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On functions of bounded n-th variation

by

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Abstract. Following Sargent [15], a definition of bounded nth variation for real valued functions is introduced and it is shown that this definition is equivalent to that of Russell [11]. Various properties of functions of generalized bounded variation are established.

1. Introduction. One approach to get a definition of functions of bounded variation of higher order is based on the concept of higher order divided differences (cf. [9], p. 24). This was followed by Russell ([10], [11]) and others (see, for example, [2]). This method was also followed in [7] to define absolute continuity of higher order. Another approach was due to Sargent ([14], [15]) who introduced the concept of *absolute continuity of higher order* which involved the notion of generalized derivatives. Sargent was concerned with the descriptive definition of the Cèsaro–Denjoy integrals which needed the concept of absolute continuity of higher order. She did not specifically mention bounded variation but her method suggested a definition of bounded variation of higher order. The two approaches are different. Therefore, it is natural to ask if these two approaches have any connection. The purpose of the present paper is to give an answer to this question. Following Sargent [15] (see also [4]) we have introduced two definitions of bounded variation of order n which are analogous to the concept of VB and VB^* of [13], pp. 221–228, and showed that on intervals these definitions are equivalent to that used by Russel [11].

2. Definitions and notation. Let f be defined in some neighbourhood of x. If there are real numbers $\alpha_0(=f(x))$, α_1 ..., α_r depending on x but not on h such that

$$f(x+h) = \sum_{i=0}^{r} \alpha_i \frac{h^i}{i!} + o(h^r),$$

then α_r is called the *Peano derivative* of f at x of order r and is denoted by $f_{(r)}(x)$. Clearly, if $f_{(r)}(x)$ exists then $f_{(i)}(x)$ exists for all i, $1 \leqslant i < r$. Also, if the ordinary rth derivative $f^{(r)}(x)$ exists, then $f_{(r)}(x)$ exists and is equal to $f^{(r)}(x)$. The converse is true for $r = 1$ only.

Let $f_{(r)}(x)$ exist. Write

$$\gamma_{(r+1)}(f, x, t) = \frac{(r+1)!}{(t-x)^{(r+1)}}\left[f(t) - \sum_{i=0}^{r} f_{(i)}(x)\cdot\frac{(t-x)^i}{i!}\right].$$

We define the four Peano derivates denoted by $\bar{f}^+_{(r+1)}(x)$, $\underline{f}^+_{(r+1)}(x)$, $\bar{f}^-_{(r+1)}(x)$ and $\underline{f}^-_{(r+1)}(x)$ and defined by the upper and lower limits of $\gamma_{(r+1)}(f, x, t)$ as $t \to x+$ or $t \to x-$ as the case may be. If $\bar{f}^+_{(r+1)}(x) = \underline{f}^+_{(r+1)}(x)$ or $\bar{f}^-_{(r+1)}(x) = \underline{f}^-_{(r+1)}(x)$ the common value is called the *right-hand* or *left-hand Peano derivative* of f at x of order $(r+1)$ and is denoted by $f^+_{(r+1)}(x)$ or $f^-_{(r+1)}(x)$ respectively.

Let $n \geqslant 1$ be a fixed positive integer and let $f^+_{(n)}(x)$ exist and be finite. Define

$$\varepsilon_n^+(f, x, t) = \begin{cases} \gamma_n(f, x, t) - f^+_{(n)}(x) & \text{if } t \neq x, \\ 0 & \text{if } t = x. \end{cases}$$

Similarly, if $f^-_{(n)}(x)$ exists and is finite define

$$\varepsilon_n^-(f, x, t) = \begin{cases} \gamma_n(f, x, t) - f^-_{(n)}(x) & \text{if } t \neq x, \\ 0 & \text{if } t = x. \end{cases}$$

Let us suppose that f is defined in $[a, b]$ and let $[c, d] \subset [a, b]$. Let $f_{(n-1)}$ exist at c and d and let $f^+_{(n)}(c)$ and $f^-_{(n)}(d)$ exist. (Of course, if $c = a$ or $b = d$ or both, the existence of $f_{(n-1)}$ will mean one-side derivative $f_{(n-1)}$ at these points.) Let

$$\bar{\omega}_n(f, [c, d]) = \max\left[\max_{0 \leqslant r \leqslant n} \varepsilon_n^+\left(f, c, c + \frac{r}{n}(d-c)\right),\right.$$
$$\left.\max_{0 \leqslant r \leqslant n}\left\{-\varepsilon_n^-\left((f, d, d - \frac{r}{n}(d-c)\right)\right\}\right],$$

$$\underline{\omega}_n(f, [c, d]) = \min\left[\min_{0 \leqslant r \leqslant n} \varepsilon_n^+\left(f, c, c + \frac{r}{n}(d-c)\right),\right.$$
$$\left.\min_{0 \leqslant r \leqslant n}\left\{-\varepsilon_n^-\left(f, d, d - \frac{r}{n}(d-c)\right)\right\}\right],$$

$$\omega_n(f, [c, d] = \bar{\omega}_n(f, [c, d]) - \underline{\omega}_n(f, [c, d]).$$

Since $\bar{\omega}_n(f, [c, d]) \geqslant 0 \geqslant \underline{\omega}_n(f, [c, d])$, we have $\omega_n(f, [c, d]) \geqslant 0$. The quantity $\omega_n(f, [c, d])$ is called the *weak oscillation* of f on $[c, d]$ of order n. Similarly, writing

$$\bar{\omega}_n^*(f, [c, d]) = \max\left[\sup_{c \leqslant t \leqslant d} \varepsilon_n^+(f, c, t),\ \sup_{c \leqslant t \leqslant d}\{-\varepsilon_n^-(f, d, t)\}\right],$$

$$\underline{\omega}_n^*(f, [c, d]) = \min\left[\inf_{c \leqslant t \leqslant d} \varepsilon_n^+(f, c, t),\ \inf_{c \leqslant t \leqslant d}\{-\varepsilon_n^-(f, d, t)\}\right],$$

$$\omega_n^*(f, [c, d]) = \bar{\omega}_n^*(f, [c, d]) - \underline{\omega}_n^*(f, [c, d]),$$

the quantity $\omega_n^*(f, [c, d])$ is called the *strong oscillation* of f on $[c, d]$ of order n. Since $\bar{\omega}_n(f, [c, d]) \leqslant \bar{\omega}_n^*(f, [c, d])$ and $\underline{\omega}_n(f, [c, d]) \geqslant \underline{\omega}_n^*(f, [c, d])$, we have

$$\omega_n(f, [c, d]) \leqslant \omega_n^*(f, [c, d]).$$

Let $E \subset [a, b]$ and let $f_{(n-1)}, f^+_{(n)}, f^-_{(n)}$ exist on E. The weak [resp. strong] variation of f on E of order n, denoted by $V_n(f, E)$ [resp. $V_n^*(f, E)$], is the upper bound of the sums $\sum \omega_n(f, [c_k, d_k])$ [resp. $\sum \omega_n^*(f, [c_k, d_k])$] where $\{[c_k, d_k]\}$ is any sequence of non-overlapping intervals whose end points belong to E. If $V_n(f, E) < \infty$ [resp. $V_n^*(f, E) < \infty$], then f is said to be of *bounded variation in the wide sense*, or simply, of bounded variation [resp. *bounded variation in the restricted sense*] of order n, briefly V_nB [resp. V_nB^*], on E and is written $f \in V_nB(E)$

$$[\text{resp. } f \in V_nB^*(E)].$$

The function f is said to be of *generalized bounded variation in the wide sense*, or simply, of generalized bounded variation [resp. *generalized bounded variation in the restricted sense*] of order n, briefly V_nBG [resp. V_nBG^*] on E, if E is the union of a countable collection of measurable sets on each of which f is V_nB [resp. V_nB^*]. If f is V_nBG [resp. V_nBG^*] on E, we write $f \in V_nBG(E)$ [resp. $f \in V_nBG^*(E)$].

Since $\omega_n(f, [c, d]) \leqslant \omega_n^*(f, [c, d])$, we have $V_nB^*(E) \subset V_nB(E)$ and

$$V_nBG^*(E) \subset V_nBG(E).$$

Let $x_0, x_1, \ldots, x_n$ be $(n+1)$ distinct points (not necessarily in linear order) in $[a, b]$. The nth divided difference of f at these points is defined by

$$Q_n(f, x_0, x_1 \ldots, x_n) = \sum_{i=0}^{n} \frac{f(x_i)}{\omega'(x_i)}$$

where

$$\omega(x) = \prod_{j=1}^{n}(x - x_j).$$

If $Q_n(f, x_0, x_1, \ldots, x_n) \geqslant 0$ for all choices of the points $x_0, x_1, \ldots, x_n$ in $[a, b]$, then f is said to be *n-convex* in $[a, b]$. Clearly, a function f is 0-convex if and only if f is non-negative, f is 1-convex if and only if f is non-decreasing, and f is 2-convex if and only if f is convex in $[a, b]$.

3. Preliminary lemmas.

LEMMA 3.1. *Let f be defined in $[a, b]$ and let $[c, d] \subset [a, b]$. Let $f_{(n-1)}$ exist at c and d and let $f^+_{(n)}(c)$, $f^-_{(n)}(d)$ exist and be finite. Then*

$$|f^-_{(n)}(d) - f^+_{(n)}(c)| \leqslant K\omega_n(f, [c, d])$$

where K is a constant depending only on n.

Proof. Writing

$$\Delta_n(x,h)=\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}f(x+rh),$$

if follows from the definitions of ε_n^+ and ε_n^- that (cf. [15], Lemma 1)

$$\Delta_n(c,h)=h^n f_{(n)}^+(c)+h^n\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}\frac{r^n}{n!}\varepsilon_n^+(f,c,c+rh),$$

$$\Delta_n(d,-h)=(-h)^n f_{(n)}^-(d)+(-h)^n\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}\frac{r^n}{n!}\varepsilon_n^-(f,d,d-rh)$$

Taking $h=\dfrac{d-c}{n}$, we have

$$\Delta_n(c,h)=(-1)^n\Delta_n(d,-h),$$

and hence

$$|f_{(n)}^-(d)-f_{(n)}^+(c)|=\Big|\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}\frac{r^n}{n!}\{\varepsilon_n^+(f,c,c+rh)-\varepsilon_n^-(f,d,d-rh)\}\Big|$$

$$\leqslant\sum_{r=0}^{n}\binom{n}{r}\frac{r^n}{n!}|\varepsilon_n^+(f,c,c+rh)-\varepsilon_n^-(f,d,d-rh)|.$$

Denoting by $\sum^+$ (resp. $\sum^-$) the summation over the terms for which

$$\varepsilon_n^+(f,c,c+rh)-\varepsilon_n^-(f,d,d-rh)$$

is positive (resp. negative) and noticing that

$$2\underline{\omega}_n(f,[c,d])\leqslant\varepsilon_n^+(f,c,c+rh)-\varepsilon_n^-(f,d,d-rh)\leqslant 2\overline{\omega}_n(f,[c,d])\quad\text{for } 0\leqslant r\leqslant n,$$

we have

$$|f_{(n)}^-(d)-f_n^+(c)|\leqslant\sum{}^+\binom{n}{r}\frac{r^n}{n!}\{2\overline{\omega}_n(f,[c,d])\}$$

$$+\sum{}^-\binom{n}{r}\frac{r^n}{n!}\{-2\underline{\omega}_n(f,[c,d])\}$$

$$\leqslant 2\sum_{r=0}^{n}\binom{n}{r}\frac{r^n}{n!}\omega_n(f,[c,d])$$

$$=K\omega_n(f,[c,d])$$

where

$$K=2\cdot\sum_{r=0}^{n}\binom{n}{r}\frac{r^n}{n!}.$$

Lemma 3.2 *Let $f\in V_nB(E)$ where $E\subset[a,b]$. Then*

(i) *if $\alpha=\inf E\in E$ then $f_{(n)}^-$ is bounded on E,*
(ii) *if $\beta=\sup E\in E$ then $f_{(n)}^+$ is bounded on E,*
(iii) *if $f_{(n)}$ exist on E then $f_{(n)}$ is bounded on E.*

Proof. Let $V_n(f,E)=M$. By Lemma 3.1 we have, for any $x\in E$

$$|f_{(n)}^-(x)|-|f_{(n)}^+(\alpha)|\leqslant|f_{(n)}^-(x)-f_{(n)}^-(\alpha)|\leqslant K\omega_n(f,[\alpha\ x])\leqslant KM.$$

Hence

$$|f_{(n)}^-(x)|\leqslant KM+|f_{(n)}^+(\alpha)|\quad\text{for all } x\in E.$$

Thus $f_{(n)}^-$ is bounded on E. Similarly,

$$|f_{(n)}^+(x)|\leqslant KM+|f_{(n)}^-(\beta)|\quad\text{for all } x\in E,$$

and so $f_{(n)}^+$ is bounded on E.

Finally, let $x_0\in E$ be fixed. Then by Lemma 3.1 we have, for any $x\in E$,

$$|f_{(n)}(x)|-|f_{(n)}(x_0)|\leqslant|f_{(n)}(x)-f_{(n)}(x_0)|\leqslant K\omega_n(f,J)\leqslant KM$$

where J is the interval with endpoints x and x_0. Hence

$$|f_{(n)}(x)|\leqslant KM+|f_{(n)}(x_0)|\quad\text{for all } x\in E,$$

and hence $f_{(n)}$ is bounded on E.

Corollary 3.3. *If $f\in V_nB([a,b])$ then $f_{(n)}^+$ and $f_{(n)}^-$ are bounded on $[a,b)$ and $(a,b]$, respectively.*

The proof follows from (i) and (ii) above.

Lemma 3.4. *Let $f_{(n)}$ exist on $E\subset[a,b]$ and let $f\in V_nB(E)$. Then $f_{(n)}$ is of bounded variation on E.*

Proof. Let $\{(C_k,d_k)\}$ be any sequence of non-overlapping intervals with endpoints in E. Then by Lemma 3.1

$$|f_{(n)}(d_k)-f_{(n)}(c_k)|\leqslant K\omega_n(f,[c_k,d_k])$$

and, since $f\in V_nB(E)$, the result follows.

Lemma 3.5. *Let f be continuous and let $f_{(n-1)}$ exist in $[a,b]$. Then the upper and lower bounds of each of $\bar f_{(n)}^+$, $f_{(n)}^+$, $\bar f_{(n)}^-$ and $f_{(n)}^-$ in $[a,b]$ are, respectively, equal to the upper and lower bounds of $n!\,Q_n(f,x_0,x_1,\dots,x_n)$ where $x_0,x_1,\dots x_n$ are any $n+1$ distinct points in $[a,b]$.*

Proof. For $n = 1$ the result is well known [13, p. 204]. So we suppose $n \geqslant 2$. Let m be the lower bound of $\overline{f}^+_{(n)}$ say and suppose that m is finite. Then the function $F(x) = f(x) - \dfrac{m(x-a)^n}{n!}$ is such that $\overline{F}^+_{(n)}(x) \geqslant 0$ for all x in $[a, b)$ and hence F is n-convex (see [3, Theorem 19]). So for any $(n+1)$ distinct points $x_0, x_1, \ldots, x_n$ in $[a, b]$, $Q_n(F, x_0, x_1 \ldots, x_n) \geqslant 0$. Considering the determinant formula for Q_n (see, for example, [6, p. 183]), we have

$$Q_n(x^n, x_0, x_1, \ldots, x_n) = 1 \quad \text{and} \quad Q_n(x^i, x_0, x_1, \ldots, x_n) = 0$$

for $0 \leqslant i \leqslant n-1$ and therefore

$$n!\,Q_n(f, x_0, x_1, \ldots, x_n) \geqslant m\,.$$

Let $\varepsilon > 0$ be arbitrary. Then there is $x_0 \in [a, b)$ such that $\overline{f}^+_{(n)}(x_0) < m+\varepsilon$. Since

$$\limsup_{t \to x_0+} \gamma_n(f, x_0, t) = \overline{f}^+_{(n)}(x_0)\,, \tag{1}$$

there is $x_1 \neq x_0$ such that $\gamma_n(f, x_0, x_1) < m+\varepsilon$.

Since by [5, Lemma 4.1]

$$\lim_{t_n \to x_0} \ldots \lim_{t_2 \to x_0} n!\,Q_n(f, x_0, x_1, t_2, \ldots, t_n) = \gamma_n(f, x_0, x_1) \tag{2}$$

by repeated application, there are distinct points $x_2, x_3, \ldots, x_n$ different from x_0, x_1 such that

$$n!\,Q_n(f, x_0, x_1, \ldots, x_n) < m+\varepsilon\,.$$

This completes the proof when m is finite.

If $m = -\infty$ then for any $N > 0$ there is $x_0 \in [a, b)$ such that $\overline{f}^+_{(n)}(x_0) < -N$. Hence by (1) and (2) there are distinct points $x_1, x_2, \ldots, x_n$ different from x_0 such that $n!\,Q_n(f, x_0, x_1 \ldots, x_n) < -N$. Thus, $\inf\ n!\,Q_n(f, x_1, x_2, \ldots, x_n) = -\infty$. If $m = \infty$, the argument is similar. The other cases follow similarly.

LEMMA 3.6. *Under the hypotheses of Lemma 3.5 if at least one of $\overline{f}^+_{(n)}$, $\underline{f}^+_{(n)}$, $\overline{f}^-_{(n)}$ and $\underline{f}^-_{(n)}$ is bounded, so are the other three and $f^{(n-1)}$ exists and is absolutely continuous.*

Proof. The first part follows from Lemma 3.5. For the second part, let $\underline{f}^+_{(n)}$ be bounded and let $|\underline{f}^+_{(n)} x| < k$ for $x \in [a, b)$. Then the function $F(x) = f(x) + k\dfrac{(x-a)^n}{n!}$ is such that $\underline{F}^+_{(n)}(x) > 0$ for $x \in [a, b)$ and so by [16, Theorem 1] $F_{(n-1)}$ is continuous and non-decreasing in $[a, b]$. Thus $F_{(n-1)}$ is the continuous derivative $F^{(n-1)}$ and $F^{(n)}$ exist a.e. So, $f_{(n-1)}$ is the continuous derivative $f^{(n-1)}$ and $f^{(n)}$ exist a.e. Since $\underline{f}^+_{(n)}$ is bounded, by Lemma 3.5 the both sided derivates $\underline{f}_{(n)}$, $\overline{f}_{(n)}$ are also bounded and so $f^{(n)}$ is $C_{n-1}P$ integrable in $[a, b]$ and $f^{(n-1)}$ is its $C_{n-1}P$ integral [1]. Since $f^{(n)}$ is bounded, it is L-integrable in $[a, b]$, and $f^{(n-1)}$ is its L-integral. This completes the proof.

LEMMA 3.7. *If $f \in V_nB([a, b])$ then $f^{(n-1)}$ exists and is absolutely continuous in $[a, b]$.*

Proof. Under the hypothesis, f is continuous in $[a, b]$. In fact, if $n = 1$ then since $f \in V_1B([a, b])$, $f^+_{(1)}$ and $f^-_{(1)}$ exist and so f is right as well as left continuous and if $n > 1$ then since $f \in V_nB([a, b])$, $f_{(1)}$ exists and so f is continuous. Therefore the result follows from Corollary 3.3 and Lemma 3.6.

LEMMA 3.8. *If $F \subset E \subset [a, b]$, then*

(i) $V_n(f, F) \leqslant V_n(f, E)$,

(ii) $V^*_n(f, F) \leqslant V^*_n(f, E)$,

and hence

(iii) $V_nB(E) \subset V_nB(F)$,

(iv) $V_nB^*(E) \subset V_nB^*(F)$.

The first part follows from definition and the second part follows from the first part.

LEMMA 3 9. *If $E \subset [a, b]$ and $a < c < b$, then*

$$V_n(f, E) \geqslant V_n(f, E \cap [a, c]) + V_n(f, E \cap [c, b])$$

and

$$V^*_n(f, E) \geqslant V^*_n(f, E \cap [a, c] + V^*_n(f, E \cap [c, b])\,.$$

Proof. Let $\{(a_k, b_k)\}$ be a sequence of non-overlapping intervals with endpoints in $E \cap [a, c]$ and let $\{(c_i, d_i)\}$ be a sequence of non overlapping intervals with endpoints in $E \cap [c, b]$.

Then

$$V_n(f, E) \geqslant \sum_k \omega_n(f, [a_k, b_k]) + \sum_i \omega_n(f, [c_i, d_i])$$

and

$$V^*_n(f, E) \geqslant \sum_k \omega^*_n(f, [a_k, b_k]) + \sum_i \omega^*_n(f, [c_i, d_i])\,.$$

Since $\{(a_k, b_k)\}$ and $\{(c_i, d_i)\}$ are arbitrary, the results follow.

LEMMA 3.10. $V_nB([a, b]) \subset V_{n-1}B^*([a, b])$, $V_nB^*([a, b]) \subset V_{n-1}B^*([a, b])$.

Proof. Let $f \in V_nB([a, b])$. Since

$$\varepsilon^+_n(f, x, t) + f^+_{(n)}(x) = \varepsilon^-_n(f, x, t) + f^-_{(n)}(x) = \gamma_n(f, x, t)$$

and

$$\frac{t-x}{n}\gamma_n(f, x, t) = \varepsilon^+_{n-1}(f, x, t) = \varepsilon^-_{n-1}(f, x, t)\,,$$

we have

$$\omega_{n-1}(f, [c, d]) < \frac{|d-c|}{n}\{\omega_n(f, [c, d]) + M\}$$

whenever $[c, d] \subset [a, b]$, M being any number greater than, the upper bounds of $|f_n^+|$ and $|f_{(n)}^-|$ which are finite by Corollary 3.3. Since $f \in V_nB([a, b])$, it follows from above that $f \in V_{n-1}B([a, b])$. The proof for other part is similar.

Applying Lemma 3.2 (iii) instead of Corollary 3.3, we get from Lemma 3.10 the following lemma.

Lemma 3.11. *If $f_{(n)}$ exist on $E \subset [a, b]$ and if $f \in V_nB(E)$ (resp. $f \in V_nB^*(E)$), then $f \in V_{n-1}B(E)$ (resp. $f \in V_{n-1}B^*(E)$).*

Lemma 3.12. *If $f_{(n)}$ exists in $[a, b]$ and if at least one of $\bar{f}_{(n+1)}^+$, $\underline{f}_{(n+1)}^+$, $\bar{f}_{(n+1)}^-$ and $\underline{f}_{(n+1)}^-$ is bounded, then $f \in V_nB^*([a, b])$.*

Proof. Let $\bar{f}_{(n+1)}^+$ be bounded and let

$$|\bar{f}_{(n+1)}^+(x)| \leqslant M \quad \text{for } x \in [a, b).$$

Then, since $f_{(1)}$ exist, f is continuous and so as in Lemma 3.6 $\bar{f}_{(n+1)}^+$ is L-integrable and

$$\int_a^x \bar{f}_{(n+1)}^+(t)\,dt = f_{(n)}(x) - f_{(n)}(a).$$

Let $[c, d] \subset [a, b]$. Then for each $x \in [c, d)$ and $t \in [c, d]$, $x \neq t$, there is, by the mean value theorem [8], a ξ between x and t such that

$$|\varepsilon_n^-(f, x, t)| = |\varepsilon_n^+(f, x, t)| = |f_{(n)}(\xi) - f_{(n)}(x)|$$
$$= |\int_x^\xi \bar{f}_{(n+1)}^+(u)\,du| \leqslant M|\xi - x| < M|d-c|.$$

Hence $\omega_n^*(f, [c, d]) \leqslant 2M(d-c)$ which shows that $f \in V_nB^*([a, b])$.

Lemma 3.13. *The spaces $V_nB(E)$, $V_nB^*(E)$, $V_nBG(E)$, $V_nBG^*(E)$ are all linear spaces.*

Proof. It can be verified that

$$\varpi_n(f+g, [c, d]) \leqslant \varpi_n(f, [c, d]) + \varpi_n(g, [c, d]),$$

and this with a similar inequality gives

$$\omega_n(f+g, [c, d]) \leqslant \omega_n(f, [c, d]) + \omega_n(g, [c, d]).$$

This shows that $f, g \in V_nB(E)$ imply $f+g \in V_nB(E)$. Also if α is any constant, then

$$\omega_n(\alpha f, [c, d]) = |\alpha|\omega_n(f, [c, d]).$$

This shows that $f \in V_nB(E)$ implies $\alpha f \in V_nB(E)$.

Let $f, g \in V_nBG(E)$. Then there are $\{E_i\}$ and $\{F_j\}$ such that $\bigcup_i E_i = E = \bigcup_j F_j$ and $f \in V_nB(E_i)$ for each i and $g \in V_nB(F_j)$ for each j. Let $E_{ij} = E_i \cap F_j$. Then $E = \bigcup_i \bigcup_j E_{ij}$. Then, from Lemma 3.8, $f, g \in V_nB(E_{ij})$ and so by the above $f+g \in V_nB(E_{ij})$, and hence $f+g \in V_nBG(E)$. The case $\alpha f \in V_nBG(E)$ for any constant α is clear.

The other cases can be proved similarly.

4. Main results.

Theorem 4.1. *If f is D^*-integrable on $[a, b]$ and $f \in V_nB^*(E)$, then $F \in V_{n+1}B^*(E)$ where*

$$F(x) = \int_a^x f(t)\,dt.$$

Proof. We have

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\,dt$$
$$= \int_x^{x+h} \Big[\sum_{r=0}^{n-1} \frac{(t-x)^r}{r!} f_{(r)}(x) + \frac{(t-x)^n}{n!}\big(f_{(n)}^+(x) + \varepsilon_n^+(f, x, t)\big)\Big]\,dt$$
$$= \sum_{r=0}^{n-1} \frac{h^{r+1}}{(r+1)!} f_{(r)}(x) + \frac{h^{n+1}}{(n+1)!} f_{(n)}^+(x)$$
$$+ \int_x^{x+h} \frac{(t-x)^n}{n!} \varepsilon_n^+(f, x, t)\,dt.$$

Since $\varepsilon_n^+(f, x, t) \to 0$ as $t \to x+$, we have $F_{(r)}^+(x) = f_{(r-1)}^+(x)$ for $1 \leqslant r \leqslant n$ and

$$F_{(n+1)}^+(x) = f_{(n)}^+(x) = f_{(n)}^+(x)$$

and so

$$\varepsilon_{n+1}^+(F, x, x+h) = \frac{(n+1)}{h^{n+1}} \int_x^{x+h} (t-x)^n \varepsilon_n^+(f, x, t)\,dt.$$

Since $f \in V_nB^*(E)$, it follows that $F \in V_{n+1}B^*(E)$.

Theorem 4.2. *If g is $(n+1)$ convex in $[a, b]$, then $g \in V_nB^*([\alpha, \beta])$ for every $[\alpha, \beta] \subset (a, b)$. If moreover, $g_{(n)}^+(a)$ and $g_{(n)}^-(b)$ exist and are finite, then*

$$g \in V_nB^*([a, b]).$$

Proof. If $n = 1$, then since g is convex in $[a, b]$ for every $[\alpha, \beta] \subset (a, b)$, $g_{(1)}^+$ and $g_{(1)}^-$ exist and are finite in $[\alpha, \beta)$ and $(\alpha, \beta]$, respectively, and are non-decreasing.

Moreover, for $x_1, x_2 \in [\alpha, \beta]$, $x_1 < x_2$,

$$\int_{x_1}^{x_2} g_{(1)}^{+}(u)\,du = \int_{x_1}^{x_2} g_{(1)}^{-}(u)\,du = g(x_2) - g(x_1)\,.$$

Let $[c, d] \subset [\alpha, \beta]$. Then, for $c \leqslant t \leqslant d$,

$$|\varepsilon_1^{+}(g, c, t)| = \left|\frac{g(t)-g(c)-(t-c)g_{(1)}^{+}(c)}{t-c}\right|$$

$$= \left|\frac{1}{t-c}\int_{c}^{t} [g_{(1)}^{+}(u) - g_1^{+}(c)]\,du\right| \leqslant g_{(1)}^{-}(d) - g_{(1)}^{+}(c)$$

and similarly $|\varepsilon_1^{-}(g, d, t)| \leqslant g_{(1)}^{-}(d) - g_{(1)}^{+}(c)$. Hence

$$\omega_1^{*}(g, [c, d]) \leqslant 2[g_1^{-}(d) - g_{(1)}^{+}(c)]\,.$$

So if $\{(c_i, d_i)\}$ is any sequence of non-overlapping subintervals of $[\alpha, \beta]$ then

$$\sum_i \omega_1^{*}(g, [c_i, d_i]) \leqslant 2\sum_i [g_{(1)}^{-}(d_i) - g_{(1)}^{+}(c_i)]$$

$$\leqslant 2\sum_i [g_1^{+}(d_i) - g_{(1)}^{+}(c_i)]$$

$$\leqslant 2\,[g_{(1)}^{+}(\beta) - g_{(1)}^{+}(\alpha)]\,.$$

Hence $g \in V_1B^{*}([\alpha, \beta])$.

If $g_{(1)}^{+}(a)$ and $g_{(1)}^{-}(b)$ are finite then by the above argument

$$\sum_i \omega_1^{*}(g, [c_i, d_i]) \leqslant 2\sum_i [g_{(1)}^{-}(d_i) - g_{(1)}^{+}(c_i)] \leqslant 6\,[g_{(1)}^{-}(b) - g_{(1)}^{+}(a)]\,.$$

Hence $g \in V_1B^{*}([a, b])$.

We suppose $n \geqslant 2$. Let $[\alpha, \beta] \subset (a, b)$. Since g is $(n+1)$ convex in $[a, b]$ by [3, Theorem 7], the $(n-1)$th derivative $g^{(n-1)}$ of g exists and is continuous in $[\alpha, \beta]$, and $g_{(n)}^{+}$ and $g_{(n)}^{-}$ exist, are finite and non-descreasing in $[\alpha, \beta]$. Thus $\varepsilon_n^{+}(g, x, t)$ and $\varepsilon_n^{-}(g, x, t)$ are defined for $x \in [\alpha, \beta]$, $t \in [\alpha, \beta]$. Since g has continuous $(n-1)$th derivative in $[\alpha, \beta]$ for any closed subinterval $[c, d] \subset [\alpha, \beta]$ and for $x \in [c, d)$, $t \in [c, d]$ with $x \neq t$ we have by the mean value theorem

(1) $$\varepsilon_n^{+}(g, x, t_,) = \frac{g^{(n-1)}(\xi) - g^{(n-1)}(x) - (\xi - x)g_{(n)}^{+}(x)}{\xi - x}$$

where ξ lies between x and t. Since $g^{(n-1)}$ is convex (cf. [3. Corollary 15(a)]) and continuous in $[\alpha, \beta]$, it is absolutely continuous there, and so

(2) $$\int_{x}^{\xi} g_{(n)}^{+} u\,du = g^{(n-1)}(\xi) - g^{(n-1)}(x)\,.$$

By the value theorem there is λ such that

(3) $$\int_{x}^{\xi} g_{(n)}^{+}(u)\,du = \lambda(\xi - x)$$

where $g_{(n)}^{+}(x) \leqslant \lambda \leqslant g_{(n)}^{+}(\xi)$ or $g_{(n)}^{+}(x) \geqslant \lambda \geqslant g_{(n)}^{+}(\xi)$ according as $x < \xi$ or $x > \xi$.

So, from (1), (2), (3),

(4) $$\varepsilon_n^{+}(g, x, t) = \lambda - g_{(n)}^{+}(x)\,.$$

Since $x \in [c, d)$, $t \in [c, d)$ and since $g_{(n)}^{+}$ and $g_{(n)}^{-}$ are non-decreasing with the property that

(5) $$g_{(n)}^{+}(x_1) \leqslant g_{(n)}^{-}(x_2), \qquad g_{(n)}^{-}(x_3) \leqslant g_{(n)}^{+}(x_3)$$

whenever $\alpha \leqslant x_1 < x_2 \leqslant \beta$, $\alpha \leqslant x_3 \leqslant \beta$ (the first part follows from Theorem 2 coupled with Theorem 6 of [3], while the second part follows from Theorem 7(b) of [3]), we have

(6) $$g_{(n)}^{+}(c) - g_{(n)}^{-}(d) \leqslant \lambda - g_{(n)}^{+}(x) \leqslant g_{(n)}^{-}(d) - g_{(n)}^{+}(c)\,.$$

From (4) and (6) we get

$$g_{(n)}^{+}(c) - g_{(n)}^{-}(d) \leqslant \varepsilon_n^{+}(g, x, t) \leqslant g_{(n)}^{-}(d) - g_{(n)}^{+}(c)\,.$$

Similarly for $x \in (c, d]$, $t \in [c, d] \subset [\alpha, \beta]$, $x \neq t$, we have

$$g_{(n)}^{+}(c) - g_{(n)}^{-}(d) \leqslant \varepsilon_n^{-}(g, x, t) \leqslant g_{(n)}^{-}(d) - g_{(n)}^{+}(c)\,.$$

So,

(7) $$\omega_n^{*}(g, [c, d] \leqslant 2[g_{(n)}^{-}(d) - g_{(n)}^{+}(c)]\,.$$

Let $\{(c_i, d_i)\}$ be any sequence of non-overlapping subintervals of $[\alpha, \beta]$. Then since $g_{(n)}^{+}$ is non-decreasing, from (5) and (7)

$$\sum_i \omega_n^{*}(g, [c_i, d_i]) \leqslant 2\sum_i [g_{(n)}^{-}(d_i) - g_{(n)}^{+}(c_i)]$$

$$\leqslant 2\sum_i [g_{(n)}^{+}(d_i) - g_{(n)}^{+}(c_i)]$$

$$\leqslant 2[g_{(n)}^{+}(\beta) - g_{(n)}^{+}(\alpha)]\,.$$

Hence $g \in V_nB^{*}([\alpha, \beta])$, proving the first part.

For the second part, if $g_{(n)}^{+}(a)$ and $g_{(n)}^{-}(b)$ are finite, then applying the above argument we have

$$\sum_i \omega_n^{*}(g, [c_i, d_i]) \leqslant 2\sum_i [g_{(n)}^{-}(d_i) - g_{(n)}^{+}(c_i)]$$

$$\leqslant 6\,[g_{(n)}^{-}(b) - g_{(n)}^{+}(a)]\,,$$

and hence $g \in V_nB^{*}([a, b])$.

THEOREM 4.3 (JORDAN DECOMPOSITION THEOREM). $V_nB([a,b]) = \{f = g-h;$ *g and h are k-convex in* $[a,b]$ *for* $k = 0, 1, \ldots (n+1)$ *and* $g_+^{(n)}(a)$, $h_+^{(n)}(a)$, $g_-^{(n)}(b)$, $h_-^{(n)}(b)$ *exist and are finite*$\} = V_nB^*([a,b])$ *where, for example,* $g_+^{(n)}(a)$ *is the right-hand derivative of* $g^{(n-1)}$ *at* a.

Proof. Let $f \in V_nB([a,b])$. Then, by Corollary 3.3, $f_{(n)}^+$ and $f_{(n)}^-$ are bounded in $[a,b)$ and $(a,b]$, respectively. Hence f is continuous in $[a,b]$ and so, by Lemma 3.6, $f^{(n-1)}$ exists and is absolutely continuous in a, b. So f^n exists and is finite a.e. in $[a,b]$. Also, by Lemma 3.8, $V_n(f,[a,x])$ is finite for all $x \in [a,b]$. Let

$$S = \{x\colon\ x \in (a,b);\ f^{(n)}(x) \text{ exists finitely}\}.$$

For $x \in S$ define

$$2p(x) = KV_n(f,[a,x]) + f^{(n)}(x),$$
$$2q(x) = KV_n(f,[a,x]) - f^{(n)}(x)$$

where K is the constant posted at Lemma 3.1.

Clearly, p and q are non-decreasing on S. For, if $x_1, x_2 \in S$, $x_1 < x_2$, then using Lemma 3.9 and Lemma 3.1 we have

$$2[p(x_2)-p(x_1)] = K[V_nf,[a,x_2]) - V_n(f,[a,x_1])] + f^{(n)}(x_2) - f^n(x_1)$$
$$\geqslant KV_n(f,[x_1,x_2]) - |f^{(n)}(x_2) - f^{(n)}(x_1)| \geqslant 0,$$

and similarly $2[q(x_2)-q(x_1)] \geqslant 0$. Extend p and q in the whole of $[a,b]$ by defining

$$p(a) = \inf_{x \in S} p(x), \quad p(\xi) = \sup_{x \in S \cap (a,\xi)} p(x) \quad \text{for } \xi \in (a,b],$$
$$q(a) = \inf_{x \in S} q(x), \quad q(\xi) = \sup_{x \in S \cap (a,\xi)} q(x), \quad \text{for } \xi \in (a,b].$$

Clearly, p and q are non-descreasing in $[a,b]$ and continuous at the points a and b. Since $f_{(n)}^+$ and $f_{(n)}^-$ are bounded in $[a,b)$ and $(a,b]$, respectively, p and q are also bounded in $[a,b]$. Let $C = \min[p(a), q(a)]$. Then, writing $u(x) = p(x) - C$ and $v(x) = q(x) - C$, u and v are non-negative non-decreasing functions on $[a,b]$ and continuous at a and b. Also, for $x \in S$,

$$f^{(n)}(x) = p(x) - q(x) = u(x) - v(x).$$

Since $f^{(n-1)}$ is absolutely continuous, it is indefinite Lebesgue integral of $f^{(n)}$, and hence we have

$$f(x) = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} u(t)dt - \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} v(t)dt + P(x-a)$$

where $P(x-a)$ is the polynomial in $(x-a)$ given by

$$P(x-a) = f(a) + (x-a)f^1(a) + \ldots + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a).$$

Writing

$$U(x) = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} u(t)dt, \qquad V(x) = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} v(t)dt,$$

the functions U and V are k-convex for $k = 0, 1, \ldots, (n+1)$ and $U^{(n-1)}$ and $V^{(n-1)}$ exist in $[a,b]$. Since u and v are continuous at a and b, $U_+^{(n)}(a)$, $U_-^{(n)}(b)$, $V_+^{(n)}(a)$, $V_-^{(n)}(b)$ exist and are finite where, for example, $U_+^{(n)}(a)$ is the right-hand derivative of $U^{(n-1)}$ at a. Breaking the polynomial P into two parts P_1 and P_2, putting in P_1 those terms of P which have positive coefficients and the rest in P_2, we see that

$$U + P_1 \quad \text{and} \quad V - P_2$$

are k-convex in $[a,b]$ for $k = 0, 1, \ldots, n+1$.

Since $f = (U+P_1) - (V-P_2)$, $V_nB([a,b]) \subset \{f\colon f = g-h;\ g$ and h are k-convex in $[a,b]$ for $k = 0, 1, \ldots (n+1)$ and $g_+^{(n)}(a)$, $h_+^{(n)}(a)$, $g_-^{(n)}(b)$, $h_-^{(n)}(b)$ exist and are finite$\}$.

Next, let $f = g-h$ where g and h are k-convex in $[a,b]$ for $k = 0, 1, \ldots, (n+1)$ and $g_+^{(n)}(a)$, $h_+^{(n)}(a)$, $g_-^{(n)}(b)$, $h_-^{(n)}(b)$ exist finitely. Then it follows from Theorem 4.2 and Lemma 3.13 that $f \in V_nB^*([a,b])$. Since $V_nB^*([a,b]) \subset V_nB([a,b])$, the proof is complete.

From the above theorem it follows that on an interval the concepts V_nB and V_nB^* are the same.

As we have already remarked, Russell [11] considered the definition of bounded k variation, where k is a positive integer, using kth divided difference. He proved that f is of bounded kth variation in $[a,b]$ if and only $f = f_1 - f_2$ where f_1 and f_2 are r-convex functions in $[a,b]$, $0 \leqslant r \leqslant k$, having finite right and left $(k-1)$th derivatives at a and b, respectively (see [11, Theorem 19] followed [12, Theorem 1]). Hence, from Theorem 4.3, is follows that $f \in V_nB^*([a,b])$ if and only if f is of bounded $(n+1)$th variation in the sense of Russell.

THEOREM 4.4. *Let* f *be measurable and let* $f_{(k)}$ *exist finitely on a measurable set* $E \subset [a,b]$. *Then there are a perfect set* $E_0 \subset E$ *such that* $\mu(E \sim E_0)$ *is arbitrarily small and two functions* g *and* h *such that*

$$f = g + h$$

where $g^{(k)}$ *exsists and is continuous on* $[a,b]$ *and* $h_{(r)}(x) = 0$ *for* $x \in E_0$, $r = 0, 1, \ldots, k$.

If, moreover, $f \in V_kB(E)$ *then* $g^{(k)}$ *is* VB^* *on* E_0.

Proof. The first part is contained in [5, Theorem 3.1] (see also [17, II, p. 73, Theorem 4.2]). Only the second part needs a proof. We give an outline of the proof of the second part, keeping all the notations of [5, Theorem 3.1].

The polynomial ω considered in the proof of [5, Theorem 3.1] may be taken as

$$\omega(x) = \int_0^x t^{k+2}(1-t)^{k+2}dt \Big/ \int_0^1 t^{k+2}(1-t)^{k+2}dt\,;$$

the polynomial satisfies the requirements

$$\omega(0) = 0, \quad \omega(1) = 1, \quad \omega^{(j)}(0) = \omega^{(j)}(1) = 0$$

for $j = 1, 2, \ldots, (k+2)$ and moreover

$$\omega^{(j)} = O(x^{k-j+3}), \quad \omega^{(j)}(x) = O((1-x)^{k-j+3})$$

for $j = 1, 2, \ldots, (k+3)$ (see [17, II, p. 74]).

Since $f \in V_kB(E)$ by Lemma 3.8, $f \in V_kB(E_0)$, and hence, by Lemma 3.4, $f_{(k)}$ is of bounded variation on E_0. Since ω is increasing in $[0, 1]$, the function λ in [5, Lemma 3.3] is VB^* on E_0 and since $g^{(k)} = \lambda$, $g^{(k)}$ is VB^* on E_0. Thus, by [5, Lemma 3.3], $g^{(k)}$ satisfies the additional property that $g^{(k)}$ is VB^* on E_0. Supposing in [5, Lemma 3.5] that $g^{(k)}$ has this property, we are to prove that $g^{(k)}$ in [5, Lemma 3.6] has this property and to do this we are to prove that the function λ in [5, Assertion (3.13) satisfies the additional property that $\lambda^{(k-r_0+1)}$ is VB^* on E_0.

Let $(x_i, x_i+\delta_i)$ be any fixed interval contiguous to E_0. Then for any point x_i+t in $(x_i, x_i+\delta_i)$ we have

$$\lambda^{(k-r_0+1)}(x_1+t) = \frac{\lambda(x_i+\delta_i)-\lambda(x_i)}{\delta_i^{k-r_0+1}}\,\omega^{(k-r_0+1)}\left(\frac{i}{\delta_i}\right).$$

Since, by the property of ω, $\dfrac{\omega^{(k-r_0+1)}(x)}{x^{r_0+2}}$ and $\dfrac{\omega^{(k-r_0+1)}(x)}{(1-x)^{r_0+2}}$ remain bounded in $[0, 1]$, there is M such that

$$|\omega^{(k-r_0+1)}(x)| \leqslant M\min[x^{r_0+2}, (1-x)^{r_0+2}]$$

for all $x \in [0, 1]$. Hence

$$|\omega^{(k-r_0+1)}(x)| \leqslant M \quad \text{for all } x \in [0, 1].$$

Also, as in Lemma 3.1

$$|\tilde{h}_{(r_0-1)}(x_i+\delta_i)-\tilde{h}_{(r_0-1)}(x_i)| \leqslant \sum_{j=0}^{r_0-1}\binom{r_0-1}{j}\frac{j^{r_0-1}}{(r_0-1)!}\left|\varepsilon_{r_0-1}\left(\tilde{h}, x_i, x_i+\frac{j}{r_0-1}\delta_i\right) - \varepsilon_{r_0-1}\left(\tilde{h}, x_i+\delta_i, x_i+\delta_i-\frac{j}{r_0-1}\delta_i\right)\right|.$$

Hence

$$|\lambda^{(k-r_0+1)}(x_i+t)| \leqslant M\left|\frac{\tilde{h}_{(r_0-1)}(x_i+\delta_i)-\tilde{h}_{(r_0-1)}(x_i)}{\delta_i^{K-r_0+1}}\right|$$
$$\leqslant L\sum_{j=0}^{r_0-1}\left|\frac{1}{\delta_i^{K-r_0+1}}\varepsilon_{r_0-1}\left(\tilde{h}, x_i, x_i+\frac{j}{r_0-1}\delta_i\right) - \frac{1}{\delta_i^{K-r_0+1}}\varepsilon_{r_0-1}\left(\tilde{h}, x_i+\delta_i, x_i+\delta_i-\frac{j}{r_0-1}\delta_i\right)\right|$$

where L is a constant. Now, since $\tilde{h}_{(r)}(x) = 0$ for $x \in E_0$, $r_0 \leqslant r \leqslant k$, we have

$$\frac{t^{r_0-1}}{(r_0-1)!}\varepsilon_{r_0-1}(\tilde{h}, x_i, x_i+t) = \frac{t^k}{k!}\varepsilon_k(\tilde{h}, x_i, x_i+t)$$

and

$$\frac{(-t)^{r_0-1}}{(r_0-1)!}\varepsilon_{r_0-1}(\tilde{h}, x_i+\delta_i, x_i+\delta_i-t) = \frac{(-t)^k}{k!}\varepsilon_K(\tilde{h}, x_i+\delta_i, x_i+\delta_i-t)$$

and therefore

$$\left|\frac{1}{\delta_i^{k-r_0+1}}\varepsilon_{r_0-1}\left(\tilde{h}, x_i, x_i+\frac{j}{r_0-1}\delta_i\right) - \frac{1}{\delta_i^{k-r_0+1}}\varepsilon_{r_0-1}\left(\tilde{h}, x_i+\delta_i, x_i+\delta_i-\frac{j\delta_i}{r_0-1}\right)\right|$$
$$\leqslant \frac{(r_0-1)!}{k!}\left(\frac{j}{r_0-1}\right)^{k-r_0+1}\left|\varepsilon_k\left(\tilde{h}. x_i, x_i+\frac{j}{r_0-1}\delta_i\right) - (-1)^{k-r_0+1}\varepsilon_k\left(\tilde{h}, x_i, +\delta_i, x_i+\delta_i-\frac{j}{r_0-1}\delta_i\right)\right|$$
$$\leqslant 2\left(\frac{j}{r_0-1}\right)^{k-r_0+1}\cdot\frac{(r_0-1)!}{k!}\,\omega_k(\tilde{h}, [x_i, x_i+\delta_i]).$$

Hence

$$|\lambda^{(k-r_0+1)}(x_i+t)| \leqslant C\omega_k(\tilde{h}, [x_i, x_i+\delta_i])$$

where C is a constant. Thus oscillation of $\lambda^{(k-r_0+1)}$ on $[x_i, x_i+\delta_i]$ does not exceed $2C\omega_k(\tilde{h}, [x_i, ,x_i+\delta_i])$. Since $f \in V_kB(E)$, by Lemma 3.8 $f \in V_kB(E_0)$. Also, since $\tilde{g}^{(k)}$ is continuous and is VB^* on E_0 (by induction hypotheses in Lemma (3.5) of [5]), it can be proved as in Theorem 4.1 that $\tilde{g} \in V_kB^*(E_0)$. Hence $\tilde{h} \in V_kB(E_0)$ and therefore the series $\sum\omega_k(\tilde{h}, [x_i, x_i+\delta_i])$ converges. Hence $\lambda^{(k-r_0+1)} \in VB^*(E_0)$. This proves that the function λ in Assertion (3.13) of [5] is such that $\lambda^{(k-r_0+1)}$ is VB^* on E_0.

Let $\tilde{\lambda}$ be an indefinite integral of $\lambda^{(k-r_0+1)}$ over $[a, b]$ of order k. Set $g = \tilde{g}+\tilde{\lambda}$, $h = \tilde{h}-\tilde{\lambda}$. Then $g^{(k)}$ is continuous in $[a, b]$ and $g^{(k)} \in VB^*(E_0)$. Also $h_r(x) = 0$ on E_0 for $r_0-1 \leqslant r \leqslant k$, completing the proof of the theorem.

THEOREM 4.5. *Let f be measurable and let $f_{(n)}$ exist on $E\subset[a,b]$. If $f\in V_nBG^*(E)$, then almost everywhere on E, $f_{(n+1)}$ and $(f_{(n)})'_{ap}$ exist and are equal.*

Proof. Since $f\in V_nBG^*(E)$, $E=\bigcup_i E_i$, where E_i is measurable for each i, and $f\in V_nB^*(E_i)$ for each i. Let i be fixed. It is sufficient to show that $f_{(n+1)}=(f_{(n)})'_{ap}$ a.e. on E_i. By Theorem 4.4 there exist a perfect set $F_i\subset E_i$ such that $\mu(E_i\sim F_i)$ is arbitrarilly small and two functions g and h such that $f=g+h$ where $g^{(n)}$ is continuous in $[a,b]$ and $g^{(n)}\in VB^*(F_i)$ and $h_{(r)}(x)=0$ for $x\in F_i$, $0\leqslant r\leqslant n$. By Theorem 4.1, $g\in V_nB^*(F_i)$. Since $f\in V_nB^*(E_i)$, $h\in V_nB^*(F_i)$. Let $\{(v_i,\delta_i)\}$ be the contiguous intervals of F_i. Let

$$M(x)=\begin{cases}0 & \text{if } x\in F_i,\\ \omega_n^*(h,v_i,\delta_i) & \text{if } x\in(v_i,\delta_i).\end{cases}$$

Since $h\in V_nB^*(F_i)$, the sum $\sum\omega_n^*(h,v_i,\delta_i)$ is convergent. So, $M\in VB^*(F_i)$. Since $g^{(n)}\in VB^*(F_i)$, M^1 and $g^{(n+1)}$ exist and are finite in a subset $G\subset F_i$ such that $\mu(G)=\mu(F_i)$. Further, since $g^{(n)}=f_{(n)}$ on F_i, it follows that

$$g^{(n+1)}(x)=f'_{(n)ap}(x)$$

at every point of G which is a point of density of G. Let $\xi\in G$ be a point of density of G. Since $\xi\in F_i$, $h_{(r)}(\xi)=0$ for $0\leqslant r\leqslant n$; hence

(1) $$h(\xi+t)=\frac{t^n}{n!}\varepsilon_n(h,\xi,\xi+t)$$

and hence

(2) $$\varepsilon_n(h,\xi,\xi+t)=0\quad\text{for }\xi+t\in F_i.$$

If $\xi<v_i<\xi+t<\delta_i$ then

$$h(\xi+t)=\frac{(\xi+t-v_i)^n}{n!}\varepsilon_n(h,v_i,\xi+t).$$

Hence by (1)

$$\begin{aligned}|\varepsilon_n(h,\xi,\xi+t)|&=\left(\frac{\xi+t-v_1}{t}\right)^n|\varepsilon_n(h,v_i,\xi+t)|\\ &\leqslant\left(\frac{\xi+t-v_i}{t}\right)^n\omega_n^*(h,v_i,\delta_i)\\ &=\left(\frac{\xi+t-v_i}{t}\right)^n[M(\xi+t)-M(\xi)].\end{aligned}$$

So

$$\left|\frac{1}{t}\varepsilon_n(h,\xi,\xi+t)\right|\leqslant\left(\frac{\xi+t-v_i}{t}\right)^n\cdot\frac{1}{t}[M(\xi+t)-M(\xi)].$$

Since $M'(\xi)$ exist and is finite, and since ξ is a point of density of F_i,

$$\frac{\varepsilon_n(h,\xi,\xi+t)}{t}\to 0\quad\text{as }\xi+t\to\xi+0$$

through the points of the complementary set of F_i. Hence by (2)

$$\frac{\varepsilon_n(h,\xi,\xi+t)}{t}\to 0\quad\text{as }t\to 0+.$$

Similarly,

$$\frac{\varepsilon_n(h,\xi,\xi+t)}{t}\to 0\quad\text{as }t\to 0-.$$

Hence $h_{(n+1)}(\xi)=0$ Thus

$$f_{(n+1)}(\xi)=g^{(n+1)}(\xi)=(f_{(n)})'_{ap}(\xi).$$

So, $f_{(n+1)}=(f_{(n)})'_{ap}$ a.e. on F_i. Since $\mu(E_i\sim F_i)$ is arbitrarily small, $f_{(n+1)}=(f_{(n)})'_{ap}$ a.e. on E_i.

THEOREM 4.6. *Let f be measurable and let $f_{(n)}$ exist on $E\subset[a,b]$. If $f\in V_nBG(E)$ then $(f_{(n)})'_{ap}$ exists a.e. on E.*

Proof. Since $f\in V_nBG(E)$, $E=\bigcup_i E_i$ such that $f\in V_nB(E_i)$ for all i. Then, by Lemma 3.4 $f_{(n)}$ is of bounded variation on E_i. Thus $f_{(n)}$ is VBG on E. By the Denjoy–Khintchine Theorem [13], p. 222, it follows that $(f_{(n)})'_{ap}$ exists a.e. on E.

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Two examples concerning small intrinsic isometries

by

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Abstract. Borsuk, and Olędzki and Spież, have given examples in certain cases of arc length preserving embeddings (intrinsic isometries) whose images have arbitrarily small diameters. We give two additional examples, one negative and one positive, and raise several specific questions concerning further possibilities.

Introduction. Basic definitions and properties are in [1], [2], and [3]. If X is a metric space with metric ϱ, then the *intrinsic metric* on X induced by ϱ is given by $\varrho_X(x, y) =$ least upper bound $\{\text{length } (L): L \text{ is an arc in } X \text{ containing } x \text{ and } y\}$. For ϱ_X to be defined it is necessary that each two points of X lie in some arc of finite length; for ϱ_X to induce the same topology as ϱ, it is necessary and sufficient that, for each $x \in X$ and each $\varepsilon > 0$, there is a neighborhood U of x in X, such that for each $y \in U$ there is an arc L of length $< \varepsilon$ in U containing both x and y. Spaces for which the two metrics are compatible are called *geometrically acceptable* (GA). All sets we consider will be GA. A mapping of X onto Y is an *intrinsic isometry* if it is a isometry with respect to the intrinsic metrics; or equivalently, if it preserves all arc lengths (Borsuk [2]). A mapping f of X into Y is an *intrinsic embedding* if $f\colon X \to f(X)$ is an intrinsic isometry; here the intrinsic metric on $f(X)$ is defined using arcs in $f(X)$.

We will say that X is *intrinsically small in* Y if, for each $\varepsilon > 0$, there is an intrinsic embedding $f\colon X \to Y$ such that $f(X)$ has diameter $< \varepsilon$ in the original metric of Y. The three previously known results concerning intrinsically small spaces are:

(1) E^n is intrinsically small in E^{n+1} (Olędzki and Spież [6]; Borsuk [1] earlier obtained E^{2n});

(2) Each 1-dimensional polytope in an E^n or in Hilbert space is intrinsically small in E^3 (Borsuk [1]);

(3) No subset of E^n containing an open set is intrinsically small in E^n (Borsuk [2]).

We will add two more examples: a certain compact 1-dimensional subset of E^2 is *not* intrinsically small in E^2; and bounded cylinders in E^3 are intrinsically small in E^3.