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The growth rate of the Dedekind Zeta-function on the critical line

by

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For Paul Erdős
on his 75th birthday

1. Introduction. Let K be an algebraic number field of degree n, and let $\zeta_K(s)$ be its Dedekind Zeta-function. Thus

$$\zeta_K(s) = \sum_A (NA)^{-s} \quad (\mathrm{Re}(s) > 1),$$

where A runs over the non-zero integral ideals of K, and NA is the absolute norm of A. The question considered in this paper is the order of magnitude of $\zeta_K(s)$ on the critical line. The trivial bound is

$$\zeta_K(\tfrac{1}{2}+it) \ll_K t^{n/4} \quad (t \geq 1),$$

where the notation $\ll_K$ indicates that the implied constant may depend on K. This follows from our Lemma 2, for example. When $K = Q$, the Dedekind Zeta-function reduces to the Riemann Zeta-function $\zeta(s)$, and one has the estimate $\zeta(\frac{1}{2}+it) \ll t^{1/6+\varepsilon}$ $(t \geq 1)$ for any fixed $\varepsilon > 0$. Indeed, the exponent can be slightly reduced. When the field K is Abelian, $\zeta_K(s)$ factorizes as a product of $\zeta(s)$ and $n-1$ Dirichlet L-functions $L(s, \chi)$. For these one can prove an estimate

$$L(\tfrac{1}{2}+it, \chi) \ll_\chi t^{1/6+\varepsilon} \quad (t \geq 1). \tag{1.1}$$

(Here also it is possible to improve the exponent 1/6 in the same way as for $\zeta(s)$.) It follows that

$$\zeta_K(\tfrac{1}{2}+it) \ll_K t^{n/6+\varepsilon} \quad (t \geq 1) \tag{1.2}$$

if K is Abelian. It would be of interest to make the dependence on K explicit. However, it is difficult to get a satisfactory uniform estimate even in the case of (1.1), and so we concentrate on the t-dependence in this paper. Our goal is to prove the bound (1.2) for all K, whether Abelian or not.

THEOREM. *Let K be an algebraic number field of degree n. Then*

$$\zeta_K(\tfrac{1}{2}+it) \ll_K t^{n/6+\varepsilon} \quad (t \geqslant 1)$$

for any fixed $\varepsilon > 0$.

We conjecture that much more is true. The Lindelöf hypothesis for $\zeta_K(s)$ states that

$$\zeta_K(\tfrac{1}{2}+it) \ll_K t^{\varepsilon} \quad (t \geqslant 1)$$

for any fixed $\varepsilon > 0$. Moreover, the Riemann hypothesis for $\zeta_K(s)$ implies the stronger estimate

$$\zeta_K(\tfrac{1}{2}+it) \ll_K \exp\left(c\frac{\log t}{\log\log t}\right) \quad (t \geqslant 3)$$

for an appropriate constant $c = c(K) > 0$. Indeed, it seems reasonable to conjecture that

$$\zeta_K(\tfrac{1}{2}+it) \ll_K \exp\left((\log t)^{1/2+\varepsilon}\right) \quad (t \geqslant 2)$$

for any fixed $\varepsilon > 0$. However, until now, the only non-trivial bound available was

$$\zeta_K(\tfrac{1}{2}+it) \ll_K t^{n/4-c/n^2\log(n+2)} \quad (t \geqslant 1),$$

due to Sokolovskii [1].

When K is non-Abelian one cannot factorise $\zeta_K(s)$ completely, and so our proof of the theorem uses an n-dimensional version of van der Corput's method for estimating exponential sums. To date all treatments of the n-dimensional form of the method have been rather cumbersome. We introduce a simplification by using weighted sums and integrals. This avoids in particular the complicated conditions previously imposed on the regions of summation and integration. We hope that our approach will be of wider use in the theory of n-dimensional exponential sums.

In future all constants implied by the notations $\ll, \gg$ and $O(\ldots)$ may depend on the field K. We shall also use vector notation. Vectors, and vector valued functions, will be written in bold type thus: $\boldsymbol{x}$, $\boldsymbol{\vartheta}(\alpha)$, $\boldsymbol{u}(\boldsymbol{x})$. They will generally be n-dimensional, except in Section 3, where some r-dimensional vectors are needed. We will denote the scalar product by $\boldsymbol{x}\cdot\boldsymbol{y}$ in the usual way, and the length of $\boldsymbol{x}$ by $|\boldsymbol{x}|$.

2. Preliminary transformations. Our first concern is to reduce the problem to one involving finite sums. This we do by the method of approximate functional equations.

LEMMA 1. *Let $t \geqslant 2$. Then*

$$|\zeta_K(\tfrac{1}{2}+it)| \leqslant 2\Big|\sum_A (NA)^{-1/2-it} W_0(NA)\Big| + O(1),$$

where $W_0(x) = W_0(x, t)$ has derivatives of all orders, and satisfies

$$\frac{d^k W_0(x)}{dx^k} \ll_k x^{-k}\min(1, t^{n/2}x^{-1}).$$

Let K have r_1 real conjugates and r_2 pairs of complex conjugates, so that $n = r_1 + 2r_2$. Let $\mathscr{D}$ be the discriminant of K, and write

$$C = 2^{-r_2}\pi^{-n/2}|\mathscr{D}|^{1/2},$$
$$G(s) = C^s\Gamma(s/2)^{r_1}\Gamma(s)^{r_2},$$

and

$$\xi_K(s) = G(s)\zeta_K(s).$$

The functional equation for $\zeta_K(s)$ then takes the form $\xi_K(s) = \xi_K(1-s)$. Moreover, $\xi_K(s)$ is holomorphic and bounded in the strip $|\mathrm{Re}(s)| \leqslant 2$, except in neighbourhoods of $s = 1$ and $s = 0$, where there are simple poles. Taking $s = \frac{1}{2}+it$ with $t \geqslant 2$, we therefore find that

$$(2.1) \qquad \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\xi_K(s+w)e^{w^2}\frac{dw}{w} = G(s)\sum_A (NA)^{-s}W_0(NA),$$

with

$$(2.2) \qquad W_0(x) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} x^{-w}\frac{G(s+w)}{G(s)}e^{w^2}\frac{dw}{w}.$$

We now move the line of integration in (2.1) to $\mathrm{Re}(w) = -1$, allowing for the poles at $w = 0$, $1-s$ and $-s$. This yields

$$\begin{aligned} G(s)\sum_A (NA)^{-s}W_0(NA) &= \xi_K(s) + O(e^{-t^2}) + \frac{1}{2\pi i}\int_{-1-i\infty}^{-1+i\infty}\xi_K(s+w)e^{w^2}\frac{dw}{w} \\ &= \xi_K(s) + O(e^{-t^2}) - \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\xi_K(1-s+u)e^{u^2}\frac{du}{u}, \end{aligned}$$

on using the functional equation and the substitution $u = -w$. Since $G(s)^{-1} \ll e^{t^2}$, it follows that

$$\sum_A (NA)^{-s}W_0(NA) + \frac{G(1-s)}{G(s)}\sum_A (NA)^{s-1}W^*(NA) = \zeta_K(s) + O(1),$$

where

$$W^*(x) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} x^{-u}\frac{G(1-s+u)}{G(1-s)}e^{u^2}\frac{du}{u}.$$

Finally we recall that $s = \frac{1}{2}+it$, whence $G(1-s) = \overline{G(s)}$ and $W^*(x) = \overline{W_0(x)}$. We thus obtain the first assertion of the lemma.

From (2.2) we have

$$\frac{d^k W_0(x)}{dx^k} = x^{-k}\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}(-1)^k x^{-w} w(w+1)\dots(w+k-1)\frac{G(s+w)}{G(s)}e^{w^2}\frac{dw}{w}. \tag{2.3}$$

If $w=\sigma+i\tau$, Stirling's formula yields

$$\frac{\Gamma(s+w)}{\Gamma(s)} \ll t^{\sigma} e^{O(|\tau|)}, \qquad \frac{\Gamma((s+w)/2)}{\Gamma(s/2)} \ll t^{\sigma/2} e^{O(|\tau|)},$$

and hence

$$\frac{G(s+w)}{G(s)} \ll t^{n\sigma/2} e^{O(|\tau|)},$$

uniformly for $-1/4 \leqslant \sigma \leqslant 1$. It follows that

$$w(w+1)\dots(w+k-1)\frac{G(s+w)}{G(s)}e^{w^2} w^{-1} \ll t^{n\sigma/2} e^{-\tau^2/2},$$

so that (2.3) is $O(x^{-k-1}t^{n/2})$. Alternatively, if $x \leqslant t^{n/2}$, we move the line of integration to $\mathrm{Re}(w)=-1/4$. The resulting expression is then

$$\ll x^{-k}(t^{n/2}x^{-1})^{-1/4} \ll x^{-k}.$$

If $k=0$ there is a pole at $w=0$, with residue $1 \ll x^{-k}$. The second part of the lemma now follows.

We next define

$$\omega_0(x) = \begin{cases} \exp\left(-\dfrac{1}{x(1-x)}\right), & 0<x<1,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$\omega(x) = \Big\{\int_0^1 \omega_0(y)\,dy\Big\}^{-1}\omega_0(x). \tag{2.4}$$

Then $\omega(x)$ has derivatives of all orders, and satisfies

$$\omega^{(k)}(x) \ll_k 1, \quad \omega(x)=0 \quad (x\leqslant 0 \text{ or } x\geqslant 1), \tag{2.5}$$

and

$$\int_{-\infty}^{\infty}\omega(x)\,dx = 1.$$

We now have

$$\sum_A (NA)^{-1/2-it} W_0(NA) = \int_0^\infty S_0(x)\,dx,$$

where

$$S_0(x) = \sum_A (NA)^{-1/2-it} W_0(NA)\,\omega(x-\log NA).$$

On substituting $x=\log y$ we obtain the following result.

Lemma 2. *Let $t\geqslant 2$. Then*

$$|\zeta_K(\tfrac12+it)| \leqslant 2\int_1^\infty |S_1(y)|\,dy + O(1),$$

where

$$S_1(y) = \sum_A (NA)^{-it} W_1(NA, t, y).$$

Here $W_1(u,t,y)$ has partial derivatives of all orders with respect to $u\in \mathbf{R}$. Moreover

$$\frac{\partial^k W_1}{\partial u^k} \ll_k y^{-3/2-k}\min(1, t^{n/2}y^{-1})$$

and $W_1=0$ unless $e^{-1}y<u<y$.

3. A smoothed sum over integer vectors. We now proceed to transform $S_1(y)$ into a finite weighted sum over n-tuples of rational integers. To this end, we firstly choose a set of representatives $A_1,\dots,A_h$ for the ideal classes, where h is the class number. Then $A\sim A_j^{-1}$ if and only if $AA_j=(\alpha)$ for some algebraic integer $\alpha\in K$. Hence

$$\sum_A{}^{(1)} (NA)^{-it} W_1(NA,t,y) = (NA_j)^{it}\sum_\alpha{}^{(2)} |N\alpha|^{-it} W_1\left(\frac{|N\alpha|}{NA_j}, t, y\right),$$

where $\sum^{(1)}$ indicates the condition $A\sim A_j^{-1}$ and $\sum^{(2)}$ runs over a set of non-associated algebraic integers $\alpha\in A_j$. Thus

$$S_1(y) = \sum_{j=1}^h (NA_j)^{it}\sum_{\alpha\in A_j}{}^{*} |N\alpha|^{-it} W_1\left(\frac{|N\alpha|}{NA_j}, t, y\right),$$

where the sum $\sum^*$ runs over exactly one α from each equivalence class of associates. The ideals A_j depend only on K and not on t. Hence, for some A_j $(=A$, say) one has

$$S_1(y) \ll \left|\sum_{\alpha\in A}{}^{*} |N\alpha|^{-it} W_1\left(\frac{|N\alpha|}{NA}, t, y\right)\right|. \tag{3.1}$$

In order to select exactly one $\alpha\in A$ from each class of associates we use the following standard device. Let the real conjugates of α be $\alpha^{(j)}$ $(1\leqslant j\leqslant r_1)$

and let the complex conjugates satisfy

$$\overline{\alpha^{(j)}} = \alpha^{(j+r_2)}, \quad r_1 < j \leqslant r_1+r_2.$$

We now set $r = r_1+r_2-1$, and define a linear mapping $\vartheta\colon K-\{0\} \to \boldsymbol{R}^r$ by taking $\vartheta(\alpha)$ to be the row vector

$$\vartheta(\alpha) = (\log|\alpha^{(1)}|, \ldots, \log|\alpha^{(r)}|). \tag{3.2}$$

If K contains exactly m roots of unity then $\vartheta(\alpha) = \vartheta(\alpha_0)$ will have exactly m solutions $\alpha \in K$, if $\alpha_0 \in K-\{0\}$ is given.

Now let $\varepsilon_1, \ldots, \varepsilon_r$ be a system of fundamental units for K, and let E be the invertible $r\times r$ matrix whose rows are $\vartheta(\varepsilon_1), \ldots, \vartheta(\varepsilon_r)$. Then if X is any fixed vector and

$$C = \{(x_1, \ldots, x_r) \in \boldsymbol{R}^r\colon\ 0 \leqslant x_i + X_i < 0\}$$

we see that the condition $\vartheta(\alpha)E^{-1} \in C$ picks out exactly m numbers $\alpha \in K-\{0\}$ from each class of associates. We shall take

$$X = (\log Y)(1, \ldots, 1)E^{-1},$$

where $Y = y^{1/n}$. We now define $S \subseteq \boldsymbol{R}^r$ to be the set

$$S = \{\vartheta(\alpha)E^{-1}\colon\ \alpha \in A-\{0\}\},$$

so that S has periods $\boldsymbol{Z}^r$. Moreover, for each $\boldsymbol{\sigma} \in S$, we define

$$f(\boldsymbol{\sigma}) = |N\alpha|^{-it}\, W_1\left(\frac{|N\alpha|}{NA}, t, y\right),$$

where $\alpha \in A-\{0\}$ satisfies $\vartheta(\alpha)E^{-1} = \boldsymbol{\sigma}$. This definition is independent of the choice of α, and produces a function which, like S, has periods $\boldsymbol{Z}^r$. We now introduce a smooth r-dimensional weight

$$\omega(x_1, \ldots, x_r) = \prod_{i=1}^{r} \omega(x_i),$$

where ω is given by (2.4). Then

$$\begin{aligned}
\sum_{\alpha\in A}^{*} |N\alpha|^{-it}\, W_1\left(\frac{|N\alpha|}{NA}, t, y\right) &= \sum_{\boldsymbol{\sigma}\in S\cap C} f(\boldsymbol{\sigma}) \\
&= \int_{\boldsymbol{R}^r} \sum_{\boldsymbol{\sigma}\in S\cap C} f(\boldsymbol{\sigma})\,\omega(\boldsymbol{\sigma}+\boldsymbol{x})\,dx_1\ldots dx_r \\
&= \sum_{\boldsymbol{n}\in\boldsymbol{Z}^r} \int_C \sum_{\boldsymbol{\sigma}\in S\cap C} f(\boldsymbol{\sigma})\,\omega(\boldsymbol{\sigma}+\boldsymbol{x}+\boldsymbol{n})\,dx_1\ldots dx_r \\
&= \sum_{\boldsymbol{n}\in\boldsymbol{Z}^r} \int_C \sum_{\boldsymbol{\sigma}\in S\cap C} f(\boldsymbol{\sigma}+\boldsymbol{n})\,\omega(\boldsymbol{\sigma}+\boldsymbol{x}+\boldsymbol{n})\,dx_1\ldots dx_r \\
&= \int_C \sum_{\boldsymbol{\sigma}\in S} f(\boldsymbol{\sigma})\,\omega(\boldsymbol{\sigma}+\boldsymbol{x})\,dx_1\ldots dx_r.
\end{aligned}$$

Thus (3.1) yields

$$S_1(y) \ll \left|\sum_{\alpha\in A} |N\alpha|^{-it}\, W_1\left(\frac{|N\alpha|}{NA}, t, y\right)\omega\big(\vartheta(\alpha)E^{-1}+\boldsymbol{x}\big)\right|,$$

for some $\boldsymbol{x} \in C$.

We shall use the linear function $\varrho(\boldsymbol{u}) = (\varrho_1(\boldsymbol{u}), \ldots, \varrho_n(\boldsymbol{u}))$, where

$$\varrho_j(\boldsymbol{u}) = \begin{cases} u_j, & j \leqslant r_1, \\ u_j + iu_{j+r_2}, & r_1 < j \leqslant r_1+r_2, \\ u_{j-r_2} - iu_j, & r_1+r_2 < j \leqslant n. \end{cases}$$

We also define a weight $W_2(\boldsymbol{u})$, for any $\boldsymbol{u} \in \boldsymbol{R}^n$, by

$$W_2(\boldsymbol{u}) = W_2(\boldsymbol{u}; A, t, y, \boldsymbol{x}) = W_1\left(\frac{|\prod \varrho_j(\boldsymbol{u})|}{NA}, t, y\right)\omega(\boldsymbol{v}E^{-1}+\boldsymbol{x}),$$

where

$$\boldsymbol{v} = (\log|\varrho_1(\boldsymbol{u})|, \ldots, \log|\varrho_r(\boldsymbol{u})|).$$

It follows that there are positive constants c_1, c_2 such that $W_2(\boldsymbol{u})$ is non-zero only when

$$c_1 Y < |\varrho_j(\boldsymbol{u})| < c_2 Y \quad (1 \leqslant j \leqslant n). \tag{3.3}$$

We denote the above region by $R(Y)$. Moreover, if we adopt the notations $\boldsymbol{k} = (k_1, \ldots, k_n)$, $k = \sum k_j$ and

$$D_{\boldsymbol{u}}^{\boldsymbol{k}} = \frac{\partial^k}{\partial^{k_1}u_1\ldots\partial^{k_n}u_n},$$

then the partial derivatives of W_2 will satisfy

$$D_{\boldsymbol{u}}^{\boldsymbol{k}} W_2 \ll_k Y^{-k} y^{-3/2}\min(1, t^{n/2}y^{-1}). \tag{3.4}$$

Now let $(\alpha_1, \ldots, \alpha_n)$ be an integral basis for A over $\boldsymbol{Z}$ and define

$$\alpha^{(j)}(\boldsymbol{x}) = \sum_{h=1}^{n} \alpha_h^{(j)} x_h \quad (1 \leqslant j \leqslant n), \tag{3.5}$$

$$N(\boldsymbol{x}) = \left|\prod_{h=1}^{n} \alpha^{(h)}(\boldsymbol{x})\right|,$$

and $\boldsymbol{u}(\boldsymbol{x}) = (u_1, \ldots, u_n)$, where

$$u_j = \begin{cases} \alpha^{(j)}(\boldsymbol{x}), & j \leqslant r_1, \\ \operatorname{Re}(\alpha^{(j)}(\boldsymbol{x})), & r_1 < j \leqslant r_1+r_2, \\ \operatorname{Im}(\alpha^{(j-r_2)}(\boldsymbol{x})), & r_1+r_2 < j \leqslant n. \end{cases} \tag{3.6}$$

We then have:

LEMMA 3. *Let $t \geqslant 2$. Then there is an ideal A of K and a function $W_2(u) = W_2(u, y)$ such that*

$$\zeta_K(\tfrac{1}{2}+it) \ll 1 + \int_1^\infty |S_3(y)|\, dy,$$

where

$$S_3(y) = \sum_{m \in Z^n} N(m)^{-it} W_2(u(m)).$$

Here $W_2(u)$ satisfies (3.4) (where $Y = y^{1/n}$) and is non-zero only in the region $R(Y)$ given by (3.3).

4. Van der Corput's method. To estimate $S_3(y)$ we first apply the van der Corput "A process". For any $h \in Z^n$ we have

$$S_3(y) = \sum_{m \in Z^n} N(m+h)^{-it} W_2(u(m+h)).$$

Thus, if $1 \leqslant H \leqslant Y$, and

$$H_1 = \# \{h \in Z^n\colon |h| \leqslant H\}, \tag{4.1}$$

we obtain

$$\begin{aligned} H_1 S_3(y) &= \sum_{|h| \leqslant H} \sum_m N(m+h)^{-it} W_2(u(m+h)) \\ &= \sum_m \sum_{|h| \leqslant H} N(m+h)^{-it} W_2(u(m+h)). \end{aligned} \tag{4.2}$$

If $W_2(u(m+h)) \neq 0$, then (3.3), (3.5) and (3.6) yield

$$\sum_{i=1}^n \alpha_i^{(j)}(m_i + h_i) \ll Y \quad (1 \leqslant j \leqslant n).$$

Since $\det(\alpha_i^{(j)}) \neq 0$, it follows that $|m+h| \ll Y$. However $|h| \leqslant H \leqslant Y$, so that only those m with $|m| \ll Y$ can contribute to (4.2). Hence Cauchy's inequality yields

$$\begin{aligned} H_1^2 |S_3(y)|^2 &\ll Y^n \sum_m \Big| \sum_{|h| \leqslant H} N(m+h)^{-it} W_2(u(m+h)) \Big|^2 \\ &\ll Y^n \sum_{|h|, |j| \leqslant H} \Big| \sum_m \Big(\frac{N(m+j)}{N(m+h)} \Big)^{it} W_2(u(m+j)) W_2(u(m+h)) \Big| \\ &\ll Y^n H_1 \sum_{|l| \leqslant 2H} \Big| \sum_m \Big(\frac{N(m+l)}{N(m)} \Big)^{it} W_2(u(m+l)) W_2(u(m)) \Big| \\ &= Y^n H_1 \sum_{|l| \leqslant 2H} |\Sigma(l, y)|, \end{aligned} \tag{4.3}$$

say. We now apply the n-dimensional Poisson summation formula. This takes the shape

$$\sum_{m \in Z^n} f(m) = \sum_{p \in Z^n} \int_{R^n} e^{2\pi i p \cdot x} f(x)\, dx_1 \ldots dx_n,$$

for any smooth function f of compact support. Thus

$$\Sigma(l, y) = \sum_{p \in Z^n} F(l, p), \tag{4.4}$$

where

$$F(l, p) = \int_{R^n} e^{2\pi i p \cdot x} \Big(\frac{N(x+l)}{N(x)} \Big)^{it} W_2(u(x+l)) W_2(u(x))\, dx_1 \ldots dx_n. \tag{4.5}$$

We can simplify this somewhat by substituting $u = u(x)$. According to (3.5) and (3.6) this is a non-singular linear transformation $u = xU$. We write

$$W_3(u) = W_3(u, l, y) = W_2(u + u(l)) W_2(u).$$

Then $W_3(u)$ has support in the region $R(Y)$, and its partial derivatives satisfy

$$D_u^k W_3 \ll_k \Delta Y^{-k}, \tag{4.6}$$

where

$$\Delta = y^{-3} \min(1, t^n y^{-2}), \tag{4.7}$$

uniformly for $|l| \leqslant 2H$. If we set $\lambda = lU$ and $\mu = p(U^T)^{-1}$, then

$$|l| \ll |\lambda| \ll |l| \tag{4.8}$$

and

$$|p| \ll |\mu| \ll |p|. \tag{4.9}$$

Moreover $e^{2\pi i p \cdot x} = e^{2\pi i \mu \cdot u}$ and

$$\frac{N(x+l)}{N(x)} = \prod_{j=1}^n \frac{|\varrho_j(u+\lambda)|}{|\varrho_j(u)|}.$$

Hence (4.5) takes the form

$$F(l, p) = |\det U|^{-1} \int_{R^n} e^{i\varphi(u)} W_3(u)\, du_1 \ldots du_n. \tag{4.10}$$

Here

$$\varphi(u) = \sum_{j=1}^{r_1} \varphi_j(u_j) + \sum_{j=r_1+1}^{r_1+r_2} \varphi_j(u_j, u_{j+r_2}), \tag{4.11}$$

where

$$\varphi_j(u) = 2\pi\mu_j u + t \log\Big| \frac{u+\lambda_j}{u} \Big| \quad (j \leqslant r_1), \tag{4.12}$$

and

$$\varphi_j(u, v) = 2\pi(\mu_j u+\mu_{j+r_2} v)+t\log\left\{\frac{(u+\lambda_j)^2+(v+\lambda_{j+r_2})^2}{u^2+v^2}\right\} \quad (r_1 < j \leqslant r_1+r_2). \tag{4.13}$$

5. $F(l, p)$ for large p. In this section we estimate the integral in (4.10) when p is large. We begin by proving the following lemma.

LEMMA 4. *Let $W\colon \mathbf{R}\to\mathbf{R}$ have support in (a, b) and let $\varphi\colon [a, b]\to\mathbf{R}$. Suppose that W and φ have derivatives of all orders and write*

$$\Phi = \inf\{|\varphi'(x)|:\ a<x<b\}. \tag{5.1}$$

Suppose further that

$$\frac{d^k W}{dx^k} \ll_k (b-a)^{-k} \quad (x\in\mathbf{R},\ k\geqslant 0) \tag{5.2}$$

and

$$\frac{d^k \varphi}{dx^k} \ll_k (b-a)^{1-k}\Phi \quad (x\in(a, b),\ k\geqslant 1). \tag{5.3}$$

Then

$$\int_a^b e^{i\varphi(x)}W(x)\,dx \ll_N (b-a)^{1-N}\Phi^{-N}$$

for any integer $N\geqslant 0$.

We prove the lemma by induction on N, the case $N=0$ being the trivial bound

$$\int_a^b e^{i\varphi(x)}W(x)\,dx \ll b-a,$$

which follows from (5.2) with $k=0$. For the general case we integrate by parts to obtain

$$\int_a^b e^{i\varphi(x)}W(x)\,dx = \int_a^b \left(i\varphi'(x)e^{i\varphi(x)}\right)\frac{W(x)}{i\varphi'(x)}\,dx = i\int_a^b e^{i\varphi(x)}\left(\frac{W(x)}{\varphi'(x)}\right)'dx. \tag{5.4}$$

By Leibniz's formula we have

$$\frac{d^{k+1}}{dx^{k+1}}\left(\frac{W}{\varphi'}\right) = \sum_{j=0}^{k+1}\binom{k+1}{j}\frac{d^j}{dx^j}\left(\frac{1}{\varphi'}\right)\frac{d^{k+1-j}}{dx^j}W. \tag{5.5}$$

One may easily show by induction that

$$\frac{d^j}{dx^j}\left(\frac{1}{\varphi'}\right) = \frac{H}{(\varphi')^{j+1}},$$

where H is a polynomial in the derivatives of φ'. Moreover, any monomial

$$\prod_h \varphi^{(h)e_h}$$

occurring in H will have

$$\sum e_h = j, \quad \sum he_h = 2j.$$

It therefore follows from (5.1) and (5.3) that

$$\frac{d^j}{dx^j}\left(\frac{1}{\varphi'}\right) \ll_j \Phi^{-1}(b-a)^{-j},$$

whence (5.2) and (5.5) yield

$$\frac{d^k}{dx^k}\left(\frac{W}{\varphi'}\right)' \ll_k \left((b-a)\Phi\right)^{-1}(b-a)^{-k}.$$

Thus the function

$$(b-a)\Phi\left(\frac{W(x)}{\varphi'(x)}\right)'$$

satisfies the hypotheses of the lemma in place of $W(x)$, and our induction assumption yields

$$\int_a^b e^{i\varphi(x)}(b-a)\Phi\left(\frac{W(x)}{\varphi'(x)}\right)'dx \ll_N (b-a)^{2-N}\Phi^{1-N}.$$

The case N of the lemma then follows from (5.4).

We turn now to the integral in (4.10), and consider the case in which $\boldsymbol{\mu}$ is large. We define

$$\nu_j = |\varrho_j(\boldsymbol{\lambda})| \quad (1\leqslant j\leqslant n).$$

According to (4.8) there will be a constant $c_3>0$ such that $\nu_j\leqslant \frac{1}{2}c_1 Y$ (with c_1 as in (3.3)) providing only that $H\leqslant c_3 Y$, which we henceforth assume. It follows that

$$|\varrho_j(\boldsymbol{u}+\boldsymbol{\lambda})| \geqslant \tfrac{1}{2}c_1 Y$$

for all $\boldsymbol{u}\in R(Y)$. The formulae (4.11), (4.12) and (4.13) now yield

$$\frac{\partial\varphi}{\partial u_j} = 2\pi\mu_j + O\left(\frac{t\nu_j}{Y^2}\right)$$

for $u \in R(Y)$. There is therefore a constant $c_4 > 0$ such that

$$\left|\frac{\partial \varphi}{\partial u_j}\right| \geqslant \pi |\mu_j|$$

for every j for which

$$|\mu_j| \geqslant c_4 t Y^{-2} v_j. \tag{5.6}$$

We will then have

$$\frac{\partial^l \varphi}{\partial u_j^l} \ll_l \frac{t v_j}{Y^{l+1}} \ll \frac{|\mu_j|}{Y^{l-1}}.$$

If there is any index j satisfying (5.6) we may use Lemma 4 to deduce that

$$\int_{-\infty}^{\infty} e^{i\varphi(u)} W_3(u)\, du_j \ll_N \Delta Y^{1-N} |\mu_j|^{-N},$$

since any line meets the region $R(Y)$ in at most two intervals. We therefore conclude that

$$\int_{R^n} e^{i\varphi(u)} W_3(u)\, du_1 \ldots du_n \ll_N \Delta Y^n (Y|\mu_j|)^{-N}.$$

6. $F(l, p)$ for small p. We now consider $F(l, p)$ when μ is small. We define

$$L_j = \min\left(\tfrac{1}{8}c_1 Y,\ (tv_j/Y^3)^{-1/2}\right) \quad (1 \leqslant j \leqslant n),$$

$$L = \prod_1^n L_j$$

and

$$W_4(x) = \prod_{j=1}^n \omega(x_j/L_j),$$

where the function ω is given by (2.4). Then

$$\int_{R^n} W_4(x)\, dx_1 \ldots dx_n = L,$$

so that

$$\begin{aligned} \int_{R^n} e^{i\varphi(u)} W_3(u)\, du_1 \ldots du_n &= L^{-1} \int_{R^n}\int_{R^n} e^{i\varphi(u)} W_3(u) W_4(u-x)\, du_1 \ldots du_n\, dx_1 \ldots dx_n \\ &= L^{-1} \int_{R^n} I(x)\, dx_1 \ldots dx_n, \end{aligned} \tag{6.1}$$

say. We proceed to estimate $I(x)$. Clearly $I(x) = 0$ unless $|x| \ll Y$. The weight

$$W_5(u) = W_5(u, x) = W_3(u) W_4(u-x)$$

satisfies

$$\frac{\partial^k W_5}{\partial u_j^k} \ll_k \Delta L_j^{-k},$$

by (2.5) and (4.6). Let $B(x)$ denote the box

$$\{u \in R^n\colon\ 0 \leqslant u_j - x_j \leqslant L_j\ (1 \leqslant j \leqslant n)\}.$$

If $u \in B(x) \cap R(Y)$ then

$$|\varrho_j(u) - \varrho_j(x)| \leqslant \tfrac{1}{4} c_1 Y$$

by the choice of L_j. We have already ensured that $|\varrho_j(\lambda)| \leqslant \frac{1}{2} c_1 Y$, by taking $H \leqslant c_3 Y$. Moreover, we have $|\varrho_j(u)| \geqslant c_1 Y$ by (3.3). It follows that

$$Y \ll |\varrho_j(u)|,\ |\varrho_j(x)|,\ |\varrho_j(u+\lambda)|,\ |\varrho_j(x+\lambda)| \ll Y \quad (u \in B(x) \cap R(Y)). \tag{6.2}$$

We also note that if x' is some other point for which $B(x') \cap R(Y) \neq \emptyset$, then

$$Y \ll |\varrho_j(x + x' + \lambda)| \ll Y \tag{6.3}$$

providing only that

$$|\varrho_j(x) - \varrho_j(x')| \leqslant \tfrac{1}{2} c_1 Y. \tag{6.4}$$

We now find that

$$\frac{\partial \varphi}{\partial u_j}(u) - \frac{\partial \varphi}{\partial u_j}(x) \ll t v_j L_j Y^{-3} \quad (u \in B(x) \cap R(Y)),$$

by (4.11), (4.12), (4.13) and (6.2). It follows that there is an absolute constant $c_5 > 0$ such that

$$\left|\frac{\partial \varphi}{\partial u_j}(x)\right| \ll \left|\frac{\partial \varphi}{\partial u_j}(u)\right| \ll \left|\frac{\partial \varphi}{\partial u_j}(x)\right| \quad (u \in B(x) \cap R(Y))$$

whenever

$$\left|\frac{\partial \varphi}{\partial u_j}(x)\right| \geqslant c_5 t v_j L_j Y^{-3}.$$

Since $L_j \ll Y$, we then also have

$$\frac{\partial^k \varphi}{\partial u_j^k}(u) \ll_k \frac{t v_j}{Y^{k+1}} \ll (t v_j L_j Y^{-3}) L_j^{1-k} \ll \left|\frac{\partial \varphi}{\partial u_j}(x)\right| L_j^{1-k}$$

for $k \geqslant 2$ and $u \in B(x) \cap R(Y)$. We now assume that there is some index for which

$$\left|\frac{\partial\varphi}{\partial u_j}(x)\right| \geqslant \Psi t v_j L_j Y^{-3} \tag{6.5}$$

with $\Psi \geqslant c_5$, and

$$v_j \geqslant \frac{64}{c_1^2} Y t^{-1}, \tag{6.6}$$

(so that $L_j = (tv_j/Y^3)^{-1/2}$). Then we may use Lemma 4 to deduce that

$$\int_{-\infty}^{\infty} e^{i\varphi(u)} W_5(u)\,du_j \ll_N \Delta L_j^{1-N}(\Psi t v_j L_j Y^{-3})^{-N} = \Delta L_j \Psi^{-N},$$

since any line meets $B(x) \cap R(Y)$ in at most two intervals. We conclude that

$$I(x) \ll_N \Delta L \Psi^{-N}$$

so that such x contribute a total

$$\ll_N \Delta Y^n \Psi^{-N}$$

to (6.1).

It remains to consider the set X of those x for which (6.5) fails whenever j satisfies (6.6). Here we use the trivial bound $I(x) \ll \Delta L$, so that the corresponding contribution to (6.1) is $O(\Delta \operatorname{mes}(X))$, where $\operatorname{mes}(X)$ is the Lebesgue measure of X. For any $x \in X$ we will have

$$\left|\frac{\partial\varphi}{\partial u_j}(x)\right| \leqslant \Psi(tv_j Y^{-3})^{1/2}$$

whenever j satisfies (6.6). Thus if x_1 and x_2 are both in X it follows that

$$\left|\frac{\partial\varphi}{\partial u_j}(x_1) - \frac{\partial\varphi}{\partial u_j}(x_2)\right| \leqslant 2\Psi(tv_j Y^{-3})^{1/2}. \tag{6.7}$$

However, if $j \leqslant r_1$, then (4.11) and (4.12) yield

$$\frac{\partial\varphi}{\partial u_j}(x_1) - \frac{\partial\varphi}{\partial u_j}(x_2) = \frac{\partial\varphi_j}{\partial u_j}(x_{1j}) - \frac{\partial\varphi_j}{\partial u_j}(x_{2j})$$
$$= -\frac{t\lambda_j}{x_{1j}(x_{1j}+\lambda_j)} + \frac{t\lambda_j}{x_{2j}(x_{2j}+\lambda_j)}$$
$$= \frac{t\lambda_j(x_{1j}-x_{2j})(x_{1j}+x_{2j}+\lambda_j)}{x_{1j}x_{2j}(x_{1j}+\lambda_j)(x_{2j}+\lambda_j)}.$$

By (6.2), (6.3) and (6.4) we have

$$\left|\frac{\partial\varphi}{\partial u_j}(x_1) - \frac{\partial\varphi}{\partial u_j}(x_2)\right| \geqslant tv_j Y^{-3} |\varrho_j(x_1) - \varrho_j(x_2)|,$$

if

$$|\varrho_j(x_1) - \varrho_j(x_2)| \leqslant \tfrac{1}{2} c_1 Y. \tag{6.8}$$

It follows that

$$|\varrho_j(x_1) - \varrho_j(x_2)| \ll \Psi\left(\frac{Y^3}{tv_j}\right)^{1/2} \tag{6.9}$$

if (6.6) and (6.8) hold, and $j \leqslant r_1$. In case $r_1 < j \leqslant r_1 + r_2$ we have $v_j = v_{j+r_2}$, so we may use (6.7) for both j and $j+r_2$. If we put $w = \varrho_j(x_1)$, $z = \varrho_j(x_2)$ and $\alpha = \varrho_j(\lambda)$ then

$$\frac{\partial\varphi}{\partial u_j}(x_1) = 2\pi\mu_j + 2t\operatorname{Re}\left(\frac{d}{dw}\log\frac{w+\alpha}{w}\right),$$
$$\frac{\partial\varphi}{\partial u_{j+r_2}}(x_1) = 2\pi\mu_{j+r_2} - 2t\operatorname{Im}\left(\frac{d}{dw}\log\frac{w+\alpha}{w}\right),$$

and similarly for x_2. Thus (6.7) yields

$$\left|\frac{d}{dw}\log\frac{w+\alpha}{w} - \frac{d}{dz}\log\frac{z+\alpha}{z}\right| \ll \Psi\left(\frac{v_j}{tY^3}\right)^{1/2}.$$

However

$$\frac{d}{dw}\log\frac{w+\alpha}{w} - \frac{d}{dz}\log\frac{z+\alpha}{z} = \frac{\alpha(w-z)(w+z+\alpha)}{wz(w+\alpha)(z+\alpha)}$$

and so, just as in the case $j \leqslant r_1$, we deduce that (6.9) holds whenever (6.6) and (6.8) also hold. If (6.6) fails we trivially have

$$|\varrho_j(x_1) - \varrho_j(x_2)| \ll Y \ll \left(\frac{Y^3}{tv_j}\right)^{1/2} \ll \Psi\left(\frac{Y^3}{tv_j}\right)^{1/2}.$$

We therefore conclude that

$$\operatorname{mes}(X) \ll \Psi^n Y^{3n/2} t^{-n/2} \prod_{j=1}^{n} v_j^{-1/2}.$$

We summarize the results of this section and the previous one.

LEMMA 5. *Let* $\lambda = lU$, $\mu = p(U^T)^{-1}$, $v_j = |\varrho_j(\lambda)|$ *and*

$$\Delta = y^{-3}\min(1, t^n y^{-2}).$$

There exist positive constants c_4, c_5 such that for any integer $N \geqslant 0$ we have

$$(6.10)\quad F(l, p) \ll_N \Delta Y^n \Psi^{-N} + \Delta \Psi^n Y^{3n/2} t^{-n/2} \prod_{j=1}^{n} v_j^{-1/2} \quad \textit{for} \quad \Psi \geqslant c_5,$$

and

$$(6.11)\quad F(l, p) \ll_N \Delta Y^n (Y|\mu_j|)^{-N} \quad \textit{if} \quad |\mu_j| \geqslant c_4 t Y^{-2} v_j.$$

7. Completion of the proof. We have now to estimate $\Sigma(l, y)$, given by (4.4). From (4.8) we have $v_j \ll H$, so that (6.11) may be used if $|\mu_j| \gg tHY^{-2}$. According to (6.11) we then have

$$F(l, p) \ll_N \Delta Y^n (Y|p|)^{-N} \quad \text{if} \quad |p| \gg tHY^{-2}.$$

Taking $N = n+1$ the contribution to (4.4) is now

$$\ll \Delta Y^{-1} \sum_{p \neq 0} |p|^{-n-1} \ll \Delta Y^{-1}.$$

When $|p| \ll tHY^{-2}$ we use (6.10). Taking $\Psi = c_5 Y^{\varepsilon/n}$, and choosing $N \geqslant n^2/\varepsilon$, we have

$$F(l, p) \ll \Delta + \Delta Y^{3n/2+\varepsilon} t^{-n/2} \prod_{j=1}^{n} v_j^{-1/2}.$$

On referring to the definitions of v_j, ϱ_j, λ, and U, we see that

$$\prod_{j=1}^{n} v_j = |N\alpha(l)|,$$

where $\alpha(l) = \alpha^{(1)}(l)$ is given by (3.5). Hence

$$\sum_p F(l, p) \ll \Delta \{1 + (tHY^{-2})^n\} \{1 + Y^{3n/2+\varepsilon} t^{-n/2} |N\alpha(l)|^{-1/2}\},$$

where the sum is for $0 < |p| \ll tHY^{-2}$. It follows that

$$\Sigma(l, y) \ll \Delta t^n H^n Y^{-2n} + \Delta t^{n/2} H^n Y^{-n/2+\varepsilon} |N\alpha(l)|^{-1/2},$$

providing that

$$(7.1)\quad \max(1, Y^2 t^{-1}) \leqslant H \leqslant c_3 Y.$$

When $l = 0$ we have the trivial bound

$$\sum(0, y) = \sum_m W_3(u(m)) \ll \Delta Y^n,$$

whence

$$(7.2)\quad \sum_{|l| \leqslant 2H} |\Sigma(l, y)| \ll \Delta \{Y^n + t^n H^{2n} Y^{-2n} + t^{n/2} H^n Y^{-n/2+\varepsilon} \sum_{0 < |l| \leqslant 2H} |N\alpha(l)|^{-1/2}\}.$$

Let B be any non-zero ideal of the field K. Any two elements $\alpha(l_1)$, $\alpha(l_2)$ which generate B differ only by a unit factor, η say. Moreover

$$|\eta| = \left|\frac{\alpha(l_2)}{\alpha(l_1)}\right| = \frac{|\alpha(l_2)|}{|N\alpha(l_1)|} \prod_{j=2}^{n} |\alpha^{(j)}(l_1)| \leqslant |\alpha(l_2)| \prod_{j=2}^{n} |\alpha^{(j)}(l_1)| \ll H^n,$$

and similarly for the other conjugates of η. In the notation (3.2) we therefore see that $|\vartheta(\eta)| \ll \log H$. However $\vartheta(\eta)$ lies in the lattice generated by $\vartheta(\varepsilon_1), \ldots, \vartheta(\varepsilon_r)$, where $\varepsilon_1, \ldots, \varepsilon_r$ are a system of fundamental units for K. Thus $\vartheta(\eta)$ can take at most $O((\log H)^r)$ values. It follows that

$$\sum_{0 < |l| \leqslant 2H} |N\alpha(l)|^{-1/2} \ll (\log H)^r \sum_{NB \ll H^n} NB^{-1/2} \ll H^{n/2} Y^{\varepsilon},$$

whence (7.2) yields

$$\sum_{|l| \leqslant 2H} |\Sigma(l, y)| \ll \Delta \{Y^n + t^n H^{2n} Y^{-2n} + t^{n/2} H^{3n/2} Y^{-n/2+2\varepsilon}\}.$$

Since H_1, given by (4.1), satisfies $H_1 \gg H^n$, we now deduce from (4.3) that

$$(7.3)\quad S_3(y) \ll \Delta^{1/2} \{H^{-n/2} Y^n + t^{n/2} H^{n/2} Y^{-n/2} + t^{n/4} H^{n/4} Y^{n/4+\varepsilon}\},$$

for H in the range (7.1). We shall choose $H = Yt^{-1/3}$, which satisfies (7.1) if $t \geqslant c_3^{-3}$ and $t^{1/3} \leqslant Y \leqslant t^{2/3}$. For such Y, (7.3) reduces to

$$(7.4)\quad S_3(y) \ll \Delta^{1/2} \{t^{n/6} Y^{n/2+\varepsilon} + t^{n/3}\} \ll \Delta^{1/2} t^{n/6} Y^{n/2+\varepsilon}$$
$$\ll t^{n/6} y^{-1+\varepsilon/n} \min(1, t^{n/2} y^{-1}),$$

in view of (4.7) and the definition $Y = y^{1/n}$. We also have the trivial bound

$$S_3(y) \ll \sum_{m \in Z^n} W_2(u(m)) \ll Y^n y^{-3/2} \min(1, t^{n/2} y^{-1}),$$

so that (7.4) also holds for $Y \leqslant t^{1/3}$. Now Lemma 3 yields

$$\zeta_K(\tfrac{1}{2}+it) \ll 1 + \int_1^{t^{2n/3}} t^{n/6} y^{-1+\varepsilon/n} \min(1, t^{n/2} y^{-1})\, dy + \int_{t^{2n/3}}^{\infty} y^{-1/2} \min(1, t^{n/2} y^{-1})\, dy \ll t^{n/6+\varepsilon/2},$$

and the theorem follows.

Reference

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