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PATH SYSTEMS IN ACYCLIC DIRECTED GRAPHS

Abstract. In this paper we prove a general result for directed acyclic graphs with fixed peaks and valleys, relating the number of vertex disjoint path systems between peaks and valleys to the number of paths between individual peaks and valleys. Furthermore, we show that our basic theorem generalizes a result of John and Sachs (see [3]) concerning generalized hexagonal systems.

1. Introduction. Let $G = (V(G), E(G))$ be an acyclic directed graph, $A = \{a_1, \ldots, a_n\}$ be a certain fixed set of sources (vertices with indegree 0), and $B = \{b_1, \ldots, b_n\}$ be a fixed set of sinks (vertices with outdegree 0). The elements of A are called *peaks*, and the elements of B *valleys*. A *path system* in G is a set $W = \{w_1, \ldots, w_n\}$ of paths such that there exists a permutation $\sigma(W) \in S_n$ so that w_i leads from a_i to $b_{\sigma(i)}$. We say that W is *disjoint* if for every i and j $(1 \leqslant i < j \leqslant n)$ w_i and w_j have disjoint sets of vertices.

Let p_{ij} be the number of paths leading from a_i to b_j. There is a simple algorithm counting p_{ij} in $O(n|E(G)|)$ steps (see Appendix). As suggested by Sachs, we investigate the relation between the numbers p_{ij} and the number of disjoint path systems. We prove a general theorem concerning this relation, give a sufficient condition for its application, and prove that every generalized hexagonal system satisfies this condition.

2. Basic theorem. Let

$$p^+ = |\{W\colon W \text{ is a disjoint path system such that } \operatorname{sgn}(\sigma(W)) = +1\}|$$

and

$$p^- = |\{W\colon W \text{ is a disjoint path system such that } \operatorname{sgn}(\sigma(W)) = -1\}|.$$

THEOREM 1. $\det(p_{ij}) = p^+ - p^-$.

Proof. Let $E(w)$ be the set of arcs used by a path w and let

$$E(W) = \bigcup_{i=1}^{n} E(w_i)$$

for a path system W. Let

$$P(E, \sigma) = \{W\colon\ E(W) = E \text{ and } \sigma(W) = \sigma\}$$

and

$$p(E, \sigma) = |P(E, \sigma)|$$

for any $E \subseteq E(G)$ and $\sigma \in S_n$. We count

$$p = \sum_{\sigma \in S_n} \sum_{E \subseteq E(G)} \operatorname{sgn}(\sigma)\, p(E, \sigma)$$

using both orders of summation:

$$\begin{aligned} p &= \sum_{\sigma \in S_n} \sum_{E \subseteq E(G)} \operatorname{sgn}(\sigma)\, p(E, \sigma) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \sum_{E \subseteq E(G)} p(E, \sigma) \\ &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) |\{W\colon\ \sigma(W) = \sigma\}| = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} p_{i\sigma(i)} \\ &= \det(p_{ij}). \end{aligned}$$

It remains to show that $p = p^+ - p^-$. Clearly,

$$p = \sum_{E \subseteq E(G)} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, p(E, \sigma).$$

Let us fix $E \subseteq E(G)$. If $E = E(W)$, where W is disjoint, then W is the unique path system such that $E = E(W)$. Hence in this case

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, p(E, \sigma) = \operatorname{sgn}(\sigma(W)).$$

To complete the proof we have to show that for any other E

(1) $$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, p(E, \sigma) = 0.$$

There are two cases possible:

(a) $E \neq E(W)$ for every path system W in G,

(b) $E = E(W)$ for some W which is not disjoint.

Obviously, (a) implies (1), so assume E satisfies (b). In this case (1) is a consequence of the following

LEMMA 1. *Let $E \subseteq E(G)$ satisfy* (b) *and let*

$$P(E) = \bigcup_{\sigma \in S_n} P(E, \sigma) = \{W\colon\ E(W) = E\}.$$

Then there exists a bijection $f\colon P(E) \to P(E)$ such that

$$\operatorname{sgn}(\sigma(f(W))) = -\operatorname{sgn}(\sigma(W)) \quad \textit{for all } W \in P(E).$$

Proof of Lemma 1. A vertex $v \in V(G)$ is called a *fork vertex* with regard to E if it is the common head of (at least) two arcs $e_1, e_2 \in E$ (v is called the *head* of the arc $e=(u, v)$); and v is called a *good fork vertex* if there are two arcs $e_1, e_2 \in E$ with a common head v and two peaks a_i, a_j satisfying the following condition: for any $W=\{w_1, \ldots, w_n\}$ such that $E(W) = E$ the path w_i contains e_1 and w_j contains e_2.

Assume that v is a fixed good fork vertex of G with regard to E and e_1, e_2, a_i, a_j are as above. Then the following function is well defined:

$$f(W) = W' = \{w'_1, \ldots, w'_n\} \quad \text{for } W \in P(E),$$

where $w'_k = w_k$ for $k \notin \{i, j\}$ and w'_i, w'_j are obtained from w_i and w_j by exchanging the segments leading from v to $b_{\sigma(W)(i)}$ and from v to $b_{\sigma(W)(j)}$.

Since $f \circ f = \text{id}$, f is a bijection. Obviously,

$$\operatorname{sgn}\big(\sigma(f(w))\big) = -\operatorname{sgn}\big(\sigma(W)\big).$$

It remains to prove that there exists a good fork vertex with regard to E in G. Let $G' = \big(V(G'), E(G')\big)$ be the digraph such that $V(G')$ is the set of fork vertices in G and $(u, v) \in E(G')$ iff there is a path w leading from u to v in G such that $E(w) \subseteq E$. Evidently, $V(G')$ is nonempty and G' is acyclic. We show that every source v of G' is a good fork vertex in G. Since v is a fork vertex in G, there exist two arcs $e_1, e_2 \in E$ with a common head v. Let W be a path system such that $E(W) = E$. Let w_i contain e_1 and w_j contain e_2. Take the peaks a_i and a_j. Assume that there is another path system W' such that $E(W') = E$ and w'_i does not contain e_1. Then there is $w'_k \in W'$ which contains e_1. This situation is shown in Fig. 1.

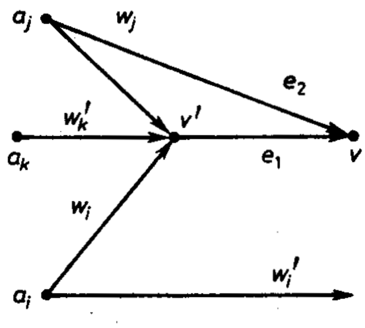

Fig. 1

Let v' be the first common vertex of w_i and w'_k. Since v' is a fork vertex in G, we have $v' \in V(G')$ and $(v', v) \in E(G')$, so v is not a source in G'. This contradiction proves that v is a good fork vertex, completing the proof of Lemma 1 and also of Theorem 1.

Remark 1. The assumption that G is a directed and acyclic graph is essential, as illustrated in Fig. 2.

3. Applications. We are interested in counting the number p of disjoint path systems in G. Clearly, $p = p^+ + p^-$. The basic theorem implies

COROLLARY 1. *Let G be such that*

$$\operatorname{sgn}(\sigma(W)) = \operatorname{sgn}(\sigma(W')) \tag{2}$$

for every two disjoint path systems W, W' in G. Then $p = |\det(p_{ij})|$.

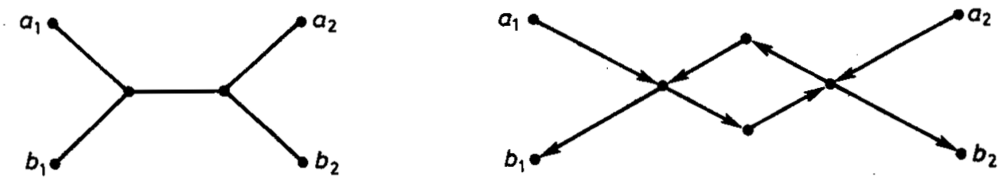

Fig. 2

EXAMPLE 1. There are graphs which satisfy not only (2) but also

$$\sigma(W) = \sigma(W') \quad \text{for every two disjoint path systems } W, W'. \tag{3}$$

For instance, this occurs when G is planar with the peaks and valleys lying on the exterior face of G so that we may walk around its boundary in such a way that we ascend exactly once from a valley to a peak. Then any path from a peak to a valley divides G into two components, and so we have (3).

Another class of graphs satisfying (3) is investigated in [1]; for the definition of a generalized hexagonal system (GHS) see Example 2 (below).

THEOREM 2 ([1]). *If G is a GHS and all peaks and valleys lie on the boundary of the exterior face, then G satisfies* (3).

There is an interesting connection to the DISJOINT CONNECTING PATHS problem for the class of graphs with property (3). The DISJOINT CONNECTING PATHS problem is defined as follows:

Instance: $G = (V(G), E(G))$, $(s_1, t_1), \ldots, (s_n, t_n)$ with $s_i, t_i \in V(G)$.

Question: Is there a disjoint path system $\{w_1, \ldots, w_n\}$ in G such that w_i leads from s_i to t_i for all i $(1 \leqslant i \leqslant n)$?

This problem is known to be NP-complete for undirected as well as for directed graphs (see [2]). However, consider G satisfying (3) and assume that the peaks and valleys are numbered in such a way that $\sigma(W) = \mathrm{id}$ for all disjoint path systems. Clearly, for these graphs the answer to the DISJOINT CONNECTING PATHS problem is "yes" if and only if $\det(p_{ij}) \neq 0$. Since we can verify the latter property in polynomial time, it is not hard to see that, for the class of graphs described at the beginning of Example 1, Theorem 1 yields a solution to the DISJOINT CONNECTING PATHS problem which can be found in polynomial time. It would be interesting to find other classes of graphs with property (3) because the NP-complete problem restricted to those classes has a polynomial time solution method, provided we can decide in polynomial time which permutation is the only possible one.

Notice that Theorem 2 does not characterize the class of all GHS satisfying (3), as illustrated in Fig. 3.

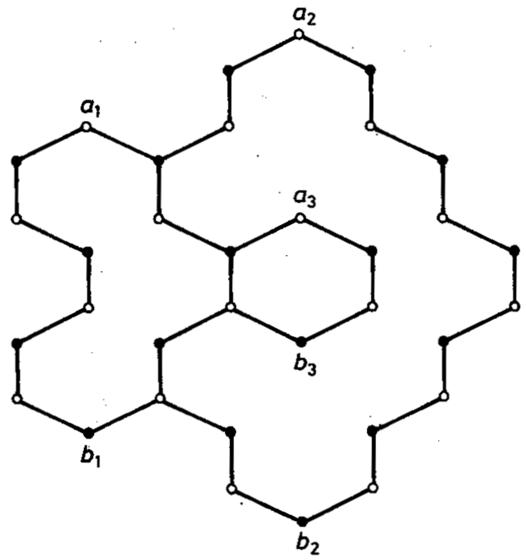

Fig. 3

Example 2. A *hexagonal system* (HS) is a 2-connected plane graph G in which every finite face is a regular hexagon of side length 1. We assume that G is drawn in the plane in such a way that one of its sides is vertical and colour the vertices of G black and white in such a way that the top vertex in every hexagon is white and the colours alternate on every path.

A *generalized hexagonal system* (GHS) is a connected subgraph G of a hexagonal system such that the length of the walk around the boundary of any finite face of G is equal to $4k+2$ for some k. For an example see Fig. 4.

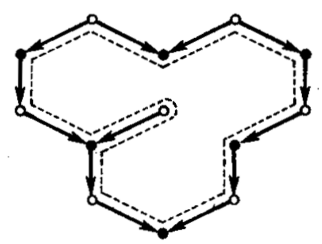

Fig. 4

We may consider G as an acyclic digraph whose every arc runs downwards. Our set of peaks will consist of all sources coloured white, and valleys are the black sinks.

Chemists are interested in counting the number of perfect matchings in GHS. It has been shown that this number is equal to the number of disjoint path systems (see [3]). Evidently, if the number of peaks is not equal to the number of valleys, then there is no disjoint path system, so there is no perfect matching as well. Thus we are only interested in the case where the number of peaks is equal to the number of valleys. In the sequel, G is called a GHSPn if it is a GHS with n peaks and n valleys, and a GHSP if it is a GHSPn for some n. The letter P stands here for parity. It has been shown that the number of perfect matchings in any GHSPn is equal to $|\det(p_{ij})|$ (see [3]). So it follows that the number of disjoint path systems in GHSPn is equal to $|\det(p_{ij})|$. The authors of [3] asked for a simple combinatorial proof of Theorem 2 which avoids the concept of a perfect matching, not appearing in its statement. We have found such a proof for HS, since by Theorem 2 condition (3) is satisfied. Now we prove that (2) holds for every GHSP, completing the solution of the problem stated above.

In the sequel, it will be more convenient not to restrict considerations to connected graphs. A graph G is called a DGHS if each component of G is a GHS, and a DGHSP if every component of G is a GHSP. If $G_1, \ldots, G_r$ are the components of G, and G_i is a GHSPn_i, then G is a DGHSP $\sum_{i=1}^{r} n_i$; see an example of a DGHSP4 in Fig. 5.

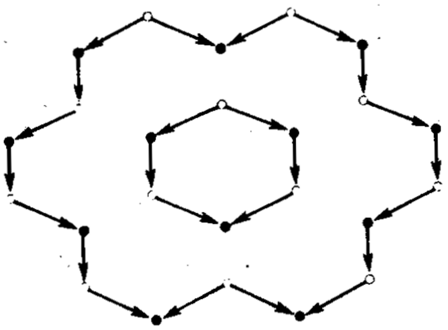

Fig. 5

Our aim is to prove the following

THEOREM 3. *If G is a GHSP, then* (2) *holds for G.*

Before proving Theorem 3, we show some lemmas. In the sequel a vertex which is either a peak or a valley is called an *extremal point*.

LEMMA 2. *Let G be a GHS. Then the length of the boundary of the exterior face of G is equal to $4k+2$ for some k iff G contains an even number of extremal points. Otherwise, the length is $4k$.*

Proof. First assume that G is a tree. During a walk around its boundary every edge is visited twice, so we have to show that

$$|E(G)| \equiv_{(\mathrm{mod}\,2)} 1$$

iff the number of extremal points of G is even. This is of course true if $|E(G)| = 1$. By adding a new edge to G, either we eliminate one extremal point or we create a new one. By induction we see that the number of extremal points of G is even iff the number of edges of G is odd.

Now assume that $G = T \cup C$, where T is a spanning tree of G and

$$E(T) \cap E(C) = \emptyset.$$

We prove the lemma by induction on $|E(C)|$. For $|E(C)| = 0$ we have just proved it.

Now observe that if $|E(C)| > 0$, then there is an edge $e \in C$ separating the exterior face from an interior one, say F. The length of its boundary is $4l+2$ for some l, since G is a GHS. By deleting e we get a new GHS G' such that the numbers of extremal points in G and G' have the same parity. Let m and m' be the lengths of the boundaries of the exterior faces of G and G', respectively. Then

$$m = (m'+1)-(4l+2-1) = m'-4l.$$

By induction hypothesis the lemma holds for G', so it holds also for G.

COROLLARY 2. *The length of the boundary of the exterior face of a GHSP is equal to* $4k+2$ *for some* k.

Now we sketch a proof of Theorem 3. If we fix a GHSP G and disjoint path systems W and W' in G we obtain three kinds of edges:

(1) edges not used by W and W',

(2) edges used by exactly one of W and W',

(3) edges used by both W and W'.

Let us call these edges *empty*, *single* and *double* in (G, W, W'), respectively. The triple (G, W, W') defines a permutation

$$\sigma(G, W, W') = \sigma(W')^{-1} \circ \sigma(W).$$

Our goal is to prove that

$$\operatorname{sgn}(\sigma(G, W, W')) = +1.$$

In order to achieve this we prove two lemmas: first that the subgraph H of G induced by the single edges is a DGHSP, and second that there are disjoint path systems W_1 and W_1' in H such that all edges of H are single in (H, W_1, W_1') and

$$\operatorname{sgn}(\sigma(G, W, W')) = \operatorname{sgn}(\sigma(H, W_1, W_1')).$$

Theorem 3 follows by observing that

$$\operatorname{sgn}(\sigma(H, W_1, W_1')) = +1$$

for any such (H, W_1, W_1').

It will be convenient to formulate our lemmas more generally. Let an SHS be any connected subgraph of an HS. The notions of SHSP, DSHS, DSHSP are defined analogously. We denote by T the set of all triples (H, W, W') such that H is an arbitrary DSHSP, W and W' are disjoint path systems in H, and there are no empty edges in (H, W, W').

LEMMA 3. *For every $(H, W, W') \in T$ there exists $(H_1, W_1, W_1') \in T$ such that H_1 is the subgraph of H induced by the single edges in (H, W, W') and*

$$\operatorname{sgn}(\sigma(H_1, W_1, W_1')) = \operatorname{sgn}(\sigma(H, W, W')).$$

Proof. Let d be a number of double edges in H. If $d = 0$, then there is nothing to prove, so assume that $d > 0$. Let π be a maximal path in H consisting of double edges. Obviously, the end points of π have degree 1 or 3 in H and all other vertices of π have degree 2. Let H_2 be the subgraph of H obtained by deleting all edges of π and all its vertices of degree 1 or 2.

CLAIM. *There exist W_2 and W_2' such that $(H_2, W_2, W_2') \in T$ and*

$$\sigma(H, W, W') = \sigma(H_2, W_2, W_2').$$

Since H_2 contains less double edges than H does, the lemma follows from the Claim by induction. It remains to prove the Claim. We distinguish three cases.

Case 1. Both ends of π have degree 1.

Then π is one component of H, one of its end points is a peak, the other is a valley. Thus π is one of the paths in both path systems. We get W_2 and W_2' by deleting this path. Obviously, W_2 and W_2' are as required.

Case 2. One end point, say a_i, of π has degree 1, the other, v, has degree 3.

Let the vertex of degree 1 be a peak (the case of a valley can be proved similarly).

Clearly, v is a peak in H_2. Exchanging the peak a_i by v in both W and W' and deleting the edges contained in π we obtain W_2 and W_2' as required.

Case 3. Both end points of π have degree 3.

Deleting π creates exactly one new peak a_{n+1} and one new valley b_{n+1}. Let

$$W = \{w_1, \ldots, w_n\} \quad \text{and} \quad W' = \{w_1', \ldots, w_n'\}.$$

Assume π is a part of the path w_i and w_j'. Deleting π breaks w_i into parts $\bar{w}_i$ leading from a_i to b_{n+1}, and $\bar{w}_{n+1}$ leading from a_{n+1} to the end of w_i; and w_j' into parts $\bar{w}_j'$ leading from a_j to b_{n+1}, and $\bar{w}_{n+1}'$ leading from a_{n+1} to the end of w_j'. We put

$$W_2 = \{w_1, \ldots, \bar{w}_i, \ldots, \bar{w}_{n+1}\}$$

and

$$W_2' = \{w_1', \ldots, \bar{w}_j', \ldots, \bar{w}_{n+1}'\}.$$

Evidently, $(H_2, W_2, W_2') \in T$.

Now we show that

$$\operatorname{sgn}(\sigma(H_2, W_2, W_2')) = \operatorname{sgn}(\sigma(H, W, W')).$$

To this end we decompose the permutation

$$\sigma(H, W, W') = c_1 \circ \ldots \circ c_p$$

into cycles and consider two cases:

(a) i and j are in the same cycle, say c_p.
Let

$$c_p = (i, m_1, m_2, \ldots, m_s, j, n_1, \ldots, n_r).$$

Then

$$\sigma(H_2, W_2, W_2') = c_1 \circ \ldots \circ c_p' \circ c_{p+1}',$$

where

$$c_p' = (i, j, n_1, \ldots, n_r) \quad \text{and} \quad c_{p+1}' = (n+1, m_1, \ldots, m_s).$$

If c_p is an even cycle, then exactly one of c_p' and c_{p+1}' is even. If c_p is an odd cycle, then either both c_p' and c_{p+1}' are even or both are odd. So we are done.

(b) i and j are in different cycles, say c_{p-1} and c_p.
Let

$$c_{p-1} = (i, m_1, \ldots, m_s) \quad \text{and} \quad c_p = (j, n_1, \ldots, n_r).$$

Then

$$\sigma(H_2, W_2, W_2') = c_1 \circ \ldots \circ c_{p-2} \circ c_{p-1}',$$

where

$$c_{p-1}' = (i, j, n_1, \ldots, n_r, n+1, m_1, \ldots, m_s).$$

If c_p and c_{p-1} have the same parity, then c_{p-1}' is an odd cycle. If one of them is odd and the other is even, then c_{p-1}' is even. This completes the proof of the Claim, and hence of the lemma.

LEMMA 4. *Let G be an SHS, each vertex of which has degree 2. Then G is an SHSP. Moreover, the number of peaks in G is odd iff $|E(G)| = 4k+2$ for some k.*

Proof. Observe that walking around G we always ascend from a valley to a peak and then descend to the next valley. Thus the numbers of peaks and valleys in G are equal. For the proof of the second part of the lemma we observe two facts: Every peak is separated from the neighbouring valley by an odd number of edges. Moreover, walking around G we pass as many edges ascending from a valley to the neighbouring peak as edges descending from a peak to the neighbouring valley.

LEMMA 5. *Let G be a GHSP, W and W' be two disjoint path systems in G and H be the subgraph of G induced by the single edges in (G, W, W'). Then H is a DGHSP.*

Proof. Each component of H is a cycle, thus H is a DSHSP by Lemma 4. We show that each cycle of H has length $4k+2$ for some k. Let H_1 be one of the cycles and let G_2 be the subgraph of G induced by all vertices belonging to H_1 or lying inside H_1. We denote by H_2 and F_2 the subgraphs of G_2 induced, respectively, by the single and nonempty edges.

CLAI .. *G_2 is a GHSP.*

Obviously, H_2 is a union of pairwise disjoint cycles, and hence it is a DSHSP. Moreover, H_2 is obtained from F_2 by deleting successively maximal paths of double edges. This operation preserves the difference between the number of peaks and the number of valleys; hence F_2 is also a DSHSP.

We show that G_2 and F_2 have the same set of peaks. Let v be a peak of G_2. If $v \in H_1$, then $v \in F_2$ since $H_1 \subseteq F_2$. If $v \notin H_1$, then v is a peak in G, hence incident to some edges in W and W', so $v \in F_2$. Since peaks remain peaks while passing to subgraphs, we have shown that v is a peak in F_2. Now assume that v^* is a peak in F_2. If it were not a peak in G_2, it would be incident with some vertical edge $e \in E(G_2)$ as shown in Fig. 6. But e_1 or e_2 must belong to W or W', and since v^* is not a peak in G, e cannot be an empty edge, contradicting the assumption that v^* is a peak in F_2.

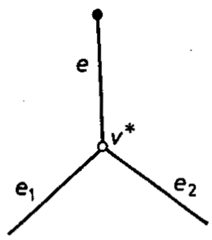

Fig. 6

The same holds for valleys, so G_2 is a DSHSP.

Every interior face of G_2 is an interior face in G, and G_2 is obviously connected, so G_2 is a GHS. Consequently, G_2 is a GHSP and the Claim is proved.

By Lemma 2 the length of the boundary of the exterior face of G_2 is equal to $4k+2$ for some k. This is exactly the length of the cycle H_1, so we are done.

Proof of Theorem 3. Let W and W' be disjoint parth systems in G and let H be the subgraph of G induced by the single edges in (G, W, W'). By Lemma 3,

$$\operatorname{sgn}(\sigma(G, W, W')) = \operatorname{sgn}(\sigma(H, W_1, W_1'))$$

for certain W_1 and W_1' such that $(H, W_1, W_1') \in T$. By Lemma 5, H is a DGHSP and, by Lemma 4, every component of H contains an odd number of peaks. So $\sigma(H, W_1, W_1')$ is a composition of odd cycles. For the example in Fig. 7 we have

$$\sigma(H, W, W') = (1, 4, 5, 3, 2)\circ(6, 8, 7).$$

In general, $\sigma(H, W_1, W_1')$ is even, so $\sigma(G, W, W')$ is even, i.e.,

$$\operatorname{sgn}(\sigma(W)) = \operatorname{sgn}(\sigma(W')).$$

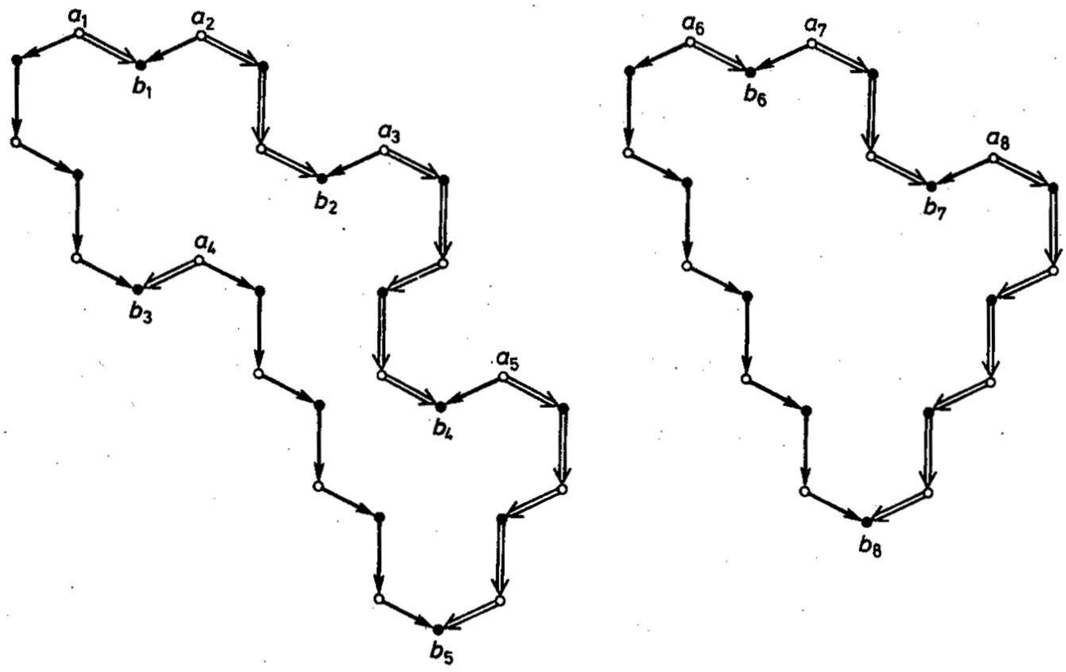

Fig. 7. → indicates W, ⇒ indicates W'.

Hence Theorem 3 holds.

Remark 2. The assumption that every component of the boundary of an interior face has length $4k+2$ is essential; see Fig. 8 where $\sigma(G, W, W') = (1, 2)$.

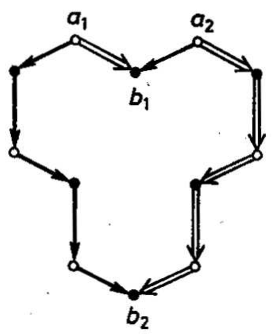

Fig. 8

Remark 3. Example 2 can be generalized as follows: Let G be a subgraph of an HS, $A = \{a_1, \ldots, a_n\}$ a set of vertices of G coloured white, and $B = \{b_1, \ldots, b_n\}$ a set of vertices of G coloured black. As before we can investigate disjoint path systems $W = \{w_1, \ldots, w_n\}$ with the corresponding permutation $\sigma(W)$ such that w_i leads from a_i to $b_{\sigma(i)}$.

Notice that none of the edges (v, a_i) and (b_j, v') (where $a_i \in A$, $b_j \in B$, $v, v' \in V$) is contained in a disjoint path system W. Moreover, a source s of G is contained in a disjoint path system if and only if $s \in A$. Similarly, a sink t of G which is not in B does not appear in any disjoint path system. This means that in order to investigate the disjoint path systems of G we may consider the subgraph G' of G obtained by repeating the following procedure as long as possible:

(1) Delete all edges (v, a_i) for $a_i \in A$.
(2) Delete all edges (b_j, v) for $b_j \in B$.
(3) Delete all sources s with $s \notin A$.
(4) Delete all sinks t with $t \notin B$.
(5) Delete all isolated vertices v such that $v \notin A \cup B$.

The set of sources of the resulting graph G' is A, and its set of sinks is B. Since every disjoint path system in G is a disjoint path system in G' and conversely, we obtain the following

Corollary 3. *If G' is a DGHSP, then the number of disjoint path systems in G between A and B is equal to* $|\det(p_{ij})|$.

Appendix. Here we describe an algorithm for counting (p_{ij}) for acyclic graphs in $O(\overline{n}|E(G)|)$ steps, thus generalizing the algorithm for GHS given in [3].

Let the graph G be given by the set of adjacency lists, i.e., for every vertex v we have a list of its successors:

$$S(v) = \{v' \colon (v, v') \in E(G)\}.$$

First we determine the indegree $In(v)$ for all vertices v. This can be done going through the adjacency lists in $O(|V(G)| + |E(G)|)$ steps. Then we define as in [3] a path vector

$$\bar{p}(v) = (p_1(v), \ldots, p_n(v)) \quad \text{for every } v \in V(G),$$

where $p_i(v)$ is the number of paths leading from a_i to v. We are interested in counting $p_{ij} = p_i(b_j)$. The algorithm proceeds as follows:

1. **for** each $a_i \in A$ and each j $(1 \leqslant j \leqslant n)$ **do** $p_j(a_i) := $ **if** $j = 1$ **then** 1 **else** 0 **endfor**
2. **for each** $v \in V(G) \setminus A$ **do** $\bar{p}(v) := 0$ **endfor**
3. put all vertices with $In(v) = 0$ into list I
4. **while** $I \neq \emptyset$ **do** remove a vertex v from I
5.   **for** $w \in S(v)$ **do** $\bar{p}(w) := \bar{p}(w) + \bar{p}(v)$
        $In(w) := In(w) - 1$
        if $In(w) = 0$ **then**
          put w into I **endif**
    endfor

endwhile

Steps 1 and 2 need $O(n|V(G)|)$ operations, step 3 needs $O(|V(G)|)$ ones. The while-loop is executed for each vertex of G exactly once, whereby steps 5–7 are executed once for each edge of G. Step 5 needs $O(n)$ operations, the other steps need $O(1)$. Together we get time complexity $O(n(|E(G)|+|V(G)|))$, which is $O(n|E(G)|)$ for graphs without isolated vertices. For graphs with bounded degree ($In(v)$, $Out(v) \leqslant d$ for every $v \in V(G)$), the execution time of the algorithm is $O(n|V(G)|)$. This applies to GHS, where $d = 2$.

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