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ON k-REGULAR GRAPHS CONTAINING $(k-1)$-REGULAR SUBGRAPHS

Abstract. A well-known open problem in graph theory is: which k-regular simple graphs contain a $(k-1)$-regular subgraph (John R. Reay, 1979). This problem is solved for k-regular simple graphs on $k+3$ vertices. Also the number of labeled k-regular simple graphs on a given number of vertices is calculated. Using this, a lower bound for the number of k-regular simple graphs with $(k-1)$-regular subgraph is established.

1. Introduction. In [1], p. 246, the following conjecture is stated: Every 4-regular simple graph contains a 3-regular subgraph. It is called the *Berge–Sauer Conjecture*; see also [5]–[7] and [10]. Recently this problem has been solved by Limin [4]. The general statement, each k-regular simple graph contains a $(k-1)$-regular subgraph, is not true in general. For instance, the complete 3-partite graph on 9 vertices $T_{3,9}$ is 6-regular but contains no 5-regular subgraph (see below). This example disproves Limin's conjecture (see [4], p. 135). So the question is: which k-regular simple graphs have a $(k-1)$-regular subgraph? In Section 2 it is shown that all k-regular simple graphs on $k+3$ vertices contain a $(k-1)$-regular subgraph on $k+2$ vertices, except precisely the k-regular complete multi-partite graphs on $k+3$ vertices. In Section 3 the number of k-regular graphs on p vertices is calculated; it serves as a tool in partly solving the main question of this paper.

2. k-regularity on $k+3$ vertices. For 2-regular simple graphs the existence of 1-regular subgraphs is obvious. It is well known that the graph itself or its complement or both are Hamiltonian provided the graph is regular. From this it follows that each k-regular simple graph on p vertices with p even and $\frac{1}{2}p \leqslant k \leqslant p-1$ contains a $(k-1)$-regular subgraph on p vertices. Clearly, each k-regular simple graph on $p = k+1$ vertices with p odd and $k \geqslant 2$ is a complete graph K_{k+1} and, therefore, contains a $(k-1)$-regular subgraph on $p-1$ vertices, namely K_k.

Let k and m be integers $\geqslant 2$. The complete m-partite graph on km vertices $T_{m,km}$ is an $(m-1)k$-regular simple graph without an $((m-1)k-1)$-regular subgraph on $km-1$ vertices.

THEOREM 1. *Let k be even $\geqslant 4$. Each k-regular simple graph G on $p = k+3$ vertices contains a $(k-1)$-regular subgraph on $p-1$ $(=k+2)$ vertices iff $G \neq T_{m,3m}$ with $k = 3m-3$ and $m = 3, 5, 7, \ldots$*

Proof. Take k even $\geqslant 4$ and let G be a k-regular simple graph on $k+3$ vertices. Then the complement $\bar{G}$ of G is a 2-regular simple graph on $k+3$ vertices. If G is not a complete multi-partite graph with precisely three vertices in all the components of $\bar{G}$, it follows that $\bar{G}$ has at least one component with 4 or more vertices. Let v_0 be a vertex in a component of $\bar{G}$ with at least 4 vertices and let v_1 and v_2 be adjacent to v_0 in $\bar{G}$. Deleting the vertex v_0 and its k edges in G gives a graph with all vertices having degree $k-1$, except v_1 and v_2. As v_1 and v_2 are not adjacent in $\bar{G}$, they are adjacent in G. So leaving out the edge $v_1 v_2$ in G yields a $(k-1)$-regular subgraph of G on $k+2$ vertices.

Note that there are always at least 4 vertices that can be taken out obtaining a $(k-1)$-regular subgraph. The necessity follows from a previous remark.

3. The number of k-regular simple graphs. By $N(p; k)$ we denote the number of k-regular simple graphs on p vertices, $k \geqslant 1$, $p \geqslant 2$. For the sake of clarity we remark that the number $N(p; k)$ does not count only the non-isomorphic graphs but all k-regular simple "configurations" on p fixed vertices. The number of all simple graphs with degree sequence $(d_1, \ldots, d_p)$ is denoted by $N(d_1, \ldots, d_p)$. Hence

$$N(p; k) = N(k, k, \ldots, k).$$

$N^*(d_1, \ldots, d_p)$ denotes the number of "locally restricted" simple graphs with degree sequence $d_1, \ldots, d_p$, which means that each labeled vertex has fixed degree (see [3]). Clearly, not all constant degree sequences are graphic, e.g., $N(p; k) = 0$ for $k \geqslant p \geqslant 2$. Define

$$N(0, \ldots, 0) = 1.$$

Erdös and Gallai (see, e.g., [1]) have given necessary and sufficient conditions for a degree sequence to be graphic. Using this we find that, for $p \geqslant k+1 \geqslant 3$, $N(p; k) = 0$ iff both p and k are odd integers. By

$$N(s_1; d_1, \ldots, s_q; d_q)$$

we mean

$$N(\underbrace{d_1, \ldots, d_1}_{s_1 \text{ times}}, \ldots, \underbrace{d_q, \ldots, d_q}_{s_q \text{ times}}),$$

where s_i is called the *order* of d_i, $1 \leqslant i \leqslant q$.

THEOREM 2. *For any positive degree sequence $(d_1, \ldots, d_q)$ with d_i of order s_i $(i = 1, \ldots, q$ and $s_1 + \ldots + s_q = p)$ the following hold:*

1. *The equality*

$$N(s_1; d_1, \ldots, s_q; d_q) = \frac{p!}{s_1! \ldots s_q!} N^*(s_1; d_1, \ldots, s_q; d_q)$$

is satisfied.

2. *For* $0 \leqslant \alpha_i \leqslant s_i$, $\alpha_1 + \ldots + \alpha_q = d_1$ *and* $\alpha_1 < s_1$:

$$\begin{aligned} &N^*(s_1; d_1, \ldots, s_q; d_q) \\ &\quad = \sum_{\alpha_1, \ldots, \alpha_q} \binom{s_1 - 1}{\alpha_1} \binom{s_2}{\alpha_2} \ldots \binom{s_q}{\alpha_q} N^*(1; 0, s_1 - \alpha_1 - 1; d_1, s_2 - \alpha_2; \\ &\quad d_2, \ldots, s_q - \alpha_q; d_q, \alpha_1; d_1 - 1, \ldots, \alpha_q; d_q - 1). \end{aligned}$$

Proof. 1. There are s_1 vertices with degree d_1 that have to be divided over p vertices. This gives $\binom{p}{s_1}$ possibilities. There are s_2 vertices with degree d_2 for the remaining $p - s_1$ vertices, so we have $\binom{p-s_1}{s_2}$ possibilities, etc. Clearly,

$$\binom{p}{s_1} \ldots \binom{p - s_1 - \ldots - s_{q-1}}{s_q} = \frac{p!}{s_1! \ldots s_q!}.$$

2. In case of N^*, the degree of each vertex is fixed. Consider

$$N^*(s_1; d_1, \ldots, s_q; d_q)$$

and some vertex v^0 with degree d_1. This vertex v^0 is connected with d_1 other vertices. Say v^0 is connected with α_1 vertices of degree d_1 $(0 \leqslant \alpha_1 < s_1)$, with α_2 vertices of degree d_2 $(0 \leqslant \alpha_2 \leqslant s_2), \ldots$, and with α_q vertices of degree d_q $(0 \leqslant \alpha_q \leqslant s_q)$, $\alpha_1 + \ldots + \alpha_q = d_1$. Taking away the d_1 edges of v^0 yields a graph with the degree sequence

$$(1; 0, s_1 - \alpha_1 - 1; d_1, s_2 - \alpha_2; d_2, \ldots, s_q - \alpha_q; d_q, \alpha_1; d_1 - 1, \ldots, \alpha_q; d_q - 1).$$

There are $\binom{p-1}{d_1}$ possibilities the vertex v^0 is connected with the other vertices. One can easily check that

$$\binom{p-1}{d_1} = \sum_{\alpha_1, \ldots, \alpha_q} \binom{s_1 - 1}{\alpha_1} \binom{s_2}{\alpha_2} \ldots \binom{s_q}{\alpha_q}$$

for each $\alpha_1, \ldots, \alpha_q \geqslant 0$ such that $\alpha_1 < s_1$, $\alpha_2 \leqslant s_2, \ldots, \alpha_q \leqslant s_q$ and $\alpha_1 + \ldots + \alpha_q = d_1$. This gives the desired result.

For k-regular simple graphs on p vertices we have $N(p; k) = N^*(p; k)$, because all vertices have the same degree.

The following theorem is a direct consequence of Theorem 2:

THEOREM 3. *We have*

$$N(p;k) = \binom{p-1}{k} N(1;0,k;k-1,p-k-1;k).$$

In the following example, $N(7;4)$ is calculated using the technique of Theorems 2 and 3:

$$\begin{aligned} N(7;4) &= N(4,4,4,4,4,4,4) = \binom{6}{4} N(4,4,3,3,3,3,0) \\ &= \binom{6}{4}\Big[N(4,2,2,2,2,0,0) + \binom{4}{3} N(3,3,2,2,2,0,0)\Big] \\ &= \binom{6}{4}\Big[N(1,1,1,1,0,0,0) + \binom{4}{3}\Big[N(3,1,1,1,0,0,0) \\ &\quad + \binom{3}{2} N(2,2,1,1,0,0,0)\Big]\Big] \\ &= \binom{6}{4}\Big[\binom{3}{1} N(1,1,0,0,0,0,0) + \binom{4}{3}\Big[N(0,0,0,0,0,0,0) \\ &\quad + \binom{3}{2}\Big[N(2,0,0,0,0,0,0) + \binom{2}{1} N(1,1,0,0,0,0,0)\Big]\Big]\Big] \\ &= 15\big[(3)(1) + 4\big[1 + 3\,[0 + (2)(1)]\big]\big] = 15\,[3 + (4)(7)] = (15)(31) \\ &= 465. \end{aligned}$$

In the Table the number $N(p;k)$ is given for $2 \leqslant p \leqslant 11$ and $1 \leqslant k \leqslant 9$. For more results see [9]. In [8] the number of labeled 4-regular graphs are calculated for $5 \leqslant p \leqslant 13$. In [9] an algorithm is developed that counts the number of labeled simple graphs.

THEOREM 4. $N(p;k) = N(p;p-k-1)$ *for* $p \geqslant k+2$.

Proof. This theorem expresses that the number of k-regular simple graphs on p vertices with $p \geqslant k+2$ equals the number of the complements of the k-regular simple graphs on p vertices, being $(p-k-1)$-regular simple graphs on p vertices. The proof is left to the reader.

THEOREM 5. $N(p;1)$ *is for even* $p \geqslant 2$ *the number of perfect matchings of the complete graph* K_p. *Moreover,*

$$N(p+2;1) = (p+1)N(p;1)$$

and

$$N(p;1) = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (p-1) = (p-1)!!.$$

TABLE

$k \backslash p$	2	3	4	5	6	7	8	9	10	11
1	1	0	3	0	15	0	105	0	945	0
2	0	1	3	12	70	456	3507	30016	286884	3026655
3	0	0	1	0	70	0	19355	0	11180820	0
4	0	0	0	1	15	456	19355	1024380	66462606	5188453830
5	0	0	0	0	1	0	3507	0	66462606	0
6	0	0	0	0	0	1	105	30016	11180820	5188453830
7	0	0	0	0	0	0	1	0	286884	0
8	0	0	0	0	0	0	0	1	945	3026655
9	0	0	0	0	0	0	0	0	1	0
									0	1

Proof. The first part is obvious. The second part follows from Theorems 2 and 3:

$$N(p+2;\, 1) = N(\underbrace{1, \ldots, 1}_{p+2\text{ times}}) = \binom{p+1}{1} N(0, 0, \underbrace{1, \ldots, 1}_{p\text{ times}}) = (p+1)\, N(p;\, 1).$$

The last part is left to the reader.

Corollary 1. *For even $p \geqslant 2$ the following holds:*

$$N(p+2;\, p) = (p+1)\, N(p;\, p-2).$$

This corollary is a direct consequence of Theorems 4 and 5.

Theorem 6. *We have*

$$pN(p;\, 2) + \binom{p}{2} N(p-2;\, 2) = N(p+1;\, 2) \quad \textit{for } p \geqslant 4.$$

Proof. We obtain

$$N(p+1;\, 2) = N(\underbrace{2, \ldots, 2}_{p+1}) = \binom{p}{2} N(\underbrace{2, \ldots, 2}_{p-2}, 1, 1, 0)$$

$$= \binom{p}{2} [(p-2)\, N(\underbrace{2, \ldots, 2}_{p-3}, 1, 1, 0, 0) + N(\underbrace{2, \ldots, 2}_{p-2}, 0, 0, 0)].$$

On the other hand,

$$N(p;\, 2) = N(\underbrace{2, \ldots, 2}_{p}) = \binom{p-1}{2} N(\underbrace{2, \ldots, 2}_{p-3}, 1, 1, 0).$$

So we find

$$N(p+1;\, 2) = \binom{p}{2} \left[\left((p-2) \Big/ \binom{p-1}{2} \right) N(p;\, 2) + N(p-2;\, 2) \right]$$

$$= pN(p;\, 2) + \binom{p}{2} N(p-2;\, 2).$$

The above theorem can also be shown in the following way. Take any set of vertices V with $|V| = p+1$ and let $v_0 \in V$. Consider $V \setminus \{v_0\}$. The number of 2-regular simple graphs on $V \setminus \{v_0\}$ equals $N(p; 2)$. A 2-regular simple graph on p vertices has p edges. Extending a 2-regular simple graph on p vertices to a 2-regular simple graph on $p+1$ vertices means taking away some edge $v_1 v_2$ and insert two new edges $v_1 v_0$ and $v_2 v_0$ (see Fig. 1). For each graph on p vertices there are p possible extensions $V \setminus \{v_0\}$ to V, making $pN(p; 2)$ 2-regular simple graphs on all of V. However, this process yields that v_0 is always in a cycle with more than 3 vertices. The number of 3-cycles that contain v_0 is $\binom{p}{2}$. As the number of 2-regular simple graphs on $p-2$ vertices is $N(p-2; 2)$, there are $\binom{p}{2} N(p-2; 2)$ graphs with v_0 in a 3-cycle. So the total number of extensions equals

$$pN(p; 2) + \binom{p}{2} N(p-2; 2),$$

and this has to be $N(p+1; 2)$.

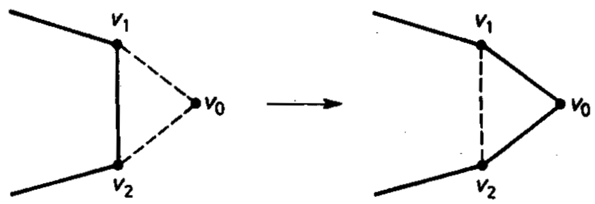

Fig. 1

Corollary 2. *We have*

$$pN(p; p-3) + \binom{p}{2} N(p-2; p-5) = N(p+1; p-2) \quad \textit{for } p \geq 6.$$

This corollary is a direct consequence of Theorems 4 and 6.

Problem. Find a recurrence relation for $N(p; k)$.

As we have seen above, not all k-regular simple graphs contain a $(k-1)$-regular subgraph, e.g., the complete multi-partite graphs do not. How many are there?

Conjecture. *The complete multi-partite graphs are the only k-regular simple graphs without $(k-1)$-regular subgraphs.*

Theorem 7. *Let m and k be integers ≥ 2. The number of $(m-1)k$-regular simple graphs on mk vertices without an $((m-1)k-1)$-regular subgraph on $mk-1$ vertices is at least*

$$\frac{(mk)!}{(k!)^m m!}.$$

Proof. In Section 2 we have seen that the complete $T_{m,km}$ graph is a graph satisfying the theorem assumptions. Its complement is a graph consisting of m complete graphs K_k. One can easily verify that the required number equals

$$\frac{1}{m!}\binom{mk}{k}\binom{(m-1)k}{k}\cdots\binom{k}{k},$$

and this leads, by a straightforward calculation, to the above formula.

Thus not all $N(9;6) = 30016$ 6-regular simple graphs on 9 vertices do have a 5-regular subgraph. Theorem 7 states that there are at least 280 without a 5-regular subgraph. By Theorem 1, the remaining 29736 graphs contain a 5-regular subgraph on 8 vertices. We investigate now whether it is possible to generate these 29736 graphs by means of extending the 3507 5-regular simple graphs on 8 vertices by adding a new vertex v_0 with its six edges together with an edge connecting (if possible) the remaining two vertices. As $N(8;5) = 3507$ and as there are 8 pairs of vertices not adjacent, there are at least $8N(8;5) = 28056$ extensions to a 6-regular simple graph on 9 vertices by adding a new vertex v_0. So there are 1680 graphs left. Looking at the complements of the 28056 graphs, v_0 is never in a 3-cycle. There are $\binom{8}{2} = 28$ 3-cycles with v_0 on 9 vertices. For each such 3-cycle there are 6 vertices left. As $N(6;2) = 70$ and as there are

$$\frac{1}{2}\binom{6}{3} = 10$$

pairs of 3-cycles on the remaining 6 vertices, there exist $70-10 = 60$ graphs which have as the complement a 3-cycle with v_0 and a connected cycle on the remaining 6 vertices. Clearly, these 60 graphs can also be obtained from a 5-regular simple graph on 8 vertices. So we obtain

$$8N(8;5)+\binom{8}{2}\left[N(6;2)-\frac{1}{2}\binom{6}{3}\right] = 29736,$$

which shows that in fact all 6-regular simple graphs on 9 vertices, except the $T_{3,9}$-graphs, can be obtained from the 5-regular simple graphs on 8 vertices by an extension argument. Using this technique it is possible to give a lower bound for the number of k-regular simple graphs on p vertices that contain a $(k-1)$-regular subgraph on $p-1$ vertices.

4. Lower bound for the number of k-regular simple graphs on p vertices with $(k-1)$-regular subgraph on $p-1$ vertices. Consider a line segment with begin point 1 and end point q. These q points divide the line segment into $q-1$ segments, denoted by $\overline{12}, \overline{23}, \ldots, \overline{q-1q}$. The segments will be marked 0 or 1 in such a way that no two adjacent segments are marked 1; the number of segments marked 1 is n with $1 \leqslant n \leqslant [q/2]$, $q \geqslant 2$. Let $D(q, n)$ be the number of possibilities the $q-1$ segments can be marked in the above described way. We shall shortly call such a marking an *$(n \times 1)$-marking*. It follows directly that $D(q, 1) = q-1$ and $D(2n, n) = 1$. We define $D(q, n) = 0$ for $n > [q/2]$.

LEMMA 1. *For $q \geqslant 4$ and $2 \leqslant n \leqslant [q/2]$ the following holds:*

$$D(q, n) = D(q-1, n) + D(q-2, n-1).$$

Moreover,

$$D(q, n) = \binom{q-n}{n} \quad \text{for } q \geqslant 2 \text{ and } 1 \leqslant n \leqslant [q/2].$$

Proof. If $\overline{q-1q}$ is marked 1, the path $1, \ldots, q-2$ has $D(q-2, n-1)$ $((n-1) \times 1)$-markings. If $\overline{q-1q}$ is marked 0, then the path $1, \ldots, q-1$ has $D(q-1, n)$ $(n \times 1)$-markings. The total number $D(q, n)$ of $(n \times 1)$-markings on $q-1$ segments is therefore equal to $D(q-2, n-1) + D(q-1, n)$. The remaining part is well known.

THEOREM 8. *For $k \leqslant p - [p/2] - 1$ and $p+k$ odd the following holds:*

$$N(p; k) \geqslant \frac{p-1}{\frac{1}{2}(p-k-1)\, N(p-k-1; 2)} \binom{\frac{1}{2}(p+k-3)}{\frac{1}{2}(p-k-3)} N(p-1; k-1).$$

Proof. Take p vertices $v_1, v_2, \ldots, v_p$. The number of $(k-1)$-regular simple graphs on $v_1, \ldots, v_{p-1}$ is $N(p-1; k-1)$. Let G be such a graph. As $p-k-1 \geqslant [p/2]$, it follows from [1], p. 54 and Ex. 4.2.10, that G^c is Hamiltonian. Let H be a Hamilton cycle in G^c. Extending a $(k-1)$-regular simple graph G on $v_1, \ldots, v_{p-1}$ to a k-regular simple graph on $v_1, \ldots, v_p$ means: connecting v_p with k vertices of G and adding a perfect matching to the remaining $p-k-1$ vertices of G. Although $p-k-1$ is even, it is not always possible that such a perfect matching exists.

We calculate the number of possibilities $p-k-1$ vertices can be connected in such a way that the new $\frac{1}{2}(p-k-1)$ edges belong to H and such that no two new edges are adjacent. Put

$$H = \overline{v_1, \ldots, v_{p-1}, v_1}.$$

Fix one edge, say $\overline{v_{p-2} v_{p-1}} \in H$. Let $n = \frac{1}{2}(p-k-1)$. The problem now reduces to: calculating the number of possibilities one can add $n-1$ new edges to $v_1, \ldots, v_{p-3}$ in such a way that the new edges belong to H and no two of them are adjacent. This means: designing an $((n-1) \times 1)$-marking to $v_1, \ldots, v_{p-3}$ in H. According to Lemma 1 it follows that

$$D(p-3, n-1) = \binom{p-n-2}{n-1}.$$

So the number of $(n\times 1)$-markings of the cycle H equals

$$\frac{p-1}{n}D(p-3,\, n-1)=\frac{p-1}{n}\binom{p-n-2}{n-1}.$$

For each $(k-1)$-regular simple graph G on $v_1, \ldots, v_{p-1}$ there is a Hamilton cycle H in G^c. So for each G there are

$$\frac{p-1}{n}\binom{p-n-2}{n-1}$$

possibilities to connect v_p with k vertices of G such that the remaining $p-k-1$ vertices have a perfect matching in H. For each $(k-1)$-regular simple graph on $p-1$ vertices we observe the following. There are at most $N(2n;\, 2)$ possibilities to add a matching to $2n$ fixed vertices of the graph. So at most $N(2n;\, 2)$ $(k-1)$-regular simple graphs on $p-1$ vertices deliver the same k-regular simple graph on p vertices by extending the original graph. As there are $N(p-1;\, k-1)$ $(k-1)$-regular simple graphs on $v_1, \ldots, v_{p-1}$, the number of "extensions" equals

$$\frac{p-1}{nN(2n;\, 2)}\binom{p-n-2}{n-1}N(p-1;\, k-1).$$

This means that, for the number of k-regular simple graphs on p vertices,

$$N(p;\, k)\geqslant \frac{p-1}{nN(2n;\, 2)}\binom{p-n-2}{n-1}N(p-1;\, k-1),$$

which yields the desired result.

The above theorem gives a lower bound for the number of k-regular simple graphs on p vertices (with $k\leqslant p-[p/2]-1$ and $p+k$ odd) that contain a $(k-1)$-regular graph on $p-1$ vertices. Question: Can there be found a better lower bound?

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