)}.$$

Conversely, let φ be a continuous linear functional on $L^1\,\mathrm{Harm}(B)$. The functional φ can be extended to a continuous functional $\tilde{\varphi}$ on $L^1(D)$ and represented by a function $m \in L^\infty(D)$. By Proposition 1, $P(m) \in \mathrm{Bl\,Harm}(D)$ and does not depend on the choice of $\tilde{\varphi}$ and m. The space $L^2\,\mathrm{Harm}(B)$ is dense in $L^1\,\mathrm{Harm}(B)$ since for every $u \in L^1\,\mathrm{Harm}(B)$, $u_r(x) = u(rx)$ tends to u in L^1 norm if $r \to 1$, $r < 1$. Thus we get a continuous mapping F from $\mathrm{Bl\,Harm}(D)$ onto $(L^1\,\mathrm{Harm}(D))^*$ such that $\ker F = 0$ and by the open mapping theorem it must be an isomorphism.

The same method of proof was used in [6] in order to prove that $\mathrm{Bl\,Hol}(D)$, the space of Bloch holomorphic functions, represents the dual of $L^1\,\mathrm{Hol}(D)$, the space of holomorphic functions from $L^1(D)$, if D is a bounded strictly pseudoconvex domain with C^4 boundary.

3. The space of Bloch harmonic functions and interpolation. In [7] the following fact was proved: Let D be a smooth bounded domain in $\boldsymbol{R}^n$. Let $A_p^s = \mathrm{Harm}_p^s(D)$ be the subspace of the Sobolev space $W_p^s(D)$ consisting of harmonic functions and let $A_\infty^s = \Lambda_s\,\mathrm{Harm}(D)$ be the space of Hölder harmonic functions, $s > 0$. Then

$$[A_{p_1}^{s_1}, A_{p_2}^{s_2}]_{[\theta]} = A_p^s, \quad s = (1-\theta)s_1+\theta s_2, \quad \frac{1}{p} = \frac{1-\theta}{p_1}+\frac{\theta}{p_2},$$

$0 \leqslant \theta \leqslant 1$, for $s \geqslant 0$, $1 < p < \infty$ or $p = \infty$, $s > 0$. Note that $A_p^0 = \mathrm{Harm}_p^0(D) = L^p\,\mathrm{Harm}(D)$.

Moreover, in [7] it was also proved that if P maps $L^\infty(D)$ onto $\mathrm{Bl\,Harm}(D)$ then we can put $A_\infty^0 = \mathrm{Bl\,Harm}(D)$ and obtain

$$[A_{p_1}^{s_1}, A_\infty^0]_{[\theta]} = A_p^s, \quad s = (1-\theta)s_1, \ p = \frac{p_1}{1-\theta}$$

for $1 < p < \infty$, $s \geqslant 0$, and

$$[A_\infty^{s_1}, A_\infty^0]_{[\theta]}^{\sim} = A_\infty^s, \quad s = (1-\theta)s_1.$$

Recall that if E_1, E_2 is an admissible pair of Banach spaces, then $[E_1, E_2]_{[\theta]}$ denotes the value of the interpolation functor, constructed by the complex interpolation method, at θ. The symbol $[E_1, E_2]_{[\theta]}^{\sim}$ denotes the completion of $[E_1, E_2]_{[\theta]}$ with respect to E_1+E_2. (If $F \subset E$ are Banach spaces then the completion of F with respect to E is the space of all $x \in E$ for which there exists $c > 0$ and a sequence $\{x_n\} \subset F$ converging to x in E such that $\sup_n \|x_n\|_F \leqslant c$. The norm of x is defined as the infimum of the constants c.)

Further information concerning interpolation can be found in [3] and [4].

Thus our Proposition 1 yields immediately

THEOREM 2. *Let B be the unit ball in $\boldsymbol{R}^n$. Let $A_p^s = \mathrm{Harm}_p^s(B)$, $s \geqslant 0$, $1 < p < \infty$, $A_\infty^s = \Lambda_s\,\mathrm{Harm}(B)$, $A_\infty^0 = \mathrm{Bl\,Harm}(B)$. Then*

$$[A_{p_1}^{s_1}, A_{p_2}^{s_2}]_{[\theta]} = A_p^s, \quad s = (1-\theta)s_1+\theta s_2, \quad \frac{1}{p} = \frac{1-\theta}{p_1}+\frac{\theta}{p_2},$$

if $\inf(p_1, p_2) < \infty$, *and*

$$[A_\infty^{s_1}, A_\infty^{s_2}]_{[\theta]}^{\sim} = A_\infty^s, \qquad s = (1-\theta)s_1 + \theta s_2.$$

In other words, the L^p, Sobolev and Hölder spaces of harmonic functions form a double interpolation scale whose vertex is the space of Bloch harmonic functions. In [5] it was proved that for every smooth bounded domain D, $\Lambda_s \operatorname{Harm}(D)$ is the dual space to $\mathring{L}^1 \operatorname{Harm}(D, [\operatorname{dist}(x, \partial D)]^s)$, the closure of $L^2 \operatorname{Harm}(D)$ in $L^1(D, [\operatorname{dist}(x, \partial D)]^s)$. If $D = B$ then it is easy to prove that $L^2 \operatorname{Harm}(B)$ is dense in $L^1 \operatorname{Harm}(B, [\operatorname{dist}(x, \partial B)]^s)$ by considering again the functions $u(rx)$, $r < 1$, which tend to u in $L^1(B, [\operatorname{dist}(x, \partial B)]^s)$ norm.

The space $L^1 \operatorname{Harm}(B, [\operatorname{dist}(x, \partial B)]^s)$ is equal to $L^1 \operatorname{Harm}(B, (1-|x|^2)^s)$. It follows from the duality theorem from [7] that the spaces $\operatorname{Harm}_p^s(B)$ and $L^q \operatorname{Harm}(B, (1-|x|^2)^{qs})$, $q = p/(p-1)$, are mutually dual. Thus Theorems 1 and 2 yield the following

COROLLARY. *The spaces* $E_p^t = L^p \operatorname{Harm}(B, (1-|x|^2)^t)$ *form a double interpolation scale whose vertex is* $L^1 \operatorname{Harm}(D)$:

$$[E_{p_1}^{t_1}, E_{p_2}^{t_2}]_{[\theta]} = E_p^t, \qquad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad t = \frac{t_1 p}{p_1} + \frac{t_2 p}{p_2},$$
$$1 \leqslant p_1, p_2 < \infty, \ t \geqslant 0.$$

(Cf. Remark 4 of [7].)

4. Weighted Sobolev spaces of harmonic functions. This part of the paper is devoted to a generalization of one of the results of Beatrous and Burbea from [1]. We shall denote by $\operatorname{Harm}_{p,q}^s(B)$, $1 < p < \infty$, $-1 \leqslant q < \infty$, $s \geqslant 0$, the weighted Sobolev space of harmonic functions defined as follows:

(a) If $q > -1$ and s is an integer then $\operatorname{Harm}_{p,q}^s(B)$ is the space of harmonic functions whose derivatives up to order s belong to $L^p(B, (1-|x|^2)^q)$.

(b) If $q > -1$ and s is a real number then $\operatorname{Harm}_{p,q}^s(B)$ is the space of harmonic functions f such that $D^\alpha f \in L^p(B, (1-|x|^2)^{q+p(1+[s]-s)})$ for $|\alpha| = [s]+1$ ($[s]$ denotes the integer part of s).

(c) If $q = -1$ then $\operatorname{Harm}_{p,q}^s(B)$ is the space of harmonic functions whose traces on ∂B belong to $W_p^s(\partial B)$.

We prove the following

THEOREM 3. (a) *If* $(q_1 - q_2)/p = s_1 - s_2$ *and* $q_1, q_2 > -1$ *then*

$$\operatorname{Harm}_{p,q_1}^{s_1}(B) = \operatorname{Harm}_{p,q_2}^{s_2}(B) = \operatorname{Harm}_p^{s_2 - q_2/p}(B).$$

(b) *If* $q_2 = -1$ *and* $(q_1+1)/p = s_1 - s_2$, $q_1 > -1$, *then*

$$\operatorname{Harm}_{p,-1}^{s_2}(B) \subset \operatorname{Harm}_{p,q_1}^{s_1}(B) \quad \text{if } 2 \leqslant p \leqslant \infty,$$
$$\operatorname{Harm}_{p,q_1}^{s_1}(B) \subset \operatorname{Harm}_{p,-1}^{s_2}(B) \quad \text{if } 1 < p \leqslant 2,$$

and the inclusions are continuous.

Proof. We shall first consider the case $q_1, q_2 \geqslant 0$.

a) In [7] it was proved that for each $s \geqslant 0$, the space $\operatorname{Harm}_p^{-s}(B)$ is equal to $L^p \operatorname{Harm}(B, (1-|x|^2)^{ps})$. Thus we have for $q_1 \geqslant 0$

$$D^{[s_1]+1} h \in L^p \operatorname{Harm}(B, (1-|x|^2)^{q_1 + p(1+[s]-s)})$$

if and only if $h \in \operatorname{Harm}_p^{s_1 - q_1/p}(B)$. The same is true for s_2 and q_2. Hence

$$\operatorname{Harm}_{p,q_1}^{s_1}(B) = \operatorname{Harm}_p^{s_1 - q_1/p}(B), \qquad \operatorname{Harm}_{p,q_2}^{s_2}(B) = \operatorname{Harm}_p^{s_2 - q_2/p}(B).$$

By the assumptions, $s_1 - q_1/p = s_2 - q_2/p$ and all these spaces are equal.

b) We begin with the following lemma.

LEMMA. *If* $p \geqslant 2$ *then* $\operatorname{Harm}^p(\partial B) = \operatorname{Harm}_{p,-1}^0(B) \subset \operatorname{Harm}_p^{1/p}(B)$.

If $1 < p \leqslant 2$ *then* $\operatorname{Harm}_p^{1/p}(B) \subset \operatorname{Harm}^p(\partial B)$ ($\operatorname{Harm}^p(\partial B) = \operatorname{Harm}_{p,-1}^0(B)$ *is the Hardy space of harmonic functions*).

Proof. Let $p \geqslant 2$. It is well known that $\operatorname{Harm}^2(\partial B) = \operatorname{Harm}_2^{1/2}(B)$, that $\operatorname{Harm}^\infty(\partial B) \subset \operatorname{Bl} \operatorname{Harm}(B)$ and that the last inclusion is continuous. Thus Theorem 2 yields that

$$\begin{aligned}
\operatorname{Harm}^{2/(1-\theta)}(B) &= [\operatorname{Harm}^2(\partial B), \operatorname{Harm}^\infty(\partial B)]_{[\theta]} \\
&\subset \operatorname{Harm}_{2/(1-\theta)}^{(1-\theta)/2}(B) \\
&= [\operatorname{Harm}_2^{1/2}(B), \operatorname{Bl} \operatorname{Harm}(B)]_{[\theta]}
\end{aligned}$$

for every $0 < \theta < 1$. By putting $\theta = (p-2)/p$ we get the statement of the lemma for $p \geqslant 2$.

Let $1 < p \leqslant 2$. From the Green formula we have for harmonic u and v

$$\begin{aligned}
\int_{\partial B} u\bar{v}\, d\sigma &= \tfrac{1}{2} \int_{\partial B} u\bar{v} \frac{\partial(|x|^2 - 1)}{\partial n}\, d\sigma \\
&= \tfrac{1}{2} \int_B u\bar{v} \Delta(|x|^2 - 1)\, dV - \tfrac{1}{2} \int_B \Delta(u\bar{v})(|x|^2 - 1)\, dV \\
&= \tfrac{1}{2} \int_B z\bar{v} \Delta(|x|^2 - 1)\, dV - \sum_{i=1}^n \int_B \frac{\partial \bar{v}}{\partial x_i} \frac{\partial u}{\partial x_i}(|x|^2 - 1)\, dV \\
&= \tfrac{1}{2} \int_B u \left[\Delta(|x|^2 - 1)\bar{v} + 2 \sum_{i=1}^n \frac{\partial \bar{v}}{\partial x_i} \frac{\partial}{\partial x_i}(|x|^2 - 1)\right] dV \\
&= \langle u, \tfrac{1}{2} \Delta[(|x|^2 - 1)v]\rangle.
\end{aligned}$$

Note that if v is harmonic then $\Delta[(|x|^2-1)v]$ is also harmonic. If $v\in \mathrm{Harm}^p(\partial B)$ and $q=p/(p-1)$, then

$$\|v\|_{\mathrm{Harm}^p(\partial B)} = \sup_{\substack{u\in \mathrm{Harm}^q(\partial B)\\ \|u\|\leqslant 1}} \left|\int_{\partial B} u\bar{v}\, d\sigma\right| = \tfrac{1}{2} \sup_{\substack{u\in \mathrm{Harm}^q(\partial B)\\ \|u\|\leqslant 1}} \langle u, \Delta[(|x|^2-1)v]\rangle.$$

Now, let $v\in \mathrm{Harm}_p^{1/p}(B)$. Theorem 1 of [7] yields that $(|x|^2-1)v\in W_p^{1+1/p}(B)$ and hence $\Delta[(|x|^2-1)v]\in \mathrm{Harm}_p^{-1+1/p}(B)$. The duality theorem from [7] implies that $\mathrm{Harm}_p^{-1+1/p}(B)$ is the dual space of $\mathrm{Harm}_q^{1/q}(B)$ and

$$\sup_{\substack{u\in \mathrm{Harm}_q^{1/q}(B)\\ \|u\|<1}} |\langle u, \Delta[|x|^2-1)v]\rangle| \leqslant c\|\Delta[(|x|^2-1)v]\|_p^{-1+1/p} \leqslant c_1\|v\|_p^{1/p}.$$

Since $1<p\leqslant 2$ we have $q=p/(p-1)\geqslant 2$ and thus $\mathrm{Harm}^q(\partial B)\subset \mathrm{Harm}_q^{1/q}(B)$ and the inclusion is continuous. Thus

$$\|v\|_{\mathrm{Harm}^p(\partial B)} = \tfrac{1}{2} \sup_{\substack{u\in \mathrm{Harm}^q(\partial B)\\ \|u\|\leqslant 1}} |\langle u, \Delta[(|x|^2-1)v]\rangle|$$

$$\leqslant c \sup_{\substack{u\in \mathrm{Harm}_q^{1/q}(B)\\ \|u\|\leqslant 1}} |\langle u, \Delta[(|x|^2-1)v]\rangle| \leqslant c_1\|v\|_p^{1/p}.$$

This implies that $\mathrm{Harm}_p^{1/p}(B)\subset \mathrm{Harm}^p(\partial B)$ and the inclusion is continuous.

Having proved the lemma we can finish the proof of part (b) of Theorem 3 in the case $q_1\geqslant 0$.

Our lemma yields immediately that

$$\mathrm{Harm}^{s_2}_{p,-1}(B)\subset \mathrm{Harm}_p^{s_2+1/p}(B) \quad \text{for } s_2>0,\ p\geqslant 2,$$

$$\mathrm{Harm}_p^{s_1+1/p}(B)\subset \mathrm{Harm}^{s_2}_{p,-1}(B) \quad \text{for } s_2>0,\ 1<p\leqslant 2.$$

We have as before

$$\mathrm{Harm}^{s_1}_{p,q_1}(B) = \mathrm{Harm}_p^{s_1-q_1/p}(B).$$

c) *Case* $0>\min(q_1,q_2)>-1$. In order to prove Theorem 3 in this case it suffices to show that

$$\mathrm{Harm}_p^{-q/p}(B)\subset L^p(B,(1-|x|^2)^q), \quad 0>q>-1,\ 1<p<\infty.$$

Put $r=-q$. If u is a bounded function from $\mathrm{Harm}_p^{r/p}(B)$ then clearly $u\in L^p(B,(1-|x|^2)^{-r})$ since $r<1$. We have the following estimate

$$\|u\|^p_{L^p(B,(1-|x|^2)^{-r})} = \int_B |u|^p(1-|x|^2)^{-r}\,dV$$

$$= \int_B |u|^p(1-|x|^2)^{1-r}\,dV + \int_B |x|^2|u|^p(1-|x|^2)^{-r}\,dV$$

$$= \int_B |u|^p(1-|x|^2)^{1-r}\,dV - \frac{1}{2(1-r)}\sum_i\int_B x_i\frac{\partial}{\partial x_i}(1-|x|^2)^{1-r}|u|^p\,dV$$

$$= \left(1+\frac{n}{2(1-r)}\right)\int_B |u|^p(1-|x|^2)^{1-r}\,dV$$

$$+\frac{1}{2(1-r)}\int_B\sum_i x_i\frac{\partial u}{\partial x_i}(1-|x|^2)^{1-r}|u|^{p-2}u\,dV$$

$$= \left(1+\frac{n}{2(1-r)}\right)\int_B \left(|u|^{p-1}(1-|x|^2)^{-r(p-1)/p}\right)\left(|u|(1-|x|^2)^{1-r/p}\right)dV$$

$$+\frac{1}{2(1-r)}\int_B\left(\sum_i x_i\frac{\partial u}{\partial x_i}(1-|x|^2)^{1-r/p}\right)\left(|u|^{p-1}(1-|x|^2)^{-r(p-1)/p}\right)dV$$

$$\leqslant \left[\left(1+\frac{n}{2(1-r)}\right)\|u(1-|x|^2)^{1-r/p}\|_{L^p(B)}\right.$$

$$\left.+\frac{1}{2(1-r)}\left\|\sum_i x_i\frac{\partial u}{\partial x_i}(1-|x|^2)^{1-r/p}\right\|_{L^p(B)}\right]\left(\int_B |u|^p(1-|x|^2)^{-r}\,dV\right)^{(p-1)/p}$$

by the Hölder inequality. Since $u\in \mathrm{Harm}_p^{r/p}(B)$ we have $\partial u/\partial x_i\in \mathrm{Harm}_p^{-1+r/p}(B)$ and thus

$$\frac{\partial u}{\partial x_i}(1-|x|^2)^{1-r/p}\in L^p(B), \qquad \left\|\frac{\partial u}{\partial x_i}(1-|x|^2)\right\|_{L^p(B)} \leqslant c\|u\|_{\mathrm{Harm}_p^{r/p}(B)}$$

by Theorem 2 of [7]. Thus the above estimate implies that

$$\|u\|^p_{L^p(B,(1-|x|^2)^{-r})} \leqslant c\|u\|_{\mathrm{Harm}_p^{r/p}(B)}\|u\|^{p-1}_{L^p(B,(1-|x|^2)^{-r})},$$

and

$$\|u\|_{L^p(B,(1-|x|^2)^{-r})} \leqslant c\|u\|_{\mathrm{Harm}_p^{r/p}(B)}$$

for smooth u from $\mathrm{Harm}_p^{r/p}(B)$. Since smooth harmonic functions are dense in $\mathrm{Harm}_p^{r/p}(B)$, our theorem is proved. Since P maps $L^p(B,(1-|x|^2)^{-r})$ onto $\mathrm{Harm}_p^{r/p}(B)$ (see [7]) we have in fact proved that $\mathrm{Harm}_p^{r/p}(B) = L^p\,\mathrm{Harm}(B,(1-|x|^2)^{-r})$.

Theorem 3 was proved by Beatrous and Burbea (see [1], Theorem 5.12) for spaces of holomorphic functions on the unit ball in C^n in a somewhat more general setting, namely for $0<p\leqslant\infty$ and not only for $1<p<\infty$.

Remark 1. Part (a) of Theorem 3 remains valid for every smooth bounded domain D. This follows from the results of [7]. In particular, the same method as in part c) of the proof of Theorem 3, and Theorem 2 of [7] yield the following

THEOREM. *If D is a smooth bounded domain, then*

$$\mathrm{Harm}_p^s(D) = L^p\,\mathrm{Harm}(D,|\varrho|^{-ps}) \quad \textit{for each } s<1/p.$$

The symbol ϱ denotes here a *defining function* for D, i.e. $\varrho \in C^\infty(\boldsymbol{R}^n)$, $D = \{x \in \boldsymbol{R}^n \colon \varrho(x) < 0\}$, $\operatorname{grad} \varrho \neq 0$ on ∂D.

Remark 2. All results of the present paper, excluding part (b) of Theorem 3, remain valid if the L^p, Sobolev and Bloch spaces of harmonic functions are replaced by the corresponding spaces of m-polyharmonic functions, i.e. functions u for which $\Delta^m u = 0$. The details will be given in a forthcoming paper *On duality and interpolation for spaces of polyharmonic functions*.

Added in proof (July 1987). Proposition 1 along with other results of this paper is valid in the case where D is any smooth bounded domain. Details will be given in our next paper.

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Weighted weak type inequalities for the ergodic maximal function and the pointwise ergodic theorem

by

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Abstract. Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and let $T\colon X \to X$ denote an invertible measure-preserving transformation. In this paper we characterize those pairs of positive functions u, v for which the maximal operator

$$Mf(x) = \sup_{k \geqslant 0} (k+1)^{-1} \sum_{i=0}^{k} |f(T^i x)|$$

is of weak type (1, 1) with respect to the measures $v\,d\mu$ and $u\,d\mu$. We also get a pointwise ergodic theorem for noninvertible T if $\mu(X) < \infty$. More precisely, we prove that $(k+1)^{-1} \sum_{i=0}^{k} f(T^i x)$ converges a.e. for every $f \in L^1(v\,d\mu)$ if and only if $\inf_{i \geqslant 0} v(T^i x) > 0$ a.e.

1. Introduction. Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and let $T\colon X \to X$ denote an invertible measure-preserving transformation. For each pair of nonnegative integers, r and k, we consider the averages

$$T_{r,k} f(x) = (r+k+1)^{-1} \sum_{i=-r}^{k} f(T^i x)$$

where f is a measurable function. Associated to these averages we have the following maximal operators:

$$f^* = \sup_{r,k \geqslant 0} T_{r,k}|f|, \qquad Mf = \sup_{k \geqslant 0} T_{0,k}|f|.$$

The maximal ergodic theorem asserts that f^* and Mf satisfy weak type inequalities

$$\mu(\{x \in X\colon f^*(x) > \lambda\}) \leqslant 2\lambda^{-1} \int_X |f|\,d\mu,$$

$$\mu(\{x \in X\colon Mf(x) > \lambda\}) \leqslant \lambda^{-1} \int_X |f|\,d\mu$$

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