4. The final norm. Let now

$$\|p\| = \operatorname*{Lim}_{n\to+\infty} |p|_{(n)},$$

for any polynomial p. We have the following properties of the limit norm $\|\cdot\|$:

PROPOSITION 5.

(a) $\|l_j q_j - q_j'\| \leqslant \varepsilon_j, \quad j \geqslant 1.$

(b) $\|l_j p\| \leqslant 2^{N_j}\|p\|.$

(c) $\|xp\| \leqslant 2\|p\|.$

(d) *The norm* $\|\cdot\|$ *is hilbertian.*

(e) *For any* $n \geqslant 1$ *and any* p *with* $d^\circ p \leqslant n$,

$$\|p\| \geqslant \prod_{k\geqslant n}(1-4^{-k})\,|p|_{(n-1)}.$$

This last property ensures of course that the limit norm is nonzero. Therefore the completion of the polynomials for $\|\cdot\|$ is a Hilbert space on which the multiplication by x is continuous. Every polynomial q with rational coefficients is hypercyclic. Indeed, let $q' \neq 0$ in H, and let $\varepsilon > 0$. We can find in the enumeration an integer j such that

$$q_j = q, \qquad \varepsilon_j < \varepsilon/2, \qquad |q_j' - q|_w < \varepsilon/2.$$

Then $\|q_j' - q\| < \varepsilon/2$, and

$$\|x^{N_j} q_j - q\| \leqslant \|x^{N_j} q_j - q_j'\| + \|q_j' - q_j\| < \varepsilon,$$

which proves our claim.

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Some remarks on Triebel spaces

by

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Abstract. Some extensions of results in the recent monograph by Triebel [13] about Triebel spaces F_{pq}^s are given. This concerns multiplication properties, dual spaces and some remarks on the spaces $F_{\infty q}^s$.

0. Introduction. Triebel spaces are a natural generalization of Sobolev–Hardy spaces. The characterization of these spaces by decompositions of Littlewood–Paley type provides a useful tool for the study of multiplication properties, dual spaces, etc.

The plan of this paper is as follows. Chapter 1 is used to fix the notation and to recall some results on Besov and Triebel spaces. In Chapter 2 multiplication properties of Triebel spaces are studied: multiplication by functions belonging to Hölder–Zygmund spaces, multiplication algebras and multiplication by the characteristic function of an interval.

Chapter 3 is devoted to some complementary results in the determination of dual spaces. The main result can be phrased as follows. Let us denote by $\mathring{F}_{pq}^s$ the closure of the Schwartz space $\mathscr{S}(\boldsymbol{R}^n)$ in F_{pq}^s. Then for $1 \leqslant p, q \leqslant \infty$ the dual of $\mathring{F}_{pq}^s$ is isomorphic to $F_{p'q'}^{-s}$, $1/p+1/p' = 1/q+1/q' = 1$. Also some extensions to weighted spaces are given. The weight may belong to the Muckenhoupt class A_∞.

Finally, Chapter 4 contains some remarks on $F_{\infty q}^s$, $1 \leqslant q \leqslant \infty$. In particular, the trace problem is solved.

1. Besov and Triebel spaces. All functions and distributions are assumed to be defined on the n-dimensional Euclidean space $\boldsymbol{R}^n$. $\mathscr{S}(\boldsymbol{R}^n)$ is the Schwartz space of rapidly decreasing functions and $\mathscr{S}'(\boldsymbol{R}^n)$ its dual, the space of tempered distributions.

The Fourier transform is defined by

$$\hat{f}(\xi) := \int e^{-ix\cdot\xi} f(x)\,dx, \qquad f \in \mathscr{S}(\boldsymbol{R}^n),$$

and extended to $\mathscr{S}'(\boldsymbol{R}^n)$ by duality. The inverse Fourier transform is

$$\check{f}(x) := (2\pi)^{-n} \int e^{ix\cdot\xi} f(\xi)\,d\xi.$$

For suitable distributions f and g, let us denote by $f*g$ the convolution of distributions.

Denote by $\phi(\boldsymbol{R}^n)$ the set of all partitions $\{\varphi_k\} \subset \mathcal{S}(\boldsymbol{R}^n)$ such that

(1) $$\operatorname{supp}\varphi_0 \subset \{\xi\colon |\xi| \leqslant 2\}, \quad \operatorname{supp}\varphi_k \subset \{\xi\colon 2^{k-1} \leqslant |\xi| \leqslant 2^{k+1}\}$$

for $k = 1, 2, \ldots$,

(2) $$|\partial^\alpha \varphi_k(\xi)| \leqslant C_\alpha 2^{-k|\alpha|}$$

for all multi-indices α and

(3) $$\sum_{k=0}^{\infty} \varphi_k(\xi) \equiv 1.$$

$\phi(\boldsymbol{R}^n)$ is not empty; see Triebel [13], Remark 2.3.1.1.

For $0 < p < \infty$, $0 < q \leqslant \infty$ and $s \in \boldsymbol{R}$ we define the *Triebel space* F^s_{pq} to be the set of all $f \in \mathcal{S}'(\boldsymbol{R}^n)$ such that

(4) $$\|f\|_{F^s_{pq}} := \|\{2^{ks}\check{\varphi}_k * f\}\|_{L^p(l^q)} = \Big\|\Big(\sum_{k=0}^{\infty} 2^{ksq}|\check{\varphi}_k * f(x)|^q\Big)^{1/q}\Big\|_{L^p} < \infty$$

(modification for $q = \infty$).

Similarly for $0 < p,\ q \leqslant \infty$ and $s \in \boldsymbol{R}$ the *Besov space* B^s_{pq} is the set of all $f \in \mathcal{S}'(\boldsymbol{R}^n)$ such that

(5) $$\|f\|_{B^s_{pq}} := \|\{2^{ks}\check{\varphi}_k * f\}\|_{l^q(L^p)} = \Big(\sum_{k=0}^{\infty} 2^{ksq}\|\check{\varphi}_k * f\|^q_{L^p}\Big)^{1/q} < \infty$$

(modification for $q = \infty$).

These spaces are independent of the choice of the partition $\{\varphi_k\} \in \phi(\boldsymbol{R}^n)$. Elementary properties are:

(6) $$B^s_{p,\min\{p,q\}} \hookrightarrow F^s_{pq} \hookrightarrow B^s_{p,\max\{p,q\}},$$

(7) $$B^{s_1}_{pq_1} \hookrightarrow B^{s_2}_{pq_2},$$

(8) $$F^{s_1}_{pq_1} \hookrightarrow F^{s_2}_{pq_2},$$

if $q_1 \leqslant q_2$ and $s_1 = s_2$, or $s_1 > s_2$.

An essential tool in Triebel [13] is the Peetre maximal function. It has the disadvantage of not giving optimal results. Therefore we will not use it. Note, however, that there exists a new maximal technique, which avoids this drawback in many situations; see Marschall [8], Chapter 4.

In this paper we use the following results.

LEMMA 1. *Let $f_k \in \mathcal{S}'(\boldsymbol{R}^n)$ and suppose that for some constants $0 < c_1 < c_2$*

$$\operatorname{supp}\hat{f}_0 \subset \{\xi\colon |\xi| \leqslant c_2\}, \quad \operatorname{supp}\hat{f}_k \subset \{\xi\colon c_1 2^k \leqslant |\xi| \leqslant c_2 2^k\}$$

for $k = 1, 2, \ldots$ Then for $0 < p,\ q \leqslant \infty$ and $s \in \boldsymbol{R}$

$$\Big\|\sum_{k=0}^{\infty} f_k\Big\|_{F^s_{pq}} \leqslant C\|\{2^{ks}f_k\}\|_{L^p(l^q)},$$

$$\Big\|\sum_{k=0}^{\infty} f_k\Big\|_{B^s_{pq}} \leqslant C\|\{2^{ks}f_k\}\|_{l^q(L^p)}. \ \blacksquare$$

The lemma follows immediately using the Nikol'skii representation; see Triebel [13], Theorem 2.5.2.

LEMMA 2. *Let $f_k \in \mathcal{S}'(\boldsymbol{R}^n)$ be distributions such that for some constant $c > 0$*

$$\operatorname{supp}\hat{f}_k \subset \{\xi\colon |\xi| \leqslant c2^k\}.$$

Then for any real number s with $s > n(\max\{1, 1/p\} - 1)$

$$\Big\|\sum_{k=0}^{\infty} f_k\Big\|_{B^s_{pq}} \leqslant C\|\{2^{ks}f_k\}\|_{l^q(L^p)}.$$

Proof. Let $\{\varphi_j\} \in \phi(\boldsymbol{R}^n)$. Then for some $l = l(c) \in \boldsymbol{N}$ one has

$$\check{\varphi}_j * \Big(\sum_{k=0}^{\infty} f_k\Big) = \sum_{k=j-l}^{\infty} \check{\varphi}_j * f_k.$$

Now for any integer k satisfying $k \geqslant j - l$ the following inequality holds with $p_1 := \min\{1, p\}$:

$$\|\check{\varphi}_j * f_k\|_{L^p} \leqslant C2^{kn(1/p_1 - 1)}\|\check{\varphi}_j\|_{L^{p_1}}\|f_k\|_{L^p} \leqslant C2^{(k-j)n(1/p_1 - 1)}\|f_k\|_{L^p}$$

(see Triebel [13], 1.5.3.3). Hence the lemma follows by summation. $\blacksquare$

Note that there is an analogous statement for weighted and unweighted Triebel spaces (see Marschall [8], Lemma 1.4 and 4.2).

We will need the following general Sobolev embedding theorem.

LEMMA 3. (a) *If $0 < p < q < \infty$ and $0 < r \leqslant \infty$ then*

$$F^{s+n(1/p-1/q)}_{p\infty} \hookrightarrow F^s_{qr}.$$

(b) *If $0 < p < q \leqslant \infty$ then*

$$F^{s+n(1/p-1/q)}_{p\infty} \hookrightarrow B^s_{qp}.$$

(c) *If $0 < p < q < \infty$ and $0 < r \leqslant \infty$ then*

$$B^{s+n(1/p-1/q)}_{pq} \hookrightarrow F^s_{qr}.$$

Proof. For (a) and (b) see Jawerth [7]. The proof of (c) is similar to that of (b). If $p < q_0 < q < q_1 < \infty$ we get from (a)

$$B^{s+n(1/p-1/q_i)}_{pp} \hookrightarrow F^s_{q_i r}.$$

Using real interpolation (see Bergh and Löfström [2]) it follows that

$$B^{s+n(1/p-1/q)}_{pq} \hookrightarrow (F^s_{q_0 r}, F^s_{q_1 r})_{\theta q}$$

where $1/q = (1-\theta)/q_0 + \theta/q_1$. Now $f \to \|\{2^{ks}\check{\varphi}_k * f\}\|_{l^r}$ is a quasilinear operator. Hence by the Marcinkiewicz interpolation theorem (see Bergh and Löfström [2])

$$(F^s_{q_0 r}, F^s_{q_1 r})_{\theta q} \hookrightarrow F^s_{qr}$$

which yields part (c). ∎

Below we shall extend part (c) of the lemma to the case $q = \infty$ (see Corollary 4).

2. Multiplication properties of Triebel spaces. For abbreviation set $f_j := \check{\varphi}_j * f$ if $f \in \mathcal{S}'(\boldsymbol{R}^n)$. If f and g belong to an appropriate Besov or Triebel space we make the following decomposition:

$$h = \sum_{j=4}^{\infty} \sum_{k=0}^{j-4} g_k f_j + \sum_{j=0}^{\infty} \sum_{k=j-3}^{j+3} g_k f_j + \sum_{k=4}^{\infty} \sum_{j=0}^{k-4} f_j g_k \tag{9}$$
$$= h_1 + h_2 + h_3.$$

If each of these sums converges in $\mathcal{S}'(\boldsymbol{R}^n)$, we call h the *product* of f and g. The convergence is usually shown by estimating h in a suitable F^s_{pq}-quasinorm. However, we shall not stress this point here, we only give the necessary estimates.

Let $g \in L^\infty$. Since the spectrum of $\sum_{k=0}^{j-4} g_k f_j$ is contained in the annulus $|\xi| \sim 2^j$, Lemma 1 yields

$$\|h_1\|_{F^s_{pq}} \leqslant C \Big\|\{2^{js} \sum_{k=0}^{j-4} g_k f_j\}\Big\|_{L^p(l^q)} \leqslant C \|g\|_{L^\infty} \|f\|_{F^s_{pq}} \tag{10}$$

for $0 < p < \infty$, $0 < q \leqslant \infty$ and $s \in \boldsymbol{R}$.

Hence h_1 is well defined for $g \in L^\infty$ and arbitrary $f \in F^s_{pq}$. Since we will assume that $g \in L^\infty$, we only have to estimate h_2 and h_3.

THEOREM 1. *Let $g \in B^r_{\infty\infty}$, $r > 0$. Then for $0 < p < \infty$, $0 < q \leqslant \infty$ and*

$$n(\max\{1, 1/p\} - 1) - r < s < r$$

the following estimate holds:

$$\|g \cdot f\|_{F^s_{pq}} \leqslant C \|g\|_{B^r_{\infty\infty}} \|f\|_{F^s_{pq}}.$$

Proof. The estimate for h_1 follows from (10).

Now observe that the spectrum of $\sum_{k=j-3}^{j+3} g_k f_j$ is contained in the ball $|\xi| \leqslant 2^{j+10}$. Hence Lemma 2 yields

$$\|h_2\|_{F^s_{pq}} \leqslant C \|h_2\|_{B^{s+r}_{p\infty}} \leqslant C \sup_j 2^{j(s+r)} \Big\| \sum_{k=j-3}^{j+3} g_k f_j \Big\|_{L^p}$$
$$\leqslant C \|g\|_{B^r_{\infty\infty}} \|f\|_{B^s_{p\infty}} \leqslant C \|g\|_{B^r_{\infty\infty}} \|f\|_{F^s_{pq}}$$

provided $s > n(\max\{1, 1/p\} - 1) - r$.

Finally, the spectrum of $\sum_{j=0}^{k-4} f_j g_k$ is contained in the annulus $|\xi| \sim 2^k$. Therefore by Lemma 1 if $s < r$

$$\|h_3\|_{F^s_{pq}} \leqslant C \Big\|\{2^{ks} \sum_{j=0}^{k-4} f_j g_k\}\Big\|_{L^p(l^q)} \leqslant C \|g\|_{B^r_{\infty\infty}} \Big\|\{2^{k(s-r)} \sum_{j=0}^{k-4} f_j\}\Big\|_{L^p(l^q)}$$
$$\leqslant C \|g\|_{B^r_{\infty\infty}} \|f\|_{F^{s-r}_{pq}}.$$

Now the theorem follows. ∎

This theorem improves Corollary 2.8.2 in Triebel [13]. For its generalization to weighted Triebel spaces see Marschall [8], Chapter 4. There pseudodifferential estimates can also be found. Another improvement concerns the case $s = r$ and $q = 2$; see Marschall [8], Chapter 11. There it is shown that for these values of the parameters the theorem remains true provided that $g \in F^r_{\infty 2}$ (for the definition of $F^s_{\infty q}$ see Chapter 3 below). Moreover, one has the following

THEOREM 2. *Let $0 < p \leqslant q < \infty$, $0 < r \leqslant \infty$ and*

$$s > \frac{n}{2}\left(\frac{1}{p} + \frac{1}{q} - 1\right).$$

Then for $s > n/q$ or $s = n/q$ and $0 < q \leqslant 1$ the following estimate holds:

$$\|g \cdot f\|_{F^s_{pr}} \leqslant C \|g\|_{F^s_{qr}} \|f\|_{F^s_{pr}}.$$

Before we prove the theorem, let us state an immediate consequence.

COROLLARY 1. *If $s > n/p$ or $s = n/p$ and $0 < p \leqslant 1$, then for $0 < r \leqslant \infty$, F^s_{pr} is a multiplication algebra.* ∎

Proof of the theorem. Note that by Lemma 3(b) we have $g \in L^\infty$. Hence in view of (10) it remains to provide the necessary estimates for h_2 and h_3.

Estimate for h_3. If $p = q$ then $f \in L^\infty$ and hence by (10)

$$\|h_3\|_{F^s_{pr}} \leqslant C \|f\|_{L^\infty} \|g\|_{F^s_{pr}}.$$

If $p < q$ let $1/p = 1/p_1 + 1/q$. Then by Hölder's inequality and Lemma 3(a) we obtain

$$\|h_3\|_{F^s_{pr}} \leqslant C \|g\|_{F^s_{qr}} \|f\|_{F^0_{p_1 1}} \leqslant C \|g\|_{F^s_{qr}} \|f\|_{F^{n/q}_{pr}}.$$

Estimate for h_2. Let $\varrho = s - n/q$. Note that $s > \frac{1}{2}n(1/p+1/q-1)$ and $s \geqslant n/q$ imply

$$s+\varrho > n(\max\{1, 1/p\}-1). \tag{11}$$

If $s > n/q$ then $g \in B^{\varrho}_{\infty\infty}$ and Lemma 2 yields

$$\|h_2\|_{F^s_{pr}} \leqslant C\|h_2\|_{B^{s+\varrho}_{p\infty}} \leqslant C\|g\|_{F^s_{qr}}\|f\|_{F^s_{pr}}.$$

If $s = n/q$ choose $p_1 < p < p_2$ such that $1/p_1 = 1/p_2 + 1/q$ and $n/q > n(1/p_1 - 1)$. Because of (11) this is possible. Then by Lemmas 3(c) and 2

$$\begin{aligned}\|h_2\|_{F^{n/q}_{pr}} &\leqslant C\|h_2\|_{B^{n(1/q+1/p_1-1/p)}_{p_1 p}} \\ &\leqslant C\sum_{l=-3}^{3}\|\{2^{jn(1/q+1/p_1-1/p)}g_{j+l}f_j\}\|_{l^p(L^{p_1})} \\ &\leqslant C\|g\|_{B^{n/q}_{q\infty}}\|f\|_{B^{n(1/p_1-1/p)}_{p_2 p}}.\end{aligned}$$

Hence Lemma 3(b) yields

$$\|h_2\|_{F^{n/q}_{pr}} \leqslant C\|g\|_{F^{n/q}_{qr}}\|f\|_{F^{n/q}_{pr}}.$$

This completes the proof of the theorem. ∎

Let us also mention the following result.

THEOREM 3. *Let $q \leqslant p$ if $0 < p \leqslant 2$ and $q \leqslant p/(p-1)$ if $2 \leqslant p < \infty$. Then for $0 < r \leqslant \infty$ and*

$$n(1/p-1) < s < n/p$$

one has the following estimate:

$$\|g\cdot f\|_{F^s_{pr}} \leqslant C(\|g\|_{L^\infty} + \|g\|_{B^{n/q}_{q\infty}})\|f\|_{F^s_{pr}}.$$

Proof. The estimate for h_1 follows from (10). For the estimation of h_3 choose $p_1 < p < p_2$ such that $1/p_1 = 1/p + 1/p_2$ and $s < n(1/p - 1/p_2)$. Then by Lemma 3 and Hölder's inequality

$$\begin{aligned}\|h_3\|_{F^s_{pr}} &\leqslant C\|h_3\|_{B^{s+n/p_2}_{p_1 p}} \\ &\leqslant C\|g\|_{B^{n/p}_{p\infty}}\|\{2^{k(s+n(1/p_2-1/p))}\sum_{j=0}^{k-4}f_j\}\|_{l^p(L^{p_2})} \\ &\leqslant C\|g\|_{B^{n/p}_{p\infty}}\|f\|_{B^{s+n(1/p_2-1/p)}_{p_2 p}} \\ &\leqslant C\|g\|_{B^{n/q}_{q\infty}}\|f\|_{F^s_{pr}}.\end{aligned}$$

Next choose $p_1 < p < p_2$ such that $1/p_1 = 1/q + 1/p_2$ and

$$s + n\left(\frac{1}{p_1} - \frac{1}{p}\right) > n(\max\{1, 1/p_1\} - 1).$$

For $q = p$ if $0 < p \leqslant 2$, resp. $q = p/(p-1)$ if $2 \leqslant p < \infty$, this choice is possible since in either case $s + n(1/p_1 - 1/p) > n(1/p_1 - 1)$.

Then by Lemmas 2 and 3 we get

$$\begin{aligned}\|h_2\|_{F^s_{pr}} &\leqslant C\|h_2\|_{B^{s+n(1/p_1-1/p)}_{p_1 p}} \leqslant C\|g\|_{B^{n/q}_{q\infty}}\|f\|_{B^{s-n(1/p-1/p_2)}_{p_2 p}} \\ &\leqslant C\|g\|_{B^{n/q}_{q\infty}}\|f\|_{F^s_{pr}}.\end{aligned}$$

This proves the theorem. ∎

This theorem is the counterpart for Triebel spaces of Remark 2.6.4.5 in Triebel [12]. Let us give two applications.

First, Proposition 3.4.1.2 in Triebel [13] can be improved as follows.

COROLLARY 2. *Let $a \in C^\infty$ be a function such that $|\partial^\alpha a(x)| \leqslant C_\alpha$ for all multi-indices α. Further, let $\varphi \in C^\infty(\boldsymbol{R}^n)$ be supported in the unit ball and set $\varphi_\tau(x) := \varphi((x-x_0)/\tau)$. Then for $0 < p < \infty$, $0 < q \leqslant \infty$ and*

$$n(1/p-1) < s < n/p$$

there exists a constant $C > 0$ such that for $0 < \tau \leqslant 1$

$$\|(a(\cdot) - a(x_0))\varphi_\tau \cdot f\|_{F^s_{pq}} \leqslant C\tau\|f\|_{F^s_{pq}}.$$

Proof. It is shown in the aforementioned proposition of Triebel that for $0 < r < \infty$

$$\|(a(\cdot) - a(x_0))\varphi_\tau\|_{B^{n/r}_{r\infty}} \leqslant C\tau.$$

Since obviously

$$\|(a(\cdot) - a(x_0))\varphi_\tau\|_{L^\infty} \leqslant C\tau,$$

the corollary follows from Theorem 3. ∎

Denote by $\varkappa_{[a,b]}$ the characteristic function of a bounded or unbounded interval $[a, b] \subset \boldsymbol{R}$.

COROLLARY 3. *For $0 < p < \infty$, $0 < q \leqslant \infty$ and $1/p - 1 < s < 1/p$*

$$\|\varkappa_{[a,b]}f\|_{F^s_{pq}(\boldsymbol{R})} \leqslant C\|f\|_{F^s_{pq}(\boldsymbol{R})}.$$

Proof. It is shown in Proposition 2.8.4, Triebel [13], that $\varkappa_{[a,b]} = g_1 + g_2$ with $g_1 \in B^{1/r}_{r\infty}$, $0 < r < \infty$, and $|\partial^k g_2(x)| \leqslant C_k$ for $k = 0, 1, 2, \ldots$ Again the corollary follows from the theorem. ∎

Note that this corollary solves the extension problem for $F^s_{pq}(\boldsymbol{R}^+)$. For details see Triebel [13], Chapter 2.9.

3. Dual spaces. Let $1 \leqslant p, q \leqslant \infty$ and $\{\varphi_j\} \in \phi(\boldsymbol{R}^n)$ be a partition. Denote by L^s_{pq} the space of all $f \in \mathscr{S}'(\boldsymbol{R}^n)$ represented by

$$f = \sum_{j=0}^{\infty}\check{\varphi}_j * f_j \tag{12}$$

such that

(13) $$\|f\|_{L^s_{pq}} := \inf \|\{2^{js} f_j\}\|_{L^p(l^q)} < \infty$$

where the infimum is taken over all representations (12). These spaces have been introduced in Triebel [12] for the study of the duals of Triebel spaces.

For $1 \leqslant p,\ q < \infty$ the Schwartz space $\mathscr{S}(\boldsymbol{R}^n)$ is dense in L^s_{pq} and if $1 < p,\ q < \infty$ then $L^s_{pq} \approx F^s_{pq}$; see Triebel [12], Proposition 2.5.1.2, Triebel [13], Proposition 2.3.4.1.

Actually the last statement is true for $1 \leqslant q \leqslant \infty$.

PROPOSITION 1. *If* $1 < p < \infty$, $1 \leqslant q \leqslant \infty$ *and* $s \in \boldsymbol{R}$ *then* $L^s_{pq} \approx F^s_{pq}$.

Proof. The inclusion $F^s_{pq} \hookrightarrow L^s_{pq}$ is obvious. For the other direction we use the Hardy–Littlewood maximal function

$$Mf(x) := \sup_{r>0} \frac{1}{mB(x, r)} \int_{B(x,r)} |f(y)|\, dy.$$

One has (see Stein [10], Theorem 3.2.2)

$$|\check{\varphi}_j * f_j(x)| \leqslant CMf_j(x).$$

Then the boundedness of the maximal function on $L^p(l^q)$ for $1 < p < \infty$ and $1 < q \leqslant \infty$ (see Fefferman and Stein [5]) implies the assertion for $1 < q \leqslant \infty$.

If $q = 1$ we use

LEMMA 4. *For* $1 \leqslant p < \infty$ *and* $1/p + 1/p' = 1$ *we have*

$$L^p(c_0)' \approx L^{p'}(l^1). \ \blacksquare$$

Here c_0 is the Banach space of all sequences converging to zero. For a proof see Edwards [5], Theorems 8.18.2 and 8.20.3.

Proof of the proposition, the case $q = 1$. Let $f = \sum_{j=0}^{\infty} \check{\varphi}_j * f_j$ be such that $\{2^{js} f_j\} \in L^p(l^1)$. We show that

$$\{2^{ks} \check{\varphi}_k * \sum_j \check{\varphi}_j * f_j\} \in L^{p'}(c_0)'.$$

By the lemma this implies $f \in F^s_{p1}$.

Since for $\{g_k\} \in L^{p'}(c_0)$

$$\langle \check{\varphi}_k * \sum_j \check{\varphi}_j * f_j,\ g_k \rangle = \langle f_j,\ \sum_{|j-k| \leqslant 3} \check{\varphi}_j * \check{\varphi}_k * g_k \rangle,$$

$$\Big| \sum_{|k-j| \leqslant 3} \check{\varphi}_j * \check{\varphi}_k * g_k(x) \Big| \leqslant C \sum_{|k-j| \leqslant 3} Mg_k(x),$$

we get

$$\Big| \langle \{2^{ks} \check{\varphi}_k * \sum_j \check{\varphi}_j * f_j\},\ \{g_k\} \rangle \Big| \leqslant C \|\{2^{js} f_j\}\|_{L^p(l^1)} \|\{Mg_k\}\|_{L^{p'}(l^\infty)}$$

$$\leqslant C \|f\|_{L^s_{p1}} \|\{g_k\}\|_{L^{p'}(l^\infty)}.$$

This yields the conclusion. $\blacksquare$

It turns out that the right choice for $F^s_{\infty q}$, $1 \leqslant q \leqslant \infty$, is

(14) $$F^s_{\infty q} := L^s_{\infty q}.$$

In particular, $F^s_{\infty\infty} \approx B^s_{\infty\infty}$. Denote by $\mathring{F}^s_{pq}$ the closure of the Schwartz space $\mathscr{S}(\boldsymbol{R}^n)$ in F^s_{pq}. Now the main result in this chapter is

THEOREM 4. *If* $1 \leqslant p,\ q \leqslant \infty$ *and* $s \in \boldsymbol{R}$ *then*

$$\mathring{F}^s_{pq}{}' \approx F^{-s}_{p'q'}$$

where $1/p + 1/p' = 1/q + 1/q' = 1$.

Proof. (a) First take $1 \leqslant p < \infty$.

Then for $1 \leqslant q < \infty$ one has $F^s_{pq}{}' \approx L^{-s}_{p'q'}$ (see Triebel [12], Theorem 2.5.1). Now for $q = \infty$ the mapping

$$f \to \{2^{ks} \check{\varphi}_k * f\}:\ \mathring{F}^s_{p\infty} \to L^p(c_0)$$

is an isometric embedding. Then by Lemma 4, Triebel's proof yields $\mathring{F}^s_{p\infty}{}' \approx L^{-s}_{p'1}$. Hence Proposition 1 and the definition of $F^s_{\infty q}$ imply the assertion for $1 \leqslant p < \infty$.

(b) Let $p = \infty$. Here we show that the norm topology on $\mathring{F}^s_{\infty q}$ is compatible with the duality $(\mathring{F}^s_{\infty q}, F^{-s}_{1q'})$, i.e. we show that the norm topology is finer than the weak topology $\sigma(\mathring{F}^s_{\infty q}, F^{-s}_{1q'})$ and weaker than the Mackey topology $\tau(\mathring{F}^s_{\infty q}, F^{-s}_{1q'})$. Then the Mackey theorem (see Edwards [5], 8.3.3) yields $\mathring{F}^s_{\infty q} \approx F^{-s}_{1q'}$.

(c) The norm topology on $\mathring{F}^s_{\infty q}$ is the topology of uniform convergence on the closed unit ball B of $F^{-s}_{1q'}$ (observe that $\mathring{F}^{-s}_{1q'}{}' \approx F^s_{\infty q}$!). Consequently, the norm topology is finer than $\sigma(\mathring{F}^s_{\infty q}, F^{-s}_{1q'})$. That it is weaker than $\tau(\mathring{F}^s_{\infty q}, F^{-s}_{1q'})$ is a consequence of the following assertion:

$$B \textit{ is } \sigma(F^{-s}_{1q'}, \mathring{F}^s_{\infty q})\textit{-compact.}$$

Proof. (i) Let us first show that B is $\sigma(F^{-s}_{1q'}, \mathring{F}^s_{\infty q})$-sequentially compact. In fact, let $f_j \in B$ be a sequence. Then $f_{j_k} \to f$ in $\mathscr{S}'(\boldsymbol{R}^n)$ for some subsequence. Now if $\{\varphi_i\} \in \phi(\boldsymbol{R}^n)$ then for fixed i

$$\check{\varphi}_i * f_{j_k} \to \check{\varphi}_i * f \quad \text{pointwise.}$$

Hence Fatou's lemma yields

$$\|f\|_{F^{-s}_{1q'}} \leqslant \liminf \|f_{j_k}\|_{F^{-s}_{1q'}} \leqslant 1$$

and therefore $f \in B$. Obviously we have $f_{j_k} \to f$ in $\sigma(F^{-s}_{1q'}, \mathring{F}^s_{\infty q})$.

(ii) $\mathring{F}^s_{\infty q}$ is separable because $\mathscr{S}(\boldsymbol{R}^n)$ is dense in it. Let $\{g_j\} \subset \mathscr{S}(\boldsymbol{R}^n)$ be dense in $\mathring{F}^s_{\infty q}$. Then

$$d(f, h) := \sum 2^{-j} \min\{1, |\langle f-h, g_j \rangle|\}$$

defines a metric on B, which induces a weaker topology than $\sigma(F^{-s}_{1q'}, \mathring{F}^s_{\infty q})$.

Now recall that any sequentially compact space with a weaker metrizable topology is actually compact, because both topologies are identical. Consequently, B is $\sigma(F^{-s}_{1q'}, \mathring{F}^{s}_{\infty q})$-compact.

This completes the proof of the theorem. ■

Remark. If $0 < p < 1$ then

$$\mathring{F}^{s}_{p\infty}{}' \approx B^{-s+n(1/p-1)}_{\infty\infty} \approx F^{-s+n(1/p-1)}_{\infty\infty}. \tag{15}$$

This is a consequence of

$$B^{s}_{pp} \hookrightarrow \mathring{F}^{s}_{p\infty} \hookrightarrow B^{s-n(1/p-1)}_{11}$$

and the known duality theory for B^{s}_{pq}; see Triebel [13], Chapter 2.11. The determination of the dual of F^{s}_{pq} for $1 < p < \infty$ and $0 < q < 1$ remains open. It is known that

$$F^{-s}_{p'\infty} \hookrightarrow F^{s}_{pq}{}' \hookrightarrow B^{-s}_{p'\infty}.$$

The conjecture is of course $F^{s}_{pq}{}' \approx F^{-s}_{p'\infty}$. ■

Let us state some consequences of Theorem 4.

Corollary 4. *For* $0 < p < \infty$

$$B^{s+n/p}_{p\infty} \hookrightarrow F^{s}_{\infty 1}.$$

Proof. We may suppose $1 < p < \infty$. But then the statement follows by duality from $\mathring{F}^{-s}_{1\infty} \hookrightarrow B^{-s-n/p}_{p'1}$. ■

We can also extend the Fourier multiplier result, Theorem 2.4.8 in Triebel [13], to the case $q = \infty$. Let φ, $\psi \in \mathscr{S}(\boldsymbol{R}^n)$ be such that

$$\operatorname{supp}\psi \subset \{\xi\colon |\xi| \leqslant 4\}, \qquad \psi(\xi) = 1 \text{ if } |\xi| \leqslant 2,$$

$$\operatorname{supp}\varphi \subset \{\xi\colon \tfrac{1}{4} \leqslant |\xi| \leqslant 4\}, \qquad \varphi(\xi) = 1 \text{ if } \tfrac{1}{2} \leqslant |\xi| \leqslant 2.$$

For abbreviation set

$$\|m\|_{h^{\varkappa}2} := \|m\psi\|_{F^{\varkappa}_{22}} + \sup_{j=1,2,\ldots} \|m(2^j\cdot)\varphi\|_{F^{\varkappa}_{22}}. \tag{16}$$

Corollary 5. *Let* $0 < p < \infty$, $s \in \boldsymbol{R}$ *and*

$$\varkappa > n(\max\{1, 1/p\} - \tfrac{1}{2}).$$

Then

$$\|\check{m} * f\|_{F^{s}_{p\infty}} \leqslant C\|m\|_{h^{\varkappa}2}\|f\|_{F^{s}_{p\infty}}.$$

Proof. For $1 < p < \infty$ the result follows by duality from the case $q = 1$. For $0 < p \leqslant 1$ one uses complex interpolation. For details see the aforementioned reference. ■

The preceding results can be extended to weighted Triebel spaces. Let

$1 < p < \infty$ and $w \geqslant 0$ be such that

$$\sup \frac{1}{mQ}\int_Q w\,dx \left(\frac{1}{mQ}\int_Q w^{-1/(p-1)}\,dx\right)^{p-1} < \infty \tag{17}$$

where the supremum is taken over all cubes $Q \subset \boldsymbol{R}^n$. Then w is said to satisfy Muckenhoupt's *A_p-condition.*

Let $w \in A_\infty := \bigcup_{1<p<\infty} A_p$. Define $F^{s}_{pq}(w)$ by

$$\|f\|_{F^{s}_{pq}(w)} := \|\{2^{js}\check{\varphi}_j * f\}\|_{L^p(w,l^q)} = \Big\|\Big(\sum_{j=0}^{\infty} 2^{jsq}|\check{\varphi}_j * f(x)|^q\Big)^{1/q}\Big\|_{L^p(w)} < \infty \tag{18}$$

(modification for $q = \infty$).

Similarly define $L^{s}_{pq}(w)$ as the space of all $f \in \mathscr{S}'(\boldsymbol{R}^n)$ which can be represented by $f = \sum \check{\varphi}_k * f_k$ in such a way that

$$\|f\|_{L^{s}_{pq}(w)} := \inf\|\{2^{ks}f_k\}\|_{L^p(w,l^q)} < \infty. \tag{19}$$

Analogously to Proposition 1 one may prove

Proposition 2. *If* $1 < p < \infty$, $1 \leqslant q \leqslant \infty$ *and* $w \in A_p$ *then* $L^{s}_{pq}(w) \approx F^{s}_{pq}(w)$. ■

Here one has to use the weighted Hardy–Littlewood maximal theorem; see Andersen and John [1]. Now using the results of Bui Huy Qui [4], we can show similarly to Theorem 4

Theorem 5. *Let* $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, w, $w^{-1/(p-1)} \in A_\infty$ *and* $1/p + 1/p' = 1/q + 1/q' = 1$. *Then*

$$\mathring{F}^{s}_{pq}(w)' \approx L^{-s}_{p'q'}(w^{-1/(p-1)}), \qquad \mathring{L}^{s}_{pq}(w)' \approx F^{-s}_{p'q'}(w^{-1/(p-1)}).\ ■$$

Obviously, $\mathring{F}^{s}_{pq}(w)$ resp. $\mathring{L}^{s}_{pq}(w)$ denotes the closure of $\mathscr{S}(\boldsymbol{R}^n)$ in $F^{s}_{pq}(w)$ resp. $L^{s}_{pq}(w)$. Since $\mathscr{S}(\boldsymbol{R}^n)$ is dense in $F^{s}_{pq}(w)$ and $L^{s}_{pq}(w)$ for p, $q < \infty$, we obtain

Corollary 6. *For* $1 < p$, $q < \infty$ *and* w, $w^{-1/(p-1)} \in A_\infty$ *the spaces* $F^{s}_{pq}(w)$ and $L^{s}_{pq}(w)$ *are reflexive.* ■

Now let $w \in A_1$, i.e. let $Mw(x) \leqslant Cw(x)$ a.e. Define bmo(w) by

$$\|f\|_{\mathrm{bmo}(w)} := \sup_{mQ \geqslant 1} \frac{1}{wQ}\int_Q |f(x)|\,dx + \sup_{mQ \leqslant 1} \frac{1}{wQ}\int_Q \Big|f - \frac{1}{wQ}\int_Q f\,dy\Big|\,dx < \infty. \tag{20}$$

Here we have set $wQ = \int_Q w\,dx$.

Let $\psi \in C^\infty_0$ be a cut-off function such that $\psi(\xi) = 0$ for $|\xi| \leqslant \frac{1}{2}$ and $\psi(\xi) = 1$ for $|\xi| \geqslant 1$. Let for $j = 1, \ldots, n$

$$r_j(D)f(x) := (2\pi)^{-n}\int e^{ix\cdot\xi}\psi(\xi)\frac{\xi_j}{|\xi|}\hat{f}(\xi)\,d\xi \tag{21}$$

be the inhomogeneous Riesz transform.

The *Hardy space* $h^1(w)$ is defined by

$$\|f\|_{h^1(w)} := \|f\|_{L^1(w)} + \sum_{j=1}^{n} \|r_j(D)f\|_{L^1(w)} < \infty. \tag{22}$$

Let cmo(w) be the closure of $\mathscr{S}(\boldsymbol{R}^n)$ in bmo(w). Since $h^1(w)' \approx$ bmo(w) (Bui Huy Qui [3]) and $h^1(w) \approx F^0_{12}(w)$ (Bui Huy Qui [4]), the method described above yields

COROLLARY 7. *For* $w \in A_1$ *the dual of* cmo(w) *is isomorphic to* $h^1(w)$. ■

The case $w \equiv 1$ of this corollary has been shown by Neri [9].

4. Remarks on $F^s_{\infty q}$. By Theorem 4, many results for F^s_{pq} extend to the case $p = \infty$. For example, one can make the following statement about Fourier multipliers. If $\varkappa > n/2$ then

$$\|\check{m} * f\|_{F^s_{\infty q}} \leqslant C\|m\|_{h^{\varkappa 2}} \|f\|_{F^s_{\infty q}}. \tag{23}$$

This follows by duality from Theorem 2.4.8, Triebel [13], if $1 < q \leqslant \infty$ and from Corollary 5 if $q = 1$. More generally, let us consider pseudodifferential operators. Define $S^m_{1\delta}(r, N)$ to consist of symbols $a(\cdot,\cdot)$ such that for all $|\alpha| \leqslant N$

$$|\partial^\alpha_\xi a(x, \xi)| \leqslant C_\alpha (1+|\xi|)^{m-|\alpha|}, \qquad \|\partial^\alpha_\xi a(\cdot, \xi)\|_{B^r_{\infty\infty}} \leqslant C_\alpha (1+|\xi|)^{m+\delta r-|\alpha|}. \tag{24}$$

We associate pseudodifferential operators to these symbols by

$$\mathrm{Op}(a) f(x) := (2\pi)^{-n} \int e^{ix\cdot\xi} a(x, \xi) \hat{f}(\xi)\, d\xi \tag{25}$$

for $f \in \mathscr{S}(\boldsymbol{R}^n)$.

Let us take $N > \frac{3}{2}n$ and $0 \leqslant \delta \leqslant 1$. Then for $1 \leqslant q \leqslant \infty$ we have

$$\mathrm{Op}(a)\colon F^{s+m}_{\infty q} \to F^s_{\infty q} \tag{26}$$

provided that $-(1-\delta)r < s < r$ and

$${}^t\mathrm{Op}(a)\colon F^s_{\infty q} \to F^{s-m}_{\infty q} \tag{27}$$

provided that $-r < s < (1-\delta)r$. Here ${}^t\mathrm{Op}(a)$ is the transpose of $\mathrm{Op}(a)$ defined by $\langle \mathrm{Op}(a)f, g\rangle = \langle f, {}^t\mathrm{Op}(a)g\rangle$ for $f, g \in \mathscr{S}(\boldsymbol{R}^n)$. (26) and (27) follow by duality from the results in Chapter 3 of Marschall [8]. Then (26) can be used to show that for $s > 0$ $F^s_{\infty q}$ is a multiplication algebra. More precisely,

$$\|g\cdot f\|_{F^s_{\infty q}} \leqslant C(\|g\|_{L^\infty} \|f\|_{F^s_{\infty q}} + \|g\|_{F^s_{\infty q}} \|f\|_{L^\infty}). \tag{28}$$

The proof is identical to the one given for $q = 2$ in Marschall [8], Chapter 11.

One topic which cannot be treated by duality is the trace operator. Decompose $\boldsymbol{R}^n = \boldsymbol{R}^{n-1} \times \boldsymbol{R}$ and $x = (x', x_n)$. On continuous functions the trace operator is defined by $\mathrm{Tr} f(x') := f(x', 0)$.

THEOREM 6. *For* $s > 0$ *and* $1 \leqslant q \leqslant \infty$

$$\mathrm{Tr}\colon F^s_{\infty q}(\boldsymbol{R}^n) \to B^s_{\infty\infty}(\boldsymbol{R}^{n-1})$$

is a retraction.

Proof. That $\mathrm{Tr}\colon F^s_{\infty q}(\boldsymbol{R}^n) \to B^s_{\infty\infty}(\boldsymbol{R}^{n-1})$ follows from $F^s_{\infty q} \subset B^s_{\infty\infty}$.

Now as a coretraction we use the operator constructed in Triebel [13], 2.7.2. Let $\{\varphi^l_k\} \in \phi(\boldsymbol{R}^l)$ be a partition of $\boldsymbol{R}^l$. We take $\psi, \psi_0 \in \mathscr{S}(\boldsymbol{R})$ to be such that

$$\operatorname{supp}\psi \subset (1, 2), \quad \operatorname{supp}\psi_0 \subset (-1, 1), \quad \check{\psi}(0) = \check{\psi}_0(0) = 1. \tag{29}$$

For $k = 1, 2, \ldots$ we set $\psi_k(\xi_n) := \psi(2^{-k}\xi_n)$. Finally, we define

$$Rf(x) := \sum_{k=0}^{\infty} 2^{-k} \check{\psi}_k(x_n) (\varphi^{n-1}_k)\check{\ } * f(x'). \tag{30}$$

Then, by (29), $\mathrm{Tr} \circ R = \mathrm{Id}$. Hence it remains to prove that

$$R\colon B^s_{\infty\infty}(\boldsymbol{R}^{n-1}) \to F^s_{\infty 1}(\boldsymbol{R}^n). \tag{31}$$

Let δ_1 be the Dirac measure on $\boldsymbol{R}$. The point is that by Corollary 4 $\delta_1 \in F^{-1}_{\infty 1}(\boldsymbol{R})$, i.e. there exists a representation

$$\delta_1 = \sum_{j=0}^{\infty} (\varphi^1_j)\check{\ } * g_j \quad \text{with} \quad \sum_{j=0}^{\infty} 2^{-j}|g_j| \in L^\infty(\boldsymbol{R}).$$

Now inserting this into (30) we obtain

$$Rf(x) = \sum_{l=-3}^{3} \sum_{k=0}^{\infty} (\varphi^{n-1}_k \otimes \psi_k\, \varphi^1_{k+l} * (f \otimes 2^{-k} g_{k+l})(x).$$

Let $\psi^{n-1}_k \in C^\infty_0(\boldsymbol{R}^{n-1})$ be equal to one in a neighbourhood of $\operatorname{supp}\varphi^{n-1}_k$ and chosen appropriately. Then we get

$$\|Rf\|_{F^s_{\infty 1}(\boldsymbol{R}^n)} \leqslant C \sum_{l=-3}^{3} \Big\| \sum_{k=0}^{\infty} 2^{ks} |((\psi^{n-1}_k)\check{\ } * f) \otimes 2^{-k} g_{k+l}| \Big\|_{L^\infty}$$
$$\leqslant C\|f\|_{B^s_{\infty\infty}}(\boldsymbol{R}^{n-1})$$

since $\sum_{k=0}^{\infty} 2^{-k}|g_k| \in L^\infty$. Hence we obtain (31). ■

For $q = 2$ the theorem is proved in Strichartz [11]. Let us also remark that the theorem is true for higher order traces as well (see Triebel [13], 2.7.2).

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Drop property equals reflexivity

by

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Abstract. We prove that in a reflexive Banach space $(X, \|\cdot\|)$ property (H) of Radon–Riesz (if $(x_n)_{n=1}^{\infty}$ is a sequence of elements in X converging weakly to an element x in X such that $\|x_n\| \to \|x\|$, then $(x_n)_{n=1}^{\infty}$ is norm-convergent to x) is equivalent to a geometric condition (the "drop property") introduced by Rolewicz: $\|\cdot\|$ has the drop property if for every closed set S disjoint with B_X (the closed unit ball of X) there exists an element $x \in S$ such that the "drop" defined by x (the convex hull of x and B_X) intersects S only at x. We also prove that a Banach space is reflexive if and only if it has an equivalent norm with drop property.

§ 1. Introduction. Let $(X, \|\cdot\|)$ be a Banach space and B_X its closed unit ball. By the *drop* $D(x, B_X)$ defined by an element $x \in X$, $x \notin B_X$, we shall mean the convex hull of the set $\{x\} \cup B_X$, $\operatorname{conv}(\{x\} \cup B_X)$. In [4], Daneš proved ("Drop Theorem") that, *for any Banach space* $(X, \|\cdot\|)$ *and every closed set* $S \subset X$ *at positive distance from* B_X, *there exists a point* $x \in S$ *such that* $D(x, B_X) \cap S = \{x\}$.

This result, as its author points out, allows to prove in a simple way certain theorems of Browder [2] and Zabreiko–Krasnosel'skii [17] which are very important in the theory of nonlinear operator equations. In [14], Rolewicz mentions a number of papers where the Daneš' result is used. Recently, Daneš has discussed the relationship between his Drop Theorem and several other results [5].

Motivated by Daneš' theorem, Rolewicz introduced in the aforesaid paper the notion of drop property for the norm in a Banach space: $\|\cdot\|$ in X has the *drop property* if for every closed set S disjoint with B_X there exists an element $x \in S$ such that $D(x, B_X) \cap S = \{x\}$. He proved that *if* X *is a uniformly convex Banach space then its norm has the drop property*, and also

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