Now

$$\lim_{x\to a}(x-a)^{-1}\int_a^x g\,d\lambda = 0\,,$$

$$\lim_{x\to a}(x-a)^{-1}\int_a^x g^2 d\lambda = \tfrac{1}{2}\varepsilon_n^2\,,$$

h is continuous at a and $2g+h+h(a)$ is bounded, so

$$\lim_{x\to a}(2g(x)+h(x)+h(a))(h(x)-h(a)) = 0\,.$$

Consequently, $\lim_{x\to a}\Phi F(x,a) = 0+\frac{1}{2}\varepsilon_n^2+0+h(a)^2 = \frac{1}{2}\varepsilon_n^2+f(a)^2$, as we wished to prove. ■

COROLLARY 6.3. *Let X be a subset of $\boldsymbol{R}$ such that $\mathscr{D}'(X)$ is a ring. Then the inner Lebesgue measure of X is* 0.

Proof. It follows from the lemma that, if $a_1, a_2, \ldots \in \boldsymbol{R}$ are pairwise distinct, then

$$\sum_{n=1}^{\infty} 2^{-n} 1_{\{a_n\}} \in \mathscr{D}'(X)\,.$$

Then according to Lemma 4.3 of [7] X cannot contain a Baire space Y with $Y\subset\overline{Y_d}$. As we have seen in 4.5 and in the proof of 4.4, this implies that the inner measure of X must be 0. ■

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Received 18 November 1986

Free subgroups of diffeomorphism groups

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Abstract. It is proved that the group $\mathrm{Diff}^k(X)$ of all C^k-diffeomorphisms of a given C^k-manifold X ($k = 1, 2, \ldots, \infty$) includes a non-trivial arcwise connected with respect to the Whitney C^k-topology free subgroup which consists (except for the identity) of diffeomorphisms which embed in no flow.

It is also proved that for each sequence of elements of $\mathrm{Diff}^k(X)$ there are diffeomorphisms arbitrarily close to the given ones which freely generate a subgroup in $\mathrm{Diff}^k(X)$.

0. Introduction. The fundamental concept of the Lie theory is to investigate topological groups by means of their one-parameter subgroups. For a classical (i.e. finite-dimensional) Lie group G, the set $L(G)$ of all one-parameter subgroups has a natural Lie algebra structure. Moreover, there is a one-one correspondence between arcwise connected subgroups of G and Lie subalgebras of $L(G)$.

The group $\mathrm{Diff}_c^\infty(X)$ of all compactly supported C^∞-diffeomorphisms of a C^∞-manifold X is a well-known model of an "infinite-dimensional Lie group" with the Lie algebra $\Gamma_c^\infty(X)$ of compactly supported C^∞-vector fields.

Despite of some resemblances, this group fails to have some properties of classical Lie groups, e.g. the image of the exponential map includes no neighbourhood of the identity (cf. [2], [4], [6]).

The main aim of this note is to show that the situation is even worse, namely that there are nontrivial arcwise connected free subgroups of $\mathrm{Diff}_c^\infty(X)$ consisting of diffeomorphisms (except for the identity, of course) which embed in no flow.

The main part of the proof can be found in § 3 of this note and the idea of the proof is the following (reduced to the case $X = \boldsymbol{R}$).

Denote by ${}^r\Phi$ a free group with r-generators. Elements of ${}^r\Phi$ can be represented by "words" $\varphi = A_{i_n}^{j_n}\ldots A_{i_1}^{j_1}$, where $i_k\in\{1,\ldots,r\}$, $j_k = \pm 1$, $j_k \neq -j_{k+1}$ providing $i_k = i_{k+1}$, and "the empty word" **1**. Set $|\varphi|$ to be the length of the "word" $\varphi\in{}^r\Phi$ and write ${}^r_n\Phi = \{\varphi\in{}^r\Phi: |\varphi|\leqslant n\}$. For a group G and $g_1,\ldots,g_r\in G$, we have the natural homomorphism ${}^r\Phi\ni\varphi\mapsto\varphi(g_1,\ldots,g_r)\in G$ given by replacing $A_{i_k}^{j_k}$ by $g_{i_k}^{j_k}$.

Let $G = \mathrm{Diff}^\infty[0,1]$ be the group of all C^∞-diffeomorphisms of the closed interval $[0,1]$. By a theorem of Kopell [4] there are $g\in G$ arbitrarily close to the identity in the natural C^∞-topology such that g^n embeds in no flow and has no fixed points in $(0,1)$ for $n = 1, 2, \ldots$ We shall call them *Kopell-diffeomorphisms*.

Take $\varphi \in {}^r_n\Phi$, $\varphi = A_{i_n}^{j_n} \dots A_{i_1}^{j_1}$. Take $g_i \in \mathrm{Diff}^\infty[0, 1]$, $i = 1, \dots, r$, such that there are closed intervals $I_1 < I_2 < \dots < I_r$ in $(0, 1)$

$$(I < J \text{ means } x < y \text{ for all } x \in I,\ y \in J)$$

such that $g_i(x) = x$ for $x < I_i$, and $g_i|_{I_i}$ is a Kopell-diffeomorphism of I_i, $i = 1, \dots, r$. Suppose $\varphi(g_1, \dots, g_r)$ embeds in a flow. Since

$$\varphi(g_1, \dots, g_r)|_{I_1} = \varphi(g_1|_{I_1}, \mathrm{id}, \dots, \mathrm{id}) = (g_1|_{I_1})^{\Sigma_1\varphi},$$

where $\Sigma_1\varphi = \sum\limits_{\substack{k=1,\dots,n\\ i_k=1}} j_k$, and since $g_1|_{I_1}$ is a Kopell-diffeomorphism, we have $\Sigma_1\varphi = 0$. For example, take

$$\varphi = A_1^1 A_1^1 A_2^1 A_1^{-1} A_3^{-1} A_2^1 A_2^1 A_1^{-1} \quad (\varphi \in {}^3_8\Phi,\ \Sigma_1\varphi = 0).$$

We can write $\varphi(g_1, g_2, g_3)$ in the form

$$\varphi(g_1, g_2, g_3) = (g_1^2 g_2 g_1^{-2})(g_1 g_3^{-1} g_1^{-1})(g_1 g_2^2 g_1^{-1}).$$

Putting

$$h_1 = g_1^2 g_2 g_1^{-2}, \quad h_2 = g_1 g_3 g_1^{-1}, \quad h_3 = g_1 g_2 g_1^{-1},$$

we get

$$\varphi(g_1, g_2, g_3) = \psi(h_1, h_2, h_3), \quad \text{where } \psi = A_1^1 A_2^{-1} A_3^1 A_3^1 \in {}^3_4\Phi.$$

The pairs (h_1, g_1), (h_2, g_2), (h_3, g_3) consist of adjoint diffeomorphisms and, being a little careful in the choice of g_i, we can assume that $\varphi(g_1, g_2, g_3) \neq \mathrm{id}$ and that there are disjoint closed subintervals J_1, J_2, J_3 of $(0, 1)$ such that $h_i(x) = x$ for $x < J_i$ and $h_i|_{J_i}$ is a Kopell-diffeomorphism, $i = 1, 2, 3$. Since $|\psi| < |\varphi|$, proceeding inductively, we get finally $\varphi(g_1, g_2, g_3) = \mathrm{id}$, a contradiction.

This example shows how to prove that for each natural n, r there are $g_1, \dots, g_r \in \mathrm{Diff}^\infty[0, 1]$ arbitrarily close to the identity such that $\varphi(g_1, \dots, g_r)$ embeds in no flow for each $\varphi \in {}^r_n\Phi$, $\varphi \neq \mathbf{1}$. Then we can use a trick (see Theorem (2.5)) to construct a curve $(0, 1) \ni t \mapsto \gamma(t) \in \mathrm{Diff}_c^\infty(\mathbf{R})$ such that $\{\gamma(t)\}_{t \in (0,1)}$ is a set of free generators of a subgroup in $\mathrm{Diff}_c^\infty(\mathbf{R})$ lying off flows. Note that all this can be done for diffeomorphisms of class C^k, $k = 1, 2, \dots$, as well.

Let us explain what "being a little careful in the choice of g_i" means in the procedure above. We simply want the intervals $\varphi(g_1, \dots, g_r)(I_i)$ to be disjoint for all $\varphi \in {}^r_n\Phi$, $\varphi \neq \mathbf{1}$, $i = 1, \dots, r$. This leads to an investigation of the maps

$$\mathop{\times}^{r} \mathrm{Diff}^k[0, 1] \ni (g_1, \dots, g_r) \mapsto \varphi(g_1, \dots, g_r)(x) \in [0, 1],$$

for $\varphi \in {}^r\Phi, x \in (0, 1)$, which is done in § 1 in a more general setting for the so-called inner mappings of topological groups of homeomorphisms. The fundamental result of § 1 is Theorem (1.9) giving some sufficient conditions asserting that the inner mappings have locally "large" images. This allows us to prove in § 2 and additional fact concerning some groups G of homeomorphisms (with $\mathrm{Diff}^k(X)$, $k = 1, 2, \dots, \infty$, as examples), namely that for each sequence $f_1, f_2, \dots$ of elements of G there are $h_i \in G$ arbitrarily close to f_i, $i = 1, 2, \dots$, such that $h_1, h_2, \dots$ freely generate a subgroup in G.

1. Inner mappings of topological groups of homeomorphisms.

(1.1) Definition. A subgroup H of the group $\mathrm{Homeo}(X)$ of all homeomorphisms of a topological space X, equipped with a group-topology such that the mapping $H \ni h \mapsto h(x) \in X$ is continuous for each $x \in X$ (i.e. the topology is finer than the topology of pointwise convergence), will be called a *topological group of homeomorphisms* of X. Given a family D of homeomorphisms of X and a subset $W \subset X$, we denote by D_W the set of all homeomorphisms from D with supports in W. A topological group H of homeomorphisms of X will be called *strongly locally transitive* (shortly SLT) if, for each $x \in X$, each neighbourhood D of the identity in H (we define neighbourhoods to be open), and each neighbourhood W of x, the set $D_W(x) = \{h(x)\colon h \in D_W\}$ includes a neighbourhood of x.

(1.2) Example. Let X be a C^k-manifold (by a *manifold* we shall mean a finite-dimensional paracompact manifold without boundary), $k = 0, 1, \dots, \infty$, and let $\mathrm{Diff}^k(X)$ be the group of all C^k-diffeomorphisms of X ($\mathrm{Diff}^0(X) = \mathrm{Homeo}(X)$) with the Whitney C^k-topology (called also the *strong* C^k*-toplogy*). Then it is easy to see that $\mathrm{Diff}^k(X)$ is a SLT topological group of homeomorphisms of X. The same is true for the group $\mathrm{Diff}_c^k(X)$ of compactly supported diffeomorphisms which is a closed normal subgroup of $\mathrm{Diff}^k(X)$.

Note that $\mathrm{Diff}^k(X)$ is a Baire space for $k = 1, 2, \dots, \infty$ (see [3] or [5]).

(1.3) Example. Let B be an open ball in $\mathbf{R}^n$ and let $\mathrm{Diff}_c^k(\bar{B})$ be the group of all C^k-diffeomorphisms of $\mathbf{R}^n$ with supports in the closure of B and the topology induced by the Whitney C^k-topology on $\mathrm{Diff}^k(\mathbf{R}^n)$, $k = 1, 2, \dots, \infty$.

It is not hard to verify that $\mathrm{Diff}_c^k(\bar{B})$ considered as a group of homeomorphisms of the open ball B has the topology of C^k-uniform convergence on B and that it is a SLT topological group of homeomorphisms of B.

One can check that the topology is completely matrizable. Observe also that $\mathrm{Diff}_c^k(\bar{B})$ is locally arcwise connected. Indeed, for each neighbourhood U of the identity in $\mathrm{Diff}_c^k(\bar{B})$ we can choose a neighbourhood V of the identity such that for each $\varphi \in V$ there is an arc in U connecting φ and the identity. Such an arc can be taken to be $[0, 1] \ni t \mapsto \varphi_t$, $\varphi_t(x) = tx + (1-t)\varphi(x)$, since it is easy to see that φ_t is really a diffeomorphism for $\varphi \in \mathrm{Diff}_c^k(\bar{B})$ sufficiently close to the identity and that φ_t is closer to the identity the closer to the identity is φ.

The same reasoning shows that the group $\mathrm{Diff}_c^k(B) \approx \mathrm{Diff}_c^k(R^n)$ is locally arcwise connected.

Observe also that, having a relatively compact coordinate chart on an n-dimensional C^k-manifold X, we can construct an embedding of $\mathrm{Diff}_c^k(\bar{B})$ into $\mathrm{Diff}_c^k(X)$.

(1.4) EXAMPLE. Let X be a connected real-analytic manifold of a positive dimension and let $\mathrm{Diff}^{\omega}(X)$ be the group of all real-analytic diffeomorphisms of X with the inductive real-analytic topology. Then $\mathrm{Diff}^{\omega}(X)$ is a topological group of homeomorphisms of X which acts transitively on X, but it is not strongly locally transitive since $(\mathrm{Diff}^{\omega}(X))_W(x) = \{x\}$ for a neighbourhood W of $x \in X$.

Let H be a (topological) group. Denote by $\overset{r}{\bigtimes} H$ the (topological) group $\underbrace{X \times \ldots \times H}_{r \text{ times}}$ with the product group structure (and the product topology). Accordingly, if $D \subset H$, then $\underbrace{D \times \ldots \times D}_{r \text{ times}}$ will be denoted by $\overset{r}{\bigtimes} D$. On the other hand, by D^r we will denote the set $\{h_1 \ldots h_r\colon h_1, \ldots, h_r \in D\}$.

Let ${}^{r}_{n}\Omega(H)$ be the set of all sequences

$$(g_{n+1}, \ldots, g_1, i_n, \ldots, i_1, j_n, \ldots, j_1),$$

where $g_1, \ldots, g_{n+1} \in H$, $i_1, \ldots, i_n \in \{1, \ldots, r\}$, and $j_1, \ldots, j_n = \pm 1$.
For $\omega \in {}^{r}_{n}\Omega(H)$ and $h = (h_1, \ldots, h_r) \in \overset{r}{\bigtimes} H$, we define

$$\omega(h) = g_{n+1} h_{i_n}^{j_n} g_n \ldots h_{i_1}^{j_1} g_1,$$

$$\omega_k(h) = \begin{cases} g_k h_{i_{k-1}}^{j_{k-1}} \ldots h_{i_1}^{j_1} & \text{if } j_k = 1, \\ h_{i_k}^{-1} g_k \ldots h_{i_1}^{j_1} g_1 & \text{if } j_k = -1, \end{cases}$$

$k = 1, \ldots, n$.
If H is a topological group of homeomorphisms of X, $x \in X$, $\omega \in {}^{r}_{n}\Omega(H)$, and $h \in \overset{r}{\bigtimes} H$, then we shall also write $\omega(h, x)$ and $\omega_k(h, x)$ instead of $(\omega(h))(x)$ and $(\omega_k(h))(x)$.

Mappings of the form $\overset{r}{\bigtimes} H \ni h \mapsto \omega(h) \in H$, where $\omega \in {}^{r}_{n}\Phi(H)$, will be called *inner mappings of rank n* of H.
Note that the definition of ω_k is of the form above in order to have

$$\omega(hu) = g_{n+1}(h_{i_n} u_{i_n})^{j_n} g_n \ldots u_{i_k}^{j_k} \omega_k(hu), \qquad k = 1, \ldots, n.$$

Hence, if we compute $\omega(hu, x)$ for $\omega \in {}^{r}_{n}\Omega(H)$, $h, u \in \overset{r}{\bigtimes} H$, H being a group of homeomorphisms of X, $x \in X$, then $u_{i_n}^{j_n}$ in the composition $\omega(hu)$ is acting at the point $\omega_k(hu, x)$.

(1.5) DEFINITION. Let H be a topological group of homeomorphisms of X, let $f = (f_1, \ldots, f_r) \in \overset{r}{\bigtimes} H$, and let $\omega \in {}^{r}_{n}\Omega(H)$. A point $x \in X$ will be called *proper* for (ω, f) if for each $k = 1, \ldots, n-1$, satisfying $i_{k+1} = i_k$ and $j_{k+1} = -j_k$, we have $f_{i_k}^{j_k} g_k \ldots f_{i_1}^{j_1} g_1(x) \in \mathrm{Int}(\mathrm{supp}(g_{k+1}))$.

(1.6) EXAMPLE. Let g be a homeomorphism of $\boldsymbol{R}$ such that $g(1) = 1$ and $g(x) > x$ for $x \neq 1$. Let $f(t) = t+1$. Consider $\omega \in {}^{1}_{2}\Omega(\mathrm{Homeo}(\boldsymbol{R}))$ such that

$\omega(h) = h^{-1}gh$. We have $g(f(0)) = f(0)$, but $f(0) \in \mathrm{Int}(\mathrm{supp}(g))$, so $0 \in \boldsymbol{R}$ is proper for (ω, f).

(1.7) EXAMPLE. Let g and f be as in (1.6) and let $u \in \mathrm{Homeo}(\boldsymbol{R})$ be such that $\mathrm{supp}(u) = (-\infty, 0]$. Consider $\eta \in {}^{1}_{4}\Omega(\mathrm{Homeo}(\boldsymbol{R}))$ such that $\eta(h) = h^{-1}g^{-1}huh^{-1}gh$. Then $f^{-1}gf(0) = 0 \in \mathrm{supp}(u)$, but $0 \notin \mathrm{Int}(\mathrm{supp}(u))$, so $0 \in \boldsymbol{R}$ is not proper for (η, f).

(1.8) Remark. Let H be a topological group of homeomorphisms of X.

(a) It is easy to see that for $\omega \in {}^{r}_{n}\Omega(H)$ and for $x \in X$ the set of all $f \in \overset{r}{\bigtimes} H$ such that x is proper for (ω, f) is open in $\overset{r}{\bigtimes} H$.

(b) If $\omega \in {}^{r}_{n}\Omega(H)$ $\omega = (\mathrm{id}, \ldots, \mathrm{id}, i_n, \ldots, i_1, j_n, \ldots, j_1)$, then each $x \in X$ is proper for (ω, f) for all $f \in \overset{r}{\bigtimes} H$ if and only if $j_{k+1} \neq -j_k$ for all $k = 1, \ldots, n-1$ satisfying $i_k = i_{k+1}$, i.e. $\omega(h) = \varphi(h)$ for φ being a "free group-word" from ${}^{r}_{n}\Phi$ (in the notation of § 2).

(1.9) THEOREM. *Let H be a* SLT *group of homeomorphisms of a Hausdorff topological space X. Let $\omega \in {}^{r}_{n}\Omega(H)$, $n \geqslant 1$, let $f \in \overset{r}{\bigtimes} H$, let $x \in X$ be proper for (ω, f), let W be a neighbourhood of the set $\{\omega_k(f, x)\colon k = 1, \ldots, n\}$, and let D be a neighbourhood of the identity in H. Then the image of the mapping*

$$\overset{r}{\bigtimes}(D_W) \ni u \mapsto \omega(fu, x) \in X$$

includes a nonempty open set.

(1.10) Remark. This open set has not to be a neighbourhood of the point $\omega(f, x)$. Indeed, one can check that, in the notation of (1.6), there is a neighbourhood D of f in $\mathrm{Homeo}(\boldsymbol{R})$ such that the image of the mapping $D \ni h \mapsto \omega(h, 0) \in \boldsymbol{R}$ contains only nonnegative reals.

(1.11) Remark. The assumption that x is proper for (ω, f) is essential in (1.9). To see this, it suffices to consider the inner mapping $\omega(h) = h^{-1}gh$ and to take f such that $f(x) \notin \mathrm{supp}(g)$. Theorem (1.9) is also not valid if we put "supp" instead of "Int supp" in Definition (1.5), since in (1.7) we have $f^{-1}gf(0) \in \mathrm{supp}(u)$, but, for a neighbourhood D of f in $\mathrm{Homeo}(\boldsymbol{R})$, the image of the mapping $D \in h \mapsto \eta(h, 0) \in \boldsymbol{R}$ consists of only one point.
Before proving Theorem (1.9), let us present the following two technical propositions which can easily be proved by induction.

(1.12) PROPOSITION. *Given $\omega \in {}^{r}_{n}\Omega(H)$, we have*

(1)
$$\omega_{k+1}(h) = \begin{cases} g_{k+1} h_{i_k} \omega_k(h) & \text{if } (j_{k+1}, j_k) = (1, 1), \\ g_{k+1} \omega_k(h) & \text{if } (j_{k+1}, j_k) = (1, -1), \\ h_{i_{k+1}}^{-1} g_{k+1} h_{i_k} \omega_k(h) & \text{if } (j_{k+1}, j_k) = (-1, 1), \\ h_{i_{k+1}}^{-1} g_{k+1} \omega_k(h) & \text{if } (j_{k+1}, j_k) = (-1, -1) \end{cases}$$

for $k = 1, \ldots, n-1$;

(2) $$\omega(h) = \begin{cases} g_{n+1}\omega_n(h) & \text{if } j_n = -1, \\ g_{n+1}h_{i_n}\omega_n(h) & \text{if } j_n = 1. \end{cases}$$

(1.13) Proposition. *Let H be a topological group of homeomorphisms of X, $x \in X$, $\omega \in {}_n^r\Omega(H)$, and $f, u \in \overset{r}{\times} H$. If $\omega_l(f, x) \notin \operatorname{supp}(u_{i_l})$ for $l = 1, \ldots, k \leqslant n-1$, then $\omega_l(f, x) = \omega(fu, x)$ for $l = 1, \ldots, k$ and*

$$\omega_{k+1}(fu, x) = \begin{cases} \omega_{k+1}(f, x) & \text{if } j_{k+1} = 1, \\ u_{i_{k+1}}^{-1}(\omega_{k+1}(f, x)) & \text{if } j_{k+1} = -1. \end{cases}$$

(1.14) Lemma. *Under the assumptions of Theorem (1.9), if x is not an isolated point of X, then there is $u \in \overset{r}{\times}(D_W)$ such that the points $\omega_l(hu, x)$, with $l \in \{1, \ldots, n\}$ satisfying $i_l = k$, are mutually different for all $k = 1, \ldots, r$.*

Proof. For $n = 1$, the lemma is trivial. Suppose it is true for n and take $\omega \in {}_{n+1}^{r}\Omega(H)$ By the inductive assumption there is $w \in \overset{r}{\times}(D_W)$ such that the points $\omega_l(fw, x)$, with $l = 1, \ldots, n$ satisfying $i_l = k$, are mutually different for all $k = 1, \ldots, r$.

Choosing w sufficiently close to the identity, we may assume additionally that $\omega_l(fw, x)$ belongs to W for $l = 1, \ldots, n+1$ and that x is proper for (ω, fw) (cf. (1.8)). Set $h = fw$ and let C be a neighbourhood of the identity in H such that $wu \in \overset{r}{\times} D$ for $u \in \overset{r}{\times} C$. Let U be a neighbourhood of $\omega_n(h, x)$ included in W and not containing $\omega_l(h, x)$ for $l = 1, \ldots, n-1$ satisfying $i_l = i_n$.

Consider $v = (v_1, \ldots, v_r) \in \overset{r}{\times} H$ such that $\operatorname{supp}(v_{i_n}) \subset U$ and $v_l = \mathrm{id}$ for $l \neq i_n$. Then, by Proposition (1.13), $\omega_l(hv, x) = \omega_l(h, x)$ for $l = 1, \ldots, n-1$ and we have the following possibilities.

(a) $(j_{n+1}, j_n) = (1, 1)$. Then $\omega_n(hv, x) = \omega_n(h, x)$ and

$$\omega_{n+1}(hv, x) = g_{n+1}h_{i_n}v_{i_n}(\omega_n(h, x)).$$

(b) $(j_{n+1}, j_n) = (-1, 1)$. Then $\omega_n(hv, x) = \omega_n(h, x)$,

$$\omega_{n+1}(hv, x) = v_{i_{n+1}}^{-1}h_{i_{n+1}}^{-1}g_{n+1}h_{i_n}v_{i_n}(\omega_n(h, x)),$$

and $\omega_n(h, x) \in \operatorname{Int}(\operatorname{supp}(h_{i_n}^{-1}g_{n+1}h_{i_n}))$, providing $i_n = i_{n+1}$, since x is proper for (ω, h).

(c) $(j_{n+1}, j_n) = (1, -1)$. Then

$$\omega_n(hv, x) = v_{i_n}^{-1}(\omega_n(h, x)), \quad \omega_{n+1}(hv, x) = g_{n+1}v_{i_n}^{-1}(\omega_n(h, x)),$$

and $\omega_n(h, x) \in \operatorname{Int}(\operatorname{supp}(g_{n+1}))$, providing $i_n = i_{n+1}$.

(d) $(j_{n+1}, j_n) = (-1, -1)$. Then $\omega_n(hv, x) = v_{i_n}^{-1}(\omega_n(h, x))$ and

$$\omega_{n+1}(hv, x) = v_{i_{n+1}}^{-1}h_{i_{n+1}}g_{n+1}v_{i_n}^{-1}(\omega_n(h, x)).$$

Now, it is easy to see that in each case we can choose $v_{i_n} \in C_U$ in such a way that $\omega_{n+1}(hv, x) \neq \omega_l(hv, x)$ for all $l = 1, \ldots, n$ satisfying $i_l = i_{n+1}$. This is possible in cases (b) and (c) for $i_n = i_{n+1}$ since we are working in the interior of the support of g_{n+1}.

Hence, the desired u can be chosen to be wv. ■

Proof of Theorem (1.9). If x is an isolated point of X, then the theorem is trivial. If x is not isolated, then by Lemma (1.14) there is $v \in \overset{r}{\times}(D_W)$ such that $\omega_l(fv, x)$, are pairwise different for those $1 \leqslant l \leqslant n$ for which $i_l = i_n$. We may also assume that $\omega_n(fv, x) \in W$. Put $h = fv$ and let U be a neighbourhood of $\omega_n(h, x)$ included in W and not containing $\omega_l(h, x)$ for those $1 \leqslant l \leqslant n-1$ for which $i_l = i_n$. Let C be a neighbourhood of the identity in H such that $v_{i_n}C \subset D$

Consider $w = (w_1, \ldots, w_r) \in \overset{r}{\times} H$ such that $\operatorname{supp}(w_{i_n}) \subset U$ and $w_l = \mathrm{id}$ for $l \neq i_n$. We have the following possibilities:

(a) $j_n = 1$. Then $\omega(hw, x) = g_{n+1}h_{i_n}w_{i_n}(\omega_n(hw, x)) = g_{n+1}h_{i_n}w_{i_n}(\omega_n(h, x))$.

(b) $j_n = -1$. Then $\omega(hw, x) = g_{n+1}(\omega_n(hw, x)) = g_{n+1}w_{i_n}^{-1}(\omega_n(h, x))$.

Now, it is easy to see that the image of the mapping $C_U \ni u \mapsto \omega(fvw_u, x) \in X$, where $w_u = (w_1, \ldots, w_r)$, $w_l = u$ if $l = i_n$ and $w_l = \mathrm{id}$ if $l \neq i_n$, has a nonempty interior. Since $vw_u \in \overset{r}{\times}(D_W)$ for $u \in C_U$, the theorem is proved. ■

2. Groups generated by families of homeomorphisms. Let ${}^r\Phi$, $r = 1, 2, \ldots, \infty$, be a free noncommutative group with r generators ("∞" denotes a countable set of generators) The elements of ${}^r\Phi$ may be represented by words $\varphi = A_{i_n}^{j_n} \ldots A_{i_1}^{j_1}$, where $i_k = 1, \ldots, r$, $j_k = \pm 1$, $n = 1, 2, \ldots$, such that $i_k \neq i_{k+1}$ if $j_k = -j_{k+1}$, $k = 1, \ldots, n-1$, and the "empty word" which will be denoted by $\mathbf{1}$.

The group operation in ${}^r\Phi$, for which $\mathbf{1}$ is the neutral element, is defined by the composition of the words:

$$(A_{i_n}^{j_n} \ldots A_{i_1}^{j_1}) \circ (A_{i'_m}^{j'_m} \ldots A_{i'_1}^{j'_1}) = A_{i_n}^{j_n} \ldots A_{i_1}^{j_1} A_{i'_m}^{j'_m} \ldots A_{i'_1}^{j'_1},$$

modulo the reduction of the given word by the formula $A_i^j A_i^{-j} = \mathbf{1}$ to the form in which $i_k \neq i_{k+1}$ if $j_k = -j_{k+1}$.

It is easy to see that $(A_{i_n}^{j_n} \ldots A_{i_1}^{j_1})^{-1} = A_{i_1}^{-j_1} \ldots A_{i_n}^{-j_n}$.

We have the natural inclusions ${}^r\Phi \hookrightarrow \Phi^{r+1}$, $r = 1, 2, \ldots$, which define the direct limit group $\varinjlim {}^r\Phi$ isomorphic with ${}^\infty\Phi$, so we can consider ${}^r\Phi$, $r = 1, 2, \ldots$, as a subgroup of ${}^\infty\Phi$. For $\varphi = A_{i_n}^{j_n} \ldots A_{i_1}^{j_1} \in {}^\infty\Phi$, we define:

(i) $|\varphi| = n$.

(ii) $|\varphi|_l = \sum_{i_k = l} |j_k|$, $l = 1, 2, \ldots$

(iii) $\Sigma_l\varphi = \sum_{i_k = l} j_k$, $l = 1, 2, \ldots$

Clearly, we assume $|\mathbf{1}| = |\mathbf{1}|_l = \Sigma_l\mathbf{1} = 0$ for $l = 1, 2, \ldots$

We have the following trivial formulas:

(a) $|\varphi| = \sum_{l=1}^{\infty} |\varphi|_l$.

(b) $|\varphi_1 \circ \varphi_2|_l \leqslant |\varphi_1|_l + |\varphi_2|_l$, $l = 1, 2, \ldots$

(c) $\Sigma_l(\varphi_1 \circ \varphi_2) = \Sigma_l \varphi_1 + \Sigma_l \varphi_2$, $l = 1, 2, \ldots$

Moreover, for $\varphi \in {}^{\infty}\Phi$ we have $\varphi \in {}^{r}\Phi$ if and only if $|\varphi|_l = 0$ for $l = r+1, r+2, \ldots$ Denote ${}^{r}\Phi \setminus \{\mathbf{1}\}$ by ${}^{r}\Phi^*$ and denote the set of all $\varphi \in {}^{r}\Phi$ such that $|\varphi| \leqslant n$ by ${}^{r}_{n}\Phi$, $r = 1, 2, \ldots, \infty$. It is easy to see that $({}^{r}_{n}\Phi)^{-1} = {}^{r}_{n}\Phi$ and $({}^{r}_{n}\Phi) \circ ({}^{r}_{m}\Phi) = {}_{n+m}^{r}\Phi$, so $\{{}^{r}_{n}\Phi\}_{n=0}^{\infty} = 0$ is a gradation of the group ${}^{r}\Phi$. For a group G and $r = 1, 2, \ldots$, we have the natural mapping ${}^{r}_{n}\Phi \ni \varphi \mapsto \varphi_G \in {}^{r}_{n}\Omega(G)$ defined by

$$(A_{i_n}^{j_n} \ldots A_{i_1}^{j_1})_G(h_1, \ldots, h_r) = h_{i_n}^{j_n} \ldots h_{i_1}^{j_1}$$

and $\mathbf{1}_G = \mathrm{id}$. If it is known what a group G is under consideration, $f = (f_1, \ldots, f_r) \in \overset{r}{\times} G$ or $f = (f_1, f_2, \ldots)$ is an infinite sequence of elements of G, $\varphi \in {}^{r}\Phi$, then we write $\varphi(f_1, \ldots, f_r)$ or $\varphi(f)$ instead of $\varphi_G(f_1, \ldots, f_r)$.

The mapping ${}^{r}\Phi \ni \varphi \mapsto \varphi(f_1, \ldots, f_r) \in G$ is a homomorphism of ${}^{r}\Phi$ onto the subgroup of G generated by the elements $f_1, \ldots, f_r$. It is clear by definitions that, for H being a topological group of homeomorphisms of X, $f \in \overset{r}{\times} H$, and $\varphi \in {}^{r}\Phi$, each $x \in X$ is proper for (φ_H, f). This fact and Theorem (1.9) imply the following theorem.

(2.1) Theorem. *Let H be a* SLT *topological group of homeomorphisms of a Hausdorff topological space X, let x be a nonisolated point of X, let D be a neighbourhood of the identity in H, let r, n be natural numbers, $f \in \overset{r}{\times} H$, and let W be a neighbourhood of the set $\{\varphi(f, x)\colon \varphi \in {}^{r}_{n}\Phi\}$. Then there is $u \in \overset{r}{\times}(D_W)$ such that for $\varphi \in {}^{r}_{n}\Phi$, $\varphi \neq \mathbf{1}$, we have $\varphi(fu, x) \neq x$. In other words, for $\varphi_1 \in {}^{r}_{n_1}\Phi$, $\varphi_2 \in {}^{r}_{n_2}\Phi$, $n_1 + n_2 \leqslant n$, $\varphi_1 \neq \varphi_2$, we have $\varphi_1(fu, x) \neq \varphi_2(fu, x)$.*

Proof. Let $\varphi_1, \ldots, \varphi_m$ be a sequence of all elements of ${}^{r}_{n}\Phi \setminus \{\mathbf{1}\}$. By Theorem (1.9) there is $u_1 \in \overset{r}{\times}(D_W)$ such that $\varphi_1(fu, x) \neq x$. We may also assume that $\{\varphi(fu_1, x)\colon \varphi \in {}^{r}_{n}\Phi\} \subset W$. Recurrently, if $u_k \in \overset{r}{\times}(D_W)$ is such that $\varphi_i(fu_k, x) \neq x$ for $1 \leqslant i \leqslant k < m$, and $\{\varphi(fu_k, x)\colon \varphi \in {}^{r}_{n}\Phi\} \subset W$, then for a neighbourhood C of the identity in H such that $w \in \overset{r}{\times} C$ implies $u_k w \in \overset{r}{\times} D$, $\{\varphi(fu_k w, x)\colon \varphi \in {}^{r}_{n}\Phi\} \subset W$ and $\varphi_i(fu_k w, x) \neq x$, $1 \leqslant i \leqslant k$, we can find, by Theorem (1.9), $w \in \overset{r}{\times}(C_W)$ such that $\varphi_{k+1}(fu_k w, x) \neq x$. Putting $u_{k+1} = u_k w$, we have then $u_{k+1} \in \overset{r}{\times}(D_W)$, $\varphi_i(fu_{k+1}, x) \neq x$ for $1 \leqslant i \leqslant k+1$ and $\{\varphi(fu_{k+1}, x)\colon \varphi \in {}^{r}_{n}\Phi\} \subset W$. ■

The theorem above allows us to distinguish a homeomorphism $\varphi_1(fu)$ from $\varphi_2(fu)$, for $\varphi_1, \varphi_2 \in {}^{r}_{u}\Phi$, $\varphi_1 \neq \varphi_2$, and some $u \in \overset{r}{\times} H$ arbitrarily close to the identity,

just looking at the images of x. An analogous statement for arbitrary lengths of the "words" φ_1 and φ_2 is the following.

(2.2) Theorem. *Let H be a* SLT *topological group of homeomorphisms of a Hausdorff topological space X. Assume additionally that H is a Baire space and X is separable. Let $x_1, x_2 \ldots$ be a sequence of nonisolated points of X, $D_1, D_2, \ldots$ a sequence of neighbourhoods of the identity in H, $f = (f_1, f_2, \ldots)$ a sequence of elements of H. Then there is a sequence $u = (u_1, u_2, \ldots)$, where $u_n \in D_n$, $n = 1, 2, \ldots$, such that $\varphi(fu, x_n) \neq x_n$ for all $\varphi \in {}^{\infty}\Phi$, $\varphi \neq \mathbf{1}$, $n = 1, 2, \ldots$*

In particular, the homeomorphisms $f_1 u_1, f_2 u_2, \ldots$ freely generate a subgroup in H.

Proof. Since H is separable, we can assume that the sequence $x_1, x_2, \ldots$ is dense in the set X_0 of all nonisolated points of X. For $\varphi \in {}^{1}\Phi$, $\varphi \neq \mathbf{1}$, and for $x \in X_0$ set

$$A_\varphi(x) = \{w \in D_1\colon \varphi(f_1 w, x) \neq x\}.$$

Obviously, $A_\varphi(x)$ is an open and, by (2.1), dense subset of D_1. Hence

$$B_1 = \bigcap_{\varphi \in {}^{1}\Phi^*} \bigcap_{n=1}^{\infty} A_\varphi(x_n)$$

is of the second category in D_1 and we can find $u_1 \in B_1$. Inductively, suppose $\varphi(f_1 u_1, \ldots, f_r u_r, x_n) \neq x_n$ for all $\varphi \in {}^{r}\Phi^*$, $n = 1, 2, \ldots$ Take $\varphi \in {}^{r+1}\Phi \setminus {}^{r}\Phi$ and put $k = |\varphi|_{r+1}$. There is $\omega \in {}^{1}_{k}\Omega(H)$ such that for all $h \in H$ we have

$$\omega(h) = \varphi(f_1 u_1, \ldots, f_r u_r, h) = g_{k+1} h^{j_k} g_k \ldots h^{j_1} g_1,$$

where $g_i = \varphi_i(f_1 u_1, \ldots, f_r u_r)$ for some $\varphi_i \in {}^{r}\Phi$, $i = 1, \ldots, k+1$, and $\varphi_i \neq \mathbf{1}$ for $i = 2, \ldots, k$. Then, by the inductive assumption, $\mathrm{supp}(g_i) \supset X_0$ for $i = 2, \ldots, k$ and, for every nonisolated point x of X and every $h \in H$, the point x is proper for (ω, h). By Theorem (1.9) we conclude now that the set

$$A_\varphi(x) = \{w \in D_{r+1}\colon \varphi(f_1 u_1, \ldots, f_r u_r, f_{r+1} w, x) \neq x\}$$

is dense (and obviously open) in D_{r+1}, and hence the set

$$B_{r+1} = \bigcap_{{}^{r+1}\Phi \setminus {}^{r}\Phi} \bigcap_{n=1}^{\infty} A_\varphi(x_n)$$

is of the second category in D_{r+1}.

Taking $u_{r+1} \in B_{r+1}$, we get $\varphi(f_1 u_1, \ldots, f_{r+1} u_{r+1}, x_n) \neq x_n$ for all $\varphi \in {}^{r+1}\Phi^*$, $n = 1, 2, \ldots$, that proves the inductive step. ■

An immediate consequence of the theorem above (cf. (1.2)) is the following corollary.

(2.3) Corollary. *Let X be a separable C^k-manifold and let $D_1, D_2, \ldots$ be a sequence of neighbourhoods of the identity in* $\mathrm{Diff}^k(X)$ *with the Whitney C^k-topology,*

$k = 1, 2, \ldots$ *Then for each sequence* $f = (f_1, f_2, \ldots)$ *of elements of* $\mathrm{Diff}^k(X)$, *there is a sequence* $u = (u_1, u_2, \ldots)$, *where* $u_i \in D_i$, $i = 1, 2, \ldots$, *such that* $\varphi(fu, x) \neq x$ *for all* $\varphi \in {}^{\infty}\Phi^*$ *and all* x *from a subset of* X *of the second category.*

In particular, the diffeomorphisms $f_1u_1, f_2u_2, \ldots$ *freely generate a subgroup* in $\mathrm{Diff}^k(X)$.

(2.4) Remark. It is obvious that, looking for $u \in \overset{r}{\times} D$ such that $\varphi(fu, x) \neq x$ for all $\varphi \in {}^r\Phi^*$, (in order to get a free group of homeomorphisms), we need a sort of the Baire property of H. We have used exactly the Baire property in (2.2) since it fits very well to the case of diffeomorphism groups, but other versions of Theorem (2.2) may also be derived.

An easy example showing that there is no version of Theorem (2.2) for H without any sort of Baire property is the following. Let X be the space of all sequences from $\{0, 1\}^N$, with almost all zeros, equipped with the metric

$$\varrho\big((a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}\big) = \sum_{n=1}^{\infty} |a_n - b_n|/2^n.$$

Let H be the groups of permutations σ of X such that $(\sigma(a))_n = a_n$ for all $a = (a_n)_{n=1}^{\infty} \in X$ and all $n \geqslant k(\sigma)$. Taking the topology of uniform convergence in H, we get a topological group of homeomorphisms of X. To see that H is SLT, consider a neighbourhood $W = \{a' \in X\colon \varrho(a, a') < \varepsilon_1\}$ of $a = (a_n)_{n=1}^{\infty}$, where ε_1 is irrational, and a neighbourhood

$$D = \{\sigma \in H\colon \sum_{n=1}^{\infty} |(\sigma(c))_n - c_n|/2^n < \varepsilon_2 \text{ for all } c = (c_n)_{n=1}^{\infty} \in X\}$$

of the identity in H. We shall show that $D_W(a)$ contains the neighbourhood

$$W' = \{a' \in X\colon \varrho(a, a') < \min(\varepsilon_1, \varepsilon_2)\}.$$

Indeed, for $a' = (a'_n)_{n=1} \in W'$ define $\sigma \in H$ by

$$\sigma(x)_n = \begin{cases} (x_n + a'_n - a_n) \pmod 2 & \text{if } x \in W, \\ x_n & \text{if } x \notin W \end{cases}$$

for all $x = (x_n)_{n=1}^{\infty} \in X$. Since W and $X \setminus W$ are both open (ε_1 is irrational), σ is really a homeomorphism, $\sigma \in D_W$, and $\sigma(a) = a'$.

On the other hand, since every element of H is of finite order, H does not include any free subgroup.

(2.5) Theorem. *Let H be a completely metrizable group of homeomorphisms of a Hausdorff topological space X. Suppose that R is a subset of H satisfying the following conditions*:

(1) *if* $f, g \in H$, $\mathrm{supp}(f) \subset W$ and $f|_W = g|_W$, *then* $f \in R$ *implies* $g \in R$,

(2) *there is a family* $\{W_i\}_{i=1}^{\infty}$ *of disjoint open subsets of* X *such that for each neighbourhood D of the identity in H and each* $r = 1, 2, \ldots$ *there are* $f_1, f_2, \ldots, f_r \in D_{W_r}$ *satisfying* $\varphi(f_1, \ldots, f_r) \in R$ *for all* $\varphi \in {}^r_r\Phi^*$.

Then there is a mapping $\gamma\colon [0, 1) \to H$, $\gamma(0) = \mathrm{id}$, *such that* $\{\gamma(t)\}_{t \in (0,1)}$ is *a set of free generators of a subgroup of H which is included in* $R \cup \{\mathrm{id}\}$.

If, additionally, the subgroups H_{W_i}, $i = 1, 2, \ldots$, *are locally arcwise connected, then γ can be taken continuous.*

Before proving this theorem, note that taking $R = H \setminus \{\mathrm{id}\}$ for a SLT topological group of homeomorphisms H, we get Corollary (2.6). Theorem (2.5) will also be used in § 3, where, for groups of diffeomorphisms, we will take R to be the set of diffeomorphisms which embed in no flow.

(2.6) Corollary. (a) *Every completely metrizable* SLT *topological group of homeomorphisms of a nondiscrete Hausdorff topological space X includes a free subgroup with continuum generators.*

(b) *For each* $k = 1, 2, \ldots, \infty$, *the group* $\mathrm{Diff}_c^k(X)$ *of compactly supported* C^k-*diffeomorphisms of a* C^k-*manifold* X (*and hence the group* $\mathrm{Diff}^k(X)$) *includes a nontrivial arcwise connected* (*with respect to the Whitney* C^k-*topology*) *free subgroup.*

Proof. To prove (a) it suffices to put $R = H \setminus \{\mathrm{id}\}$ and to make use of Theorems (2.1) and (2.5).

To prove (b), consider an embedding $j\colon \mathrm{Diff}_c^k(\bar{B}) \to \mathrm{Diff}_c^k(X)$, where B is an open ball in $\boldsymbol{R}^{\dim(X)}$, given by a relatively compact coordinate chart in X. Let $\{W_i\}_{i=1}^{\infty}$ be a family of disjoint open balls in B. Since $\mathrm{Diff}_c^k(\bar{B})$ is a completely metrizable SLT topological group of homeomorphisms of B, and since $(\mathrm{Diff}_c^k(\bar{B}))|_{W_i} \approx \mathrm{Diff}_c^k(\boldsymbol{R}^n)$ is locally arcwise connected (see (1.3)), putting $R = \mathrm{Diff}_c^k(\bar{B}) \setminus \{\mathrm{id}\}$ we can use Theorem (2.5) to obtain a continuous mapping $\gamma\colon [0, 1) \to \mathrm{Diff}_c^k(\bar{B})$ such that $\gamma(0) = \mathrm{id}$ and $\{\gamma(t)\}_{t \in (0,1)}$ is a set of free generators of a subgroup in $\mathrm{Diff}_c^k(\bar{B})$. Then, $\{j(\gamma(t))\}_{t \in (0,1)}$ generates an arcwise connected free subgroup in $\mathrm{Diff}_c^k(X)$. ∎

Proof of Theorem (2.5). Let ϱ be a complete metric on H. Enumerate all increasing finite sequences of rationals from $(0, 1)$ of even length by $(c_i)_{i=1}^{\infty}$. Write $c_i = (p_1^i, \ldots, p_{2k(i)}^i)$. We can construct recurrently a sequence of mappings

$$\gamma_n\colon [0, 1) \to H_{W_n}, \quad n = 1, 2, \ldots, \quad \gamma(0) = \mathrm{id},$$

such that

(i) $\sup_{t \in (0,1)} \{\varrho(\gamma_n(t) \ldots \gamma_1(t), \gamma_{n-1}(t) \ldots \gamma_1(t))\} < 1/2^n$,

(ii) $\gamma_n(t) = f_{n_l}$ for $t \in [p_{2l-1}^n, p_{2l}^n]$, $l = 1, \ldots, k(n)$, where $f_{n_1}, \ldots, f_{n_{k(n)}}$ are elements of H_{W_n} such that $\varphi(f_{n_1}, \ldots, f_{n_{k(n)}}) \in R$ for all $\varphi \in {}^{k(n)}_{n}\Phi^*$.

Let $\gamma(t)$ be the limit point in H of the sequence $\Gamma_n(t) = \gamma_n(t) \ldots \gamma_1(t)$. Since the topology of H is finer than the topology of pointwise convergence,

$$\mathrm{supp}(\gamma(t)) \subset \bigcup_{i=1}^{\infty} W_i$$

and $\gamma(t)|_{W_i} = \gamma_i(t)$. We shall show that $\{\gamma(t)\}_{t\in(0,1)}$ are free generators of a subgroup in H included in $R\cup\{\mathrm{id}\}$.

It suffices to prove that, for $t_1, \ldots, t_n \in (0, 1)$, $t_1 < \ldots < t_n$, we have

$$\gamma(t_{r(1)})^{m_1} \ldots \gamma(t_{r(s)})^{m_s} \in R$$

for all $m_1, \ldots, m_s \in \mathbf{Z}\setminus\{0\}$ and for all $r(1), \ldots, r(s) \in \{1, \ldots, n\}$, $r(i) \neq r(i+1)$. Take $j \geqslant |m_1|+\ldots+|m_s|$ such that $k(j) = n$ and $t_i \in [p^j_{2i-1}, p^j_{2i}]$, $i = 1, \ldots, n$.

Then

$$(\gamma(t_{r(1)})^{m_1} \ldots \gamma(t_{r(s)})^{m_s})|_{W_j} = \gamma_j(t_{r(1)})^{m_1} \ldots \gamma_j(t_{r(s)})^{m_s} = f^{m_1}_{j_{r(1)}} \ldots f^{m_s}_{j_{r(s)}} = \varphi(f_{j_1}, \ldots, f_{j_n}),$$

where φ is an element of ${}^n_j\Phi^*$. The right hand term belongs to R by the construction, so $\gamma(t_{r(1)})^{m_1} \ldots \gamma(t_{r(s)})^{m_s} \in R$ by assumption (1).

If the subgroups H_{W_n}, $n = 1, 2, \ldots$, are locally arcwise connected, then the mappings γ_n (and hence Γ_n) can be chosen continuous. Since the mappings Γ_n are uniformly convergent, the mapping γ is also continuous and the group generated by $\{\gamma(t)\}_{t\in(0,1)}$ is arcwise connected. ∎

3. Diffeomorphisms outside flows. One-parameter subgroups of a topological group G are defined to be continuous homomorphisms from the additive group of reals into G. One-parameter subgroups of groups of diffeomorphisms are also called *flows*. It is well known that each flow in $\mathrm{Diff}^k_c(X)$, $k = 1, \ldots, \infty$, (with the Whitney C^k-toplogy) is generated by a C^{k-1}-vector field on X (see [1]). It is also well-known that every neighbourhood of the identity in $\mathrm{Diff}^k_c(X)$ contains diffeomorphisms which embed in no flow. This is one of the main differences between the finite-dimensional Lie groups and the groups of diffeomorphisms (see [2]). Since each flow generates an arcwise connected commutative subgroup in $\mathrm{Diff}^k_c(X)$, Corollary (2.6) implies the following.

(3.1) THEOREM. *For X being a C^k-manifold, $k = 1, 2, \ldots, \infty$, the group $\mathrm{Diff}^k_c(X)$ includes a nontrivial arcwise connected subgroup which contains no nontrivial flow.*

(3.2) Remark. Theorem (3.1) shows that $\mathrm{Diff}^\infty_c(X)$ includes arcwise connected subgroups which are not Lie subgroups in any sense, i.e. that the theorem of Yamabe [7] is no longer valid for such "infinite dimensional Lie groups".

Our aim in this section is to prove a stronger version of Theorem (3.1), namely:

(3.3) THEOREM. *Given a C^k-manifold X, $k = 1, 2, \ldots, \infty$, there is a continuous mapping γ: $[0, 1) \to \mathrm{Diff}^k_c(X)$, $\gamma(0) = \mathrm{id}$, such that $\{\gamma(t)\}_{t\in(0,1)}$ is a set of free generators of a subgroup of $\mathrm{Diff}^k_c(X)$ which contains only (except for the identity, of course) diffeomorphisms which embed in no flow.*

(3.4) Remark. Theorem (3.3) is not valid for groups of homeomorphisms. For example, every orientation-preserving homeomorphism of the interval (0, 1) embeds in a one-parameter subgroup of Homeo(0, 1) with the compact-open topology.

Let us show this first for homeomorphisms f with no fixed points. Let $x_0 \in (0, 1)$. For $t \in [0, 1]$, set $x_t = x_0 + t(f(x_0) - x_0)$. Consider now the mapping φ: $\mathbf{R} \to (0, 1)$ defined by $\varphi(t) = f^{E(t)}(x_{\{t\}})$, where $E(t)$ is the integer part of t and $\{t\} = t - E(t)$. It is easy to see that, since f has no fixed points, φ is a homeomorphism. Define now ψ_s: $(0, 1) \to (0, 1)$ by putting $\psi_s(\varphi(t)) = \varphi(t+s)$. One can easily verify that $s \mapsto \psi_s$ is a flow in Homeo(0, 1) and that $\psi_1 = f$.

Now let h be an arbitrary orientation-preserving homeomorphism of the interval (0, 1). Let $\{I_n\}_{n\in A}$ be the set of all maximal h-invariant open subintervals of (0, 1) consisting of nonstationary points of h. We still know that $h|_{I_n}$ embeds in a flow h^t_n in $\mathrm{Diff}^k(I_n)$ such that $h^1_n = h|_{I_n}$. Hence, h embeds in the flow

$$h^t(x) = \begin{cases} x & \text{if } x \notin \bigcup_{n\in A} I_n, \\ h^t_n(x) & \text{if } x \in I_n. \end{cases}$$

Our approach to the proof of Theorem (3.3) will start with an investigation of the group $\mathrm{Diff}^k(J)$, $k = 1, 2, \ldots, \infty$, of diffeomorphisms of a closed interval J, i.e. the group of those homeomorphisms of J which can be extended to C^k-diffeomorphisms of a larger open interval, with the natural C^k-topology.

(3.5) DEFINITION. A diffeomorphism $g \in \mathrm{Diff}^k([a, b])$ such that

(i) g has no fixed points in (a, b),

(ii) g^n embeds in no one-parameter subgroup of $\mathrm{Diff}^k([a, b])$ for all $n \in \mathbf{Z}\setminus\{0\}$,

will be called a *Kopell-diffeomorphism* (*K-diffeomorphism*).

Observe that the set of all K-diffeomorphisms is invariant with respect to the action of inner automorphisms of $\mathrm{Diff}^k([a, b])$. Following Kopell [3], we shall prove:

(3.6) THEOREM. *Each neighbourhood of the identity in* $\mathrm{Diff}^k([a, b])$,

$$k = 1, 2, \ldots, \infty,$$

contains a K-diffeomorphism.

Proof. We can take $[a, b]$ to be $[0, 1] = J$. Let D be a neighbourhood of the identity in $\mathrm{Diff}^k(J)$. Take a C^k-vector field V on J which is of the form $\lambda_1 t\partial/\partial t$ in a neighbourhood of 0 and of the form $\lambda_2(t-1)\partial/\partial t$ in a neighbourhood of 1, $\lambda_1 < 0$, $\lambda_2 > 0$; V does not vanish in (0, 1) and is so close to 0 in the C^k-topology that $h \doteq \mathrm{Exp}(V) \in D$, where $t \mapsto \mathrm{Exp}(tV)$ is the flow generated by V.

Take $x_0 \in (0, 1)$ such that $\mathrm{Exp}(tV)(x) = e^{t\lambda_1}x$ for $x \in [0, x_0]$ and $t \geqslant 0$. Choose $\hat\beta \in \mathrm{Diff}^k([h(x_0), x_0])$ having the C^k-contact with the identity at x_0 and $h(x_0)$, $\hat\beta \neq \mathrm{id}$.

Then there is exactly one $\beta \in \mathrm{Homeo}(J)$ which satisfies $\beta|_{[h(x_0), x_0]} = \hat\beta$, $\beta|_{[x_0, 1]} = \mathrm{id}$, and $\beta h^n(x) = h^n\beta(x)$ for $x \in [h(x_0), x_0]$, $n \in \mathbf{N}$. Put $g = \beta^{-1}h\beta$. It is easy to see that $g \in \mathrm{Diff}^k(J)$, $g|_{[0, x_0]} = h|_{[0, x_0]}$, $g|_{[x_0, h^{-1}(x_0)]} = \hat\beta^{-1}h|_{[x_0, h^{-1}(x_0)]}$, $g|_{[h^{-1}(x_0), 1]} = h|_{[h^{-1}(x_0), 1]}$.

Assume that $\hat{\beta}$ is so close to the identity that $g \in D$. We shall show that g is a K-diffeomorphism.

First observe that the linear local diffeomorphism defined in a neighbourhood of 0 by $f(x) = e^{\lambda}x$ embeds in a unique local flow as its value at 1, namely, embeds in the flow $\varphi_t(x) = e^{\lambda t}x$. Indeed, suppose that ψ_t is a local flow of C^1-diffeomorphisms such that $\psi_1 = f$ in a neighbourhood of 0. We can assume that $\lambda \leqslant 0$, considering f or f^{-1}. Since for all $n = 1, 2, \ldots$ and $t \in [-1, 1]$ we have $f^{-n}\psi_t f^n = \psi_t$ in a neighbourhood U of 0, we get $e^{-n\lambda}\psi_t(e^{n\lambda}x) = \psi_t(x)$ for $n = 1, 2, \ldots, t \in [-1, 1]$, and $x \in U$. Differentiating with respect to x, we obtain

$$\psi_t'(e^{n\lambda}x) = \psi_t'(x) \quad \text{for } n = 1, 2, \ldots,,$$

and hence

$$\psi_t'(0) = \psi_t'(x) \quad \text{for } t \in [-1, 1] \text{ and } x \in U,$$

so that $\psi_t(x) = a_t x$ and it is easy to see now that $a_t = e^{\lambda t}$.

Now suppose that g^n, for a natural n, embeds in a C^k-flow. Since g^n equals h^n (and thus is linear) in a neighbourhood of 1, we can apply the result proved to conclude that this flow is unique in a neighbourhood of 1. But $g^n(x) < x$ for $x \in (0, 1)$, so this flow is unique on $[0, 1]$, and hence it has to be equal to $\beta^{-1}\mathrm{Exp}(ntV)\beta$.

Similarly, by the uniqueness in a neighbourhood of 0, we conclude that

$$\beta^{-1}\mathrm{Exp}(ntV)\beta(x) = \mathrm{Exp}(ntV)(x) \quad \text{for } x \in [0, x_0],\ t \geqslant 0.$$

Hence

$$e^{nt\lambda_1}\beta(x) = \beta(e^{nt\lambda_1}x) \quad \text{for } x \in [0, x_0],\ t \geqslant 0.$$

Differentiating with respect to x, we get

$$\beta'(x) = \beta'(e^{nt\lambda_1}x) \quad \text{for } x \in (0, x_0],\ t \geqslant 0,$$

that implies $\beta'(x) = \beta'(x')$ for $x', x \in [0, x_0]$. But β' is not constant on $[h(x_0), x_0]$ since $\beta|_{[h(x_0), x_0]} = \hat{\beta}$ and $\hat{\beta} \neq \mathrm{id}$, a contradiction.

To use K-diffeomorphisms in constructing the mapping γ from Theorem (3.3), we shall need a modification of Theorem (2.1). Let us introduce a notation.

For homeorphisms h_1, h_2 of a topological space X, for $U \subset X$, and for $x \in X$, we denote by $U_n^x(h_1, h_2)$ the set of all $\varphi = A_{i_k}^{j_k} \ldots A_{i_1}^{j_1} \in {}_n^2\Phi$ such that for all $l = 1, \ldots, k-1$ we have $h_{i_l}^{j_l} \ldots h_{i_1}^{j_1}(x) \in U$ providing $i_{l+1} = 1$.

We shall use the convention $\mathbf{1} \in U_n^x(h_1, h_2)$ for all natural n, all $x \in X$, and all h_1, h_2.

We shall omit an easy proof of the following technical proposition.

(3.7) Proposition. (a) *If* $\varphi_2 \in U_n^x(h_1, h_2)$ *and* $\varphi_1 \in U_m^{\varphi_1(h_1, h_2, x)}(h_1, h_2)$, *then* $\varphi_1 \circ \varphi_2 \in U_{n+m}^x(h_1, h_2)$.

(b) *If* $\varphi \in U_n^x(h_1, h_2)$, *then* $\varphi^{-1} \in (h_1(U) \cup h^{-1}(U))_n^{\varphi(h_1, h_2, x)}(h_1, h_2)$.

(c) *If* U *is open, then* $U_n^x(h_1, h_2) \subset U_n^x(h_1, h_2')$ *for* h_2' *sufficiently close to* h_2 *in the topology of the pointwise convergence.*

(d) *If* U *is open, there is a neighbourhood* W *of* x *such that* $h_1' h_1^{-1}|_U = \mathrm{id}$ *implies* $U_n^x(h_1, h_2) = U_n^x(h_1', h_2)$, *and* $\varphi(h_1, h_2, y) = \varphi(h_1', h_2, y)$ *for* $\varphi \in U_n^x(h_1, h_2)$, $y \in W$.

To each $\varphi \in {}^r\Phi$ and each element h of a group G we may assign $\hat{\varphi}_h \in {}_{|\varphi| = |\varphi|_1}^{r-1}\Omega(G)$, putting $\hat{\varphi}_h(f_1, \ldots, f_{r-1}) = \varphi(h, f_1, \ldots, f_{r-1})$. If h is a homeomorphism of a topological space, then we denote by $\mathrm{supp}_k(h)$ the set $\bigcap_{i=1}^{k} \mathrm{supp}(h^i)$.

Immediately by the definitions, we get the following.

(3.8) Proposition. *If* $\varphi \in U_n^x(h, g)$, *and* $U \subset \mathrm{supp}_n(h)$, *then* x *is proper for* $(\hat{\varphi}_h, g)$.

(3.9) Theorem. *Let* H *be a* SLT *topological group of homeomorphisms of a Hausdorff topological space* X, $h, g \in H$, *let* U *be an open* h*-invariant subset of* $\mathrm{supp}_{2n}(h)$, *and let* x *be a nonisolated point of* X *such that* $x \notin U$ *or* x *is not a periodic point of* h *with the period* $\leqslant 2n$. *Let* D *be a neighbourhood of the identity in* H, *and let* W *be a nieghbourhood of the set* $\{\varphi(h, g, x) \colon \varphi \in {}_{2n}^{2}\Phi\}$. *Then there is* $u \in D_W$ *such that* $\varphi_1(h, gu, x) \neq \varphi_2(h, gu, x)$ *for* $\varphi_1, \varphi_2 \in U_n^x(h, gu)$, $\varphi_1 \neq \varphi_2$.

Proof. Put $B = \{w \in D \colon \varphi(h, gw, x) \in W \text{ for all } \varphi \in {}_{2n}^{2}\Phi\}$. B is a neighbourhood of the identity in H. It suffices to prove the following inductive step:

If there is $w \in B_W$ such that for some $\varphi_1, \ldots, \varphi_r \in U_n^x(h, gw)$ the points $\varphi_i(h, gw, x)$, $i = 1, \ldots, r$, are pairwise different and $\{\varphi_1, \ldots, \varphi_r\} \neq U_n^x(h, gw)$, then there is $u \in B_W$ such that for some $\varphi_1', \ldots, \varphi_{r+1}' \in U_n^x(h, gu)$ the points $\varphi_i'(h, gu, x)$, $i = 1, \ldots, r+1$, are pairwise different.

Assume that we have $\varphi_{r+1} \in U_n^x(h, gw)$ different from φ_i, $i = 1, \ldots, r$, such that $\varphi_{r+1}(h, gw, x) = \varphi_j(h, gw, x)$ for some $j \in \{1, \ldots, r\}$. Let C be such a neighbourhood of the identity in H that for each $w' \in C$ we have $ww' \in B$,

$$U_n^x(h, gww') \supset U_n^x(h, gw),$$

and the points $\varphi_i(h, gww', x)$ are pairwise different for $i = 1, \ldots, r$ and

$$i = 1, \ldots, j-1, j+1, \ldots, r+1.$$

Since $h(U) = U$, by Proposition (3.7) (a) and (b) we have

$$\varphi = \varphi_j^{-1} \circ \varphi_{r+1} \in U_{2n}^x(h, gw),$$

$\varphi(h, gw, x) = x$, $\varphi \neq 1$, and $\hat{\varphi}_h \in {}_{|\varphi|_2}^{1}\Omega(H)$. If $|\varphi|_2 = 0$, then $\varphi(h, gw) = h^{\pm|\varphi|_1}$, that is impossible since $x \notin U$ or x is not a periodic point of h with the period $\leqslant 2n$. Hence, $|\varphi|_2 > 0$, and since $U \subset \mathrm{supp}_{2n}(h)$, x is proper for $(\hat{\varphi}_h, gw)$ by Proposition (3.8). It is easy to see that $(\hat{\varphi}_h)_k(gw, x) = \eta_k(h, gw, x)$ for some $\eta_k \in {}_{2n}^{2}\Phi$, $k = 1, \ldots, |\varphi|_2$, so by Theorem (1.9) there is $w' \in C_W$ such that $\hat{\varphi}_h(gww', x) \neq x$. It suffices to put now $u = ww'$, and $\varphi_i' = \varphi_i$, $i = 1, \ldots, r+1$. ■

Now, let $I = (a, b)$ be an open interval, and D a neighbourhood of the identity in $\mathrm{Diff}^k(I)$, $k = 1, 2, \ldots, \infty$, and $x_0 \in (a, b)$. Let $f \in D$ be such that f can be trivially extended to a C^k-diffeomorphism of $\boldsymbol{R}$ (i.e. the mapping $g\colon \boldsymbol{R} \to \boldsymbol{R}$, such that $g(x) = x$ if $x \notin (a, b)$, and $g(x) = f(x)$ if $x \in (a, b)$, is a C^k-diffeomorphism of $\boldsymbol{R}$), $f|_{(a, x_0)} = \mathrm{id}$ and $f|_{(x_0, b)}$ has no fixed points. Put $U = (x_0, b)$. By Theorem (3.9), for each natural n there is $g \in D$ such that g can trivially be extended to a C^k-diffeomorphism of $\boldsymbol{R}$, and that $\varphi_1(f, g, x_0) \neq \varphi_2(f, g, x_0)$ for $\varphi_1, \varphi_2 \in U_n^{x_0}(f, g)$, $\varphi_1 \neq \varphi_2$.

By Proposition (3.7)(d), there is a neighbourhood W of x_0 such that $hf^{-1}|_U = \mathrm{id}$ implies $U_n^{x_0}(f, g) = U_n^{x_0}(h, g)$ and $\varphi(f, g, y) = \varphi(h, g, y)$ for $\varphi \in U_n^{x_0}(f, g)$ and $y \in W$. We may take then y_0, $y_0 < x_0$, so close to x_0 that, for $h \in D$, $\mathrm{supp}(h) \subset [y_0, b)$, $hf^{-1}|_U = \mathrm{id}$, we have $\varphi_1(h, g, x_0) > \varphi_2(h, g, y_0)$ if $\varphi_1(h, g, x_0) > \varphi_2(h, g, x_0)$. Moreover, by Theorem (3.6) h may be chosen such that there is a closed interval $J \subset (y_0, x_0)$ such that $h(J) = J$ and $h|_J$ is a K-diffeomorphism. Thus we get the following.

(3.10) Corollary. *For each neighbourhood D of the identity in $\mathrm{Diff}^k(I)$, where $I = (a, b)$ is an open interval, $k = 1, 2, \ldots, \infty$, and for each natural n there are $y_0, x_0 \in I$, $y_0 < x_0$, and $h, g \in D$ satisfying the following conditions:*

(1) *g and h can be trivially extended to C^k-diffeomorphisms of $\boldsymbol{R}$;*

(2) *$h|_{(a, y_0)} = \mathrm{id}$, and x_0 is the only fixed point of $h|_{[x_0, b)}$;*

(3) *there is a closed interval $J \subset (y_0, x_0)$ such that $h(J) = J$ and $h|_J$ is a K-diffeomorphism;*

(4) *for each φ_1 $\varphi_2 \in (x_0, b)_n^{x_0}(h, g)$, $\varphi_1 \neq \varphi_2$, we have*

$$\varphi_1(h, g, x_0) \neq \varphi_2(h, g, x_0),$$

and $\varphi_1(h, g, x_0) > \varphi_2(h, g, y_0)$ if $\varphi_1(h, g, x_0) > \varphi_2(h, g, x_0)$.

(3.11) Lemma. *Let $N > k$ be natural numbers. Let $g, h \in \mathrm{Diff}^s(I)$, $s = 1, 2, \ldots, \infty$, satisfy conditions (1)–(4) from (3.10) for $n = 4^k N$, and let*

$$\varphi(h, g) = (g_1 h^{r_1} g_1^{-1}) \ldots (g_l h^{r_l} g_l^{-1}),$$

where

(i) $\sum_{i=1}^{l} |r_i| \leqslant N - k$,

(ii) *$g_i = \varphi_i(h, g)$ for some $\varphi_i \in (x_0, b)_n^{x_0}(h, g)$, $i = 1, \ldots, l$.*

Suppose also that there is no closed interval $K \subset I$ such that $\varphi(h, g)(K) = K$ and $\varphi(h, g)|_K$ is a K-diffeomorphism.

Then $\varphi(h, g) = (g_1' h^{r_1'} (g_1')^{-1}) \ldots (g_m' h^{r_m'} (g_m')^{-1})$, where

(iii) *$\sum_{i=1}^{m} |r_i'| \leqslant N - (k+1)$, and*

(iv) *$g_i' = \varphi_i'(h, g)$ for some $\varphi_i' \in (x_0, b)_{4n}^{x_0}(h, g)$, $i = 1, \ldots, m$.*

Proof. Choose i_0 such that $g_{i_0}(x_0) \leqslant g_i(x_0)$, $i = 1, \ldots, l$. By assumptions, $g_i(x_0) = g_j(x_0)$ implies $g_i = g_j$, and $g_{i_0}(x_0) < g_i(x_0)$ implies $g_{i_0}(x_0) < g_i(y_0)$, so that

$$\varphi(h, g)_{|(g_{i_0}(y_0), g_{i_0}(x_0))} = g_{i_0} h^{\sum_{g_{i_0} = g_i} r_i} g^{-1}_{i_0|(g_{i_0}(y_0), g_{i_0}(x_0))}.$$

If $\sum_{g_{i_0} = g_i} r_i \neq 0$, then $\varphi(h, g)|_{g_{i_0}(J)}$ is a K-diffeomorphism. Thus $\sum_{g_{i_0} = g_i} r_i = 0$, and $\varphi(h, g)$ can be written in the form

$$((g_{i_0} h^{p_1} g_{i_0}^{-1})(g_{i_1} h^{r_{i_1}} g_{i_1}^{-1})(g_{i_0} h^{-p_1} g_{i_0}^{-1})) \ldots ((g_{i_0} h^{p_m} g_{i_0}^{-1})(g_{i_m} h^{r_{i_m}} g_{i_m}^{-1})(g_{i_0} h^{-p_m} g_{i_0}^{-1})),$$

where $(i_1, \ldots, i_m)$ is formed from $(1, \ldots, l)$ by removing the elements with the indexes i such that $g_i = g_{i_0}$, and where $p_j = \sum_{\substack{i < i_j \\ g_i = g_{i_0}}} r_i$. In particular $|p_j| \leqslant \sum_{i=1}^{l} |r_i| \leqslant N$.

Putting $g_j' = g_{i_0} h^{p_j} g_{i_0}^{-1} g_{i_j}$ and $r_j' = r_{i_j}$, $j = 1, \ldots, m$, we get the required form for $\varphi(h, g)$.

It is easy to see that $\sum_{i=1}^{m} |r_i'| = \sum_{\substack{i=1 \\ g_i \neq g_{i_0}}}^{l} |r_i| \leqslant N - (k+1)$. Put $U = (x_0, b)$. Since $\varphi_{i_j} \in U_n^{x_0}(h, g)$. $\varphi_{i_0}(h, g, x_0) = g_{i_0}(x_0) < g_{i_j}(x_0) = \varphi_{i_j}(h, g, x_0)$ for $j = 1, \ldots, m$, and since $\varphi_{i_0}^{-1} \in U_n^{g_{i_0}(x_0)}(h, g)$, we have $\varphi_{i_0}^{-1} \in U_n^{g_{i_j}(x_0)})h, g)$, that implies

$$\varphi_{i_9}^{-1} \circ \varphi_{i_j} \in U_{2n}^{x_0}(h, g) \quad \text{and} \quad \varphi_{i_0}^{-1} \circ \varphi_{i_j}(h, g, x_0) = g_{i_0}^{-1} g_{i_j}(x_0) > x_0,$$

$j = 1, \ldots, m$. Hence $h^{p_j} g_{i_0}^{-1} g_{i_j}(x_0) > x_0$, and there is $\varphi_j' \in U_{3n+N}^{x_0}(h, g) \subset U_{4n}^{x_0}(h, g)$ such that $\varphi_j'(h, g) = g_{i_0} h^{p_j} g_{i_0} g_{i_j}(x_0) = g_j'$. ■

(3.12) Theorem. *If $\varphi \in {}_N^2\Phi$, $\varphi \neq 1$, $\Sigma_2 \varphi = 0$, and $g, h \in \mathrm{Diff}^s(I)$, $s = 1, 2, \ldots, \infty$, satisfy conditions (1)–(4) from (3.10) for $n = 4^N N$, then there is a closed interval $K \subset I$ such that $\varphi(h, g)(K) = K$ and $\varphi(h, g)|K$ is a K-diffeomorphism, or $\varphi(h, g) = \mathrm{id}$.*

Proof. Suppose a contrary. Since $\Sigma_2 \varphi = 0$, $\varphi(h, g)$ can be written in the form $(g^{k_1} h^{r_1} g^{-k_1}) \ldots (g^{k_l} h^{r_l} g^{-k_l})$ with $\sum_{i=1}^{l} |r_i| \leqslant N$, $|r_i| > 0$, and $|k_i| \leqslant N$, $i = 1, \ldots, l$. Moreover $g^{k_i} = \varphi_i(h, g)$ for $\varphi_i = \underbrace{A_2^1 \ldots A_2^1}_{k_i - \text{times}} \in {}_{k_i}^2\Phi$ and $\varphi_i \in (x_0, b)_N^{x_0}(h, g)$, $i = 1, \ldots, l$. Then Lemma (3.11) gives us an inductive step to prove that $\varphi(h, g) = \mathrm{id}$. ■

Proof of Theorem (3.3). Since flows preserve supports of included diffeomorphisms, and since there is an embedding $j\colon \mathrm{Diff}_c^k(\bar{B}) \to \mathrm{Diff}_c^k(X)$ for B being an open ball in $\boldsymbol{R}^m$ (cf. (1.3)), it suffices to prove that we can find a continuous mapping $\gamma\colon [0, 1) \to \mathrm{Diff}_c^k(\bar{B})$, $\gamma(0) = \mathrm{id}$, such that $\{\gamma(t)\}_{t \in (0,1)}$ is a set of free generators of a subgroup in $\mathrm{Diff}_c^k(\bar{B})$ consisting of diffeomorphisms (except for the identity) which embed in no flow of diffeomorphisms of $\bar{B}$. (Note that there is more flows in $\mathrm{Diff}^k(\bar{B})$ than in $\mathrm{Diff}_c^k(\bar{B})$.) Let R be the set of all such diffeomorphisms from

$\mathrm{Diff}_c^k(\bar{B})$. It is easy to see that R satisfies (1) from Theorem (2.5). To see that R satisfies also (2) from (2.5), take a neighbourhood D of the identity in $\mathrm{Diff}_c^k(\bar{B})$, an open subset W of B, and a natural number r. Let C be a neighbourhood of the identity such that $C^{2r-1} \subset D$. Take an open subset of the form $I \times V$, where I is an open interval included in W with its closure.

By Corollary (3.10), there are $\hat{h}, \hat{g}$ $\mathrm{Diff}^k(I)$ arbitrarily close to the identity having properties (1)–(4) from (3.10) for $n = (r^2+r)4^{r^2+r}$. Since $(x_0, b)_n^x(\hat{h}, \hat{g}) = {}_n^2\Phi$ for x sufficiently close to b, by (3.9) $\hat{h}$ and $\hat{g}$ may also be chosen in such a way that $\varphi(\hat{h}, \hat{g}) \neq \mathrm{id}$ for $\varphi \in {}_n^2\Phi^*$ and so close to the identity that the diffeomoprhisms $\tilde{h}(t, x) = (\hat{h}(t), x)$ and $\tilde{g}(t, x) = (\hat{g}(t), x)$ of $I \times V$ can be extended to diffeomorphisms $h, g \in C_W$.

Put $f_i = g^{i-1}hg^{1-i}$, $i = 1, \ldots, r$, and take $\varphi \in {}_r^r\Phi$, $\varphi = A_{i_k}^{j_k} \ldots A_{i_1}^{j_1}$. Then $f_i \in D_W$, $i = 1, \ldots, r$, and

$$\begin{aligned}\varphi(f_1, \ldots, f_r) &= (g^{i_k-1}h^{j_k}g^{1-i_k}) \ldots (g^{i_1-1}h^{j_1}g^{1-i_1}) \\ &= g^{i_k-1}h^{j_k}g^{i_{k-1}-i_k}h^{j_{k-1}} \ldots h^{j_1}g^{1-i_1}.\end{aligned}$$

Since $i_l \neq i_{l+1}$, $\varphi(f_1, \ldots, f_r) = \psi(h, g)$ for some $\psi \in {}_N^2\Phi$, $\Sigma_2\psi = 0$, where

$$N \leqslant k+(k+1)(r-1) < r^2+r\,.$$

$\psi(h, g)|_{I\times V}$ has the form $\psi(h, g)(t, x) = (\psi(\hat{h}, \hat{g})(t), x)$ and by Theorem (3.12), there is a closed interval $[a, b] = K \subset I$ such that $\psi(\hat{h}, \hat{g})(K) \subset K$ and $\psi(\hat{h}, \hat{g})|_K$ is a K-diffeomorphism or $\psi(h, g) = \mathrm{id}$. The last possibility we have excluded.

Now observe that $\varphi(f_1, \ldots, f_r) = \psi(h, g)$ embeds in no flow in $\mathrm{Diff}^k(\bar{B})$. For, suppose $s \mapsto u_s$ is a flow in $\mathrm{Diff}^k(\bar{B})$ such that $u_1 = \psi(h, g)$. Since

$$\lim_{n\to+\infty} (\psi(h, g))^n(t, x) = \big(\lim_{a\to+\infty} \psi(\hat{h}, \hat{g})(t), x\big) = (a, x)$$

or $\lim\limits_{n\to-\infty} (\psi(h, g))^n(t, x) = (a, x)$ for $(t, x) \in K \times V$ $(\psi(\hat{h}, \hat{g})|_{(a,b)}$ has no fixed points) and since $\psi(h, g)^n u_s = u_s\psi(h, g)^n$ for $n \in \mathbf{Z}$, $s \in \mathbf{R}$, we have $u_s(t, x) = (\hat{u}_s(t), x)$ for $(t, x) \in K \times V$ and $\hat{u}_s$ being a flow in $\mathrm{Diff}^k(K)$, that is impossible since $\hat{u}_1 = \psi(\hat{h}, \hat{g})|_K$ is a K-diffeomorphism.

This shows that R has property (2) from (2.5) and Theorem (3.3) follows now directly by Theorem (2.5), since $\mathrm{Diff}_c^k(\bar{B})$ is completely metrizable and locally arcwise connected SLT topological group of homeomorphisms of B. ■

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Received 3 October 1985;
in revised form 24 November 1986 and 2 February 1987