Compact-covering numbers

by

George Baloglou (Lawrence, KS) and W. W. Comfort (Middletown, CT)

Abstract. A compact cover of a topological space X is a family $\mathscr{K}$ of compact subsets of X such that $X = \bigcup \mathscr{K}$; and $\varkappa(X)$, the compact-covering number of X, is the least cardinal which arises as the cardinality of a compact cover of X. Here we study the cardinal function $\varkappa$, with emphasis on its behavior with respect to products.

The anticipated equality $\varkappa(\prod_{i\in I} X_i) = \prod_{i\in I} \varkappa(X_i)$, from which follows the implication $\varkappa(X_i) \leqslant \varkappa(Y_i) \Rightarrow \varkappa(\prod_{i\in I} X_i) \leqslant \varkappa(\prod_{i\in I} Y_i)$, is equivalent to GCH; assuming MA+$\neg$CH it is not difficult to find (Tychonoff) spaces X and Y with $\varkappa(Y) < \varkappa(X)$ and $\varkappa(X^\omega) < \varkappa(Y^\omega)$.

Writing $X \simeq Y$ if $\varkappa(X^\beta) = \varkappa(Y^\beta)$ for all cardinals β, we find several conditions sufficient to ensure that $X \simeq Y$ whenever $\varkappa(X) = \varkappa(Y)$; in general, however, we know of no upper bound on the number of $\simeq$-inequivalent spaces Y for which $\varkappa(Y) = \varkappa(X)$.

§ 1. Introduction. Working in ZFC, we follow the standard notation and terminology of Set-theoretic Topology; our basic references are [9] and [13]. For an ordinal ξ, the set ξ with the discrete and the order topologies is denoted ξ and $\langle\xi\rangle$, respectively. Unless otherwise stated, no separation axioms are assumed.

Compact-covering numbers (defined as in the abstract) have been useful in a variety of mathematical settings, including for example locally compact topological groups (see [6] (§ 3)) and separable, metrizable spaces (see [7] (§ 8)). It is the theory of products of compact-covering numbers, however, that seems to generate the most interesting questions for both the topologist and the set-theorist.

The best known compact-covering number is the "ubiquitous" cardinal $\boldsymbol{d}$, the compact-covering number of the space ω^ω of irrational numbers. The cardinal $\boldsymbol{d}$ has been introduced by Katětov [16] and studied in depth by Hechler [11], [12]. For a variety of topological results related to $\boldsymbol{d}$ and to other "small uncountable cardinals" reflecting combinatorial properties of the integers, the reader is referred to [10] and [7]; some of the relations among these cardinals are investigated in [10], [7] and [19].

The Baire category theorem for $\boldsymbol{R}$ implies that $\boldsymbol{d} > \omega$; hence $\boldsymbol{d} = \omega_1 = 2^\omega$ under CH. Further, it is known (see [11] or 2.3 below) that $\mathrm{cf}(\boldsymbol{d}) > \omega$, and Hechler has proved [11], [12] that the inequalities $\omega_1 \leqslant \mathrm{cf}(\boldsymbol{d}) \leqslant \boldsymbol{d} \leqslant 2^\omega$ are the only possible restrictions on $\boldsymbol{d}$ (in ZFC). For example, there are models of ZFC where $\boldsymbol{d} = \omega_1 < 2^{\omega}$ (see [13], p. 261). Yet, there do exist (in ZFC) subsets P of $\boldsymbol{R}$ (Bernstein sets) such that $\varkappa(P) = 2^{\omega}$ [7]; see [4] for details and for an example (provided to us by Rae Shortt) of measurable $P \subset \boldsymbol{R}$ such that $\varkappa(P) = 2^{\omega}$.

Assuming Martin's axiom together with the denial of the continuum hypothesis (MA+$\neg$CH), $\boldsymbol{R}$ can be covered by no less than 2^{ω} nowhere dense sets [18]; consequently, $\boldsymbol{d} = 2^{\omega} > \omega_1$ under MA+$\neg$CH. This result (first derived implicitly by Hechler [12] — see also [4] and [13], pp. 261, 586) yields the following (easy, yet surprising) example, which has largely motivated our work.

1.1. THEOREM. *Assuming* MA+$\neg$CH, *there exist* (*Tychonoff*) *spaces* X, Y *such that* $\varkappa(Y) < \varkappa(X)$ *and* $\varkappa(Y^{\omega}) > \varkappa(X^{\omega})$.

Proof. Let $Y = \omega$ and $X = \langle\omega_1\rangle$. Clearly, $\varkappa(Y) = \omega < \omega_1 = \varkappa(X)$ and $\varkappa(X^{\omega}) = \omega_1$ (see 2.6 below); then as indicated above from MA+$\neg$CH we have

$$\varkappa(Y^{\omega}) = \boldsymbol{d} = 2^{\omega} > \omega_1 = \varkappa(X^{\omega}).$$

Among the other cardinal functions, the ones that are most closely related to $\varkappa(X)$ are the familiar Lindelöf degree $L(X)$ (= least α such that every open cover of X has a subcover of cardinality not exceeding α) and the compact weight $\mathrm{kw}(X)$; the latter — studied by Katětov [16], [17], Arhangel'skiĭ [3] and van Douwen [7] — is defined as the least cardinality of a k-base of X. (A k-base of X is a family $\mathscr{C}$ of compact subsets of X with the property that for every compact $K \subset X$ there is $C \in \mathscr{C}$ such that $K \subset C$.)

It is easy to see that $L(X) \leqslant \varkappa(X) \leqslant \mathrm{kw}(X)$. The equalities $\varkappa(X) = L(X)$, $\varkappa(X) = \mathrm{kw}(X)$ hold for locally compact spaces and for certain other classes of spaces; see Section 4 below. It is worth mentioning here that Katětov [17], and independently Galvin, have shown that $\mathrm{kw}(\boldsymbol{Q}) = \boldsymbol{d}$ (see also [7]).

Most of the results proved here appear in the doctoral dissertation [4], written by the first-listed author under the supervision of the second-listed author. We are pleased to thank Eric K. van Douwen, Fred Galvin, Anthony W. Hager, Thomas J. Peters, Lewis C. Robertson and Johannes Vermeer for helpful remarks and conversations.

§ 2. Concerning the numbers $\varkappa(\prod_{i\in I} X_i)$. In the following two simple lemmas, we observe that continuous functions and closed subspaces do not raise the compact-covering number and we obtain upper and lower bounds for $\varkappa(\prod_{i\in I} X_i)$.

2.1. LEMMA. *Let X and Y be spaces, and suppose either that*

(a) *there is a continuous function from X onto Y, or*

(b) *Y is a closed subspace of X.*

Then $\varkappa(Y) \leqslant \varkappa(X)$.

2.2. LEMMA. *Let $\{X_i: i \in I\}$ be a set of non-empty spaces and $X = \prod_{i\in I} X_i$. Then*

(a) $\varkappa(X) \leqslant \prod_{i\in I} \varkappa(X_i)$;

(b) $\sup\{\varkappa(X_i): i \in I\} \leqslant \varkappa(X)$;

(c) *if* $|I| < \omega$, *then* $\varkappa(X) = \prod_{i\in I} \varkappa(X_i)$;

(d) *if each X_i is non-compact, then* $\varkappa(X) > |I|$.

Proof. We only prove 2.2(d). If $\varkappa(X) \leqslant |I|$ there are a compact cover $\mathscr{K}$ of X and a (not necessarily one-to-one) function $i \to K_i$ from I onto $\mathscr{K}$. Since X_i is not compact there is $p_i \in X_i \setminus \pi_i[K_i]$. This defines a point

$$p = \langle p_i: i \in I\rangle \in X.$$

Since $X = \bigcup \mathscr{K}$ there is $i \in I$ such that $p \in K_i \in \mathscr{K}$. It follows that

$$p_i = \pi_i(p) \in \pi_i[K_i],$$

a contradiction.

We see next that when the spaces X_i of 2.2(d) are homeomorphic, the inequality $\varkappa(X) > |I|$ can be strengthened to $\mathrm{cf}(\varkappa(X)) > |I|$; this result generalizes the inequality $\mathrm{cf}(\boldsymbol{d}) > \omega$ and is of independent interest.

2.3. THEOREM. *Let X be a non-compact space and $\beta \geqslant \omega$. Then* $\mathrm{cf}(\varkappa(X^{\beta})) > \beta$.

Proof. Define $Y = X^{\beta}$ and $\gamma = \mathrm{cf}(\varkappa(Y))$, and suppose $\gamma \leqslant \beta$. Since Y^{γ} is homeomorphic to Y, there is a compact cover $\mathscr{K}$ of Y^{γ} such that

$$|\mathscr{K}| = \varkappa(Y^{\gamma}) = \varkappa(Y).$$

Since $\mathscr{K}$ is infinite and $\gamma = \mathrm{cf}(|\mathscr{K}|)$ we may write

$$\mathscr{K} = \bigcup_{\eta<\gamma} \mathscr{K}_{\eta}$$

with each $|\mathscr{K}_{\eta}| < |\mathscr{K}|$. Since $\{\pi_{\eta}[K]: K \in \mathscr{K}_{\eta}\}$ is not a cover of Y, there is

$$p(\eta) \in Y \setminus \bigcup \{\pi_{\eta}[K]: K \in \mathscr{K}_{\eta}\}.$$

This defines a point $p = \langle p(\eta): \eta < \gamma\rangle \in Y^{\gamma}$. Since $Y^{\gamma} = \bigcup \mathscr{K}$ and $\mathscr{K} = \bigcup_{\eta<\gamma} \mathscr{K}_{\eta}$, there are $\eta < \gamma$ and $K \in \mathscr{K}_{\eta}$ such that $p \in K$. It then follows that

$$p(\eta) = \pi_{\eta}(p) \in \pi_{\eta}[K];$$

this contradiction completes the proof.

2.4. Remark. It is tempting to try to strengthen 2.2(d) and 2.3 into the statement that $\mathrm{cf}(\varkappa(\prod_{i\in I} X_i)) > |I|$ for every set $\{X_i: i \in I\}$ of non-compact spaces. To see that this cannot be done, let β and γ be any cardinals whatever with $\gamma > 1$

and $\beta \geqslant \omega$, choose $\lambda > \gamma^\beta$ with $\mathrm{cf}(\lambda) = \omega$, and let $X = \lambda \times \gamma^\beta$; it is easy to see that $\varkappa(X) = \lambda$, so that $\mathrm{cf}(\varkappa(X)) = \omega < \beta$.

We need two more lemmas before we present the main result of this section.

2.5. LEMMA. *Let $\{X_i: i \in I\}$ be a set of non-compact spaces and $X = \prod_{i \in I} X_i$, and define*

$$\alpha = \sup\{\varkappa(X_i): i \in I\}, \quad J = \{i \in I: \varkappa(X_i) = \alpha\},$$
$$\delta = \sup\{\varkappa(X_i): i \in I \setminus J\}.$$

If either $|J| \geqslant \mathrm{cf}(\alpha)$ or $\delta = \alpha$, then $\varkappa(X) > \alpha$.

Proof. From 2.2(b) we have $\varkappa(X) \geqslant \alpha$. We assume $\varkappa(X) = \alpha$, we choose a compact cover $\mathscr{K}$ of X such that $|\mathscr{K}| = \alpha$, and we write

$$\mathscr{K} = \bigcup_{\eta < \mathrm{cf}(\alpha)} \mathscr{K}_\eta$$

with each $|\mathscr{K}_\eta| < \alpha$. If $|J| \geqslant \mathrm{cf}(\alpha)$ we let $\{i(\eta): \eta < \mathrm{cf}(\alpha)\}$ be a faithfully indexed subset of J, and if $\delta = \alpha$ we let $\{i(\eta): \eta < \mathrm{cf}(\alpha)\}$ be a faithfully indexed subset of $I \setminus J$ such that

$$\varkappa(X_{i(\eta)}) > |\mathscr{K}_\eta|.$$

In either case $\{\pi_{i(\eta)}[K]: K \in \mathscr{K}_\eta\}$ is not a cover of $X_{i(\eta)}$, so there is

$$p(\eta) \in X_{i(\eta)} \setminus \cup \{\pi_{i(\eta)}[K]: K \in \mathscr{K}_\eta\}.$$

There also exist $p = \langle p_i: i \in I\rangle \in X$ such that $p_{i(\eta)} = p(\eta)$ for $\eta < \mathrm{cf}(\alpha)$, and as in the proof of 2.3 there are $\eta < \mathrm{cf}(\alpha)$ and $K \in \mathscr{K}_\eta$ such that $p \in K$. It then follows that

$$p(\eta) = \pi_{i(\eta)}(p) \in \pi_{i(\eta)}[K],$$

a contradiction.

2.6. LEMMA. *Let α be an infinite cardinal and $0 < \beta < \mathrm{cf}(\alpha)$. Then $\varkappa(\langle\alpha\rangle^\beta) = \mathrm{cf}(\alpha)$.*

Proof. From $\varkappa(\langle\alpha\rangle) = \mathrm{cf}(\alpha)$ and 2.2(b) follows $\varkappa(\langle\alpha\rangle^\beta) \geqslant \mathrm{cf}(\alpha)$. Now let $\alpha = \sup\{\alpha_\xi: \xi < \mathrm{cf}(\alpha)\}$ with each $\alpha_\xi < \alpha$ and for each $\xi < \mathrm{cf}(\alpha)$ define

$$K_\xi = \langle \alpha_\xi + 1\rangle^\beta = \{x \in \langle\alpha\rangle^\beta: x_\eta \leqslant \alpha_\xi \text{ for all } \eta < \beta\}.$$

Since $\beta < \mathrm{cf}(\alpha)$ and $\sup\{\alpha_\xi: \xi < \mathrm{cf}(\alpha)\} = \alpha$, for every $x = \langle x_\eta: \eta < \beta\rangle \in \langle\alpha\rangle^\beta$ there is $\xi < \mathrm{cf}(\alpha)$ such that $x \in K_\xi$. Thus $\langle\alpha\rangle^\beta = \bigcup_{\xi < \mathrm{cf}(\alpha)} K_\xi$, and since each K_ξ is compact we have $\varkappa(\langle\alpha\rangle^\beta) \leqslant \mathrm{cf}(\alpha)$, as required.

2.7. THEOREM. *The following are equivalent.*

(a) *The generalized continuum hypothesis* (GCH).

(b) *Every set $\{X_i: i \in I\}$ of spaces satisfies $\varkappa(\prod_{i \in I} X_i) = \prod_{i \in I} \varkappa(X_i)$.*

(c) *Every space X and cardinal β satisfy $\varkappa(X^\beta) = (\varkappa(X))^\beta$.*

(d) *Every infinite cardinal α satisfies $\varkappa(\langle\alpha^+\rangle^\alpha) = 2^\alpha$.*

Proof. (a) $\Rightarrow$ (b). For notational simplicity set $\prod_{i \in I} X_i = X$. Using 2.2(c) and Tychonoff's product theorem, if necessary, we may and do assume that each X_i is non-compact.

With a view to invoking 2.5 we define

$$\alpha = \sup\{\varkappa(X_i): i \in I\}, \quad J = \{i \in I: \varkappa(X_i) = \alpha\}, \quad \text{and}$$
$$\delta = \sup\{\varkappa(X_i): i \in I \setminus J\},$$

and we consider three cases separately.

Case 1. $\alpha \leqslant |I|$. From 2.2(a) and 2.2(d) we then have

$$|I|^+ \leqslant \varkappa(X) \leqslant \prod_{i \in I} \varkappa(X_i) \leqslant \alpha^{|I|} \leqslant |I|^{|I|} = 2^{|I|} = |I|^+,$$

so that $\varkappa(X) = \prod_{i \in I} \varkappa(X_i) = |I|^+$.

Case 2. $|I| < \alpha$, and either $|J| \geqslant \mathrm{cf}(\alpha)$ or $\delta = \alpha$. From 2.5 we then have

$$\alpha < \varkappa(X) \leqslant \prod_{i \in I} \varkappa(X_i) \leqslant \alpha^{|I|} \leqslant (2^\alpha)^{|I|} = 2^\alpha = \alpha^+,$$

so that $\varkappa(X) = \prod_{i \in I} \varkappa(X_i) = \alpha^+$.

Case 3. $|I| < \alpha$, and $|J| < \mathrm{cf}(\alpha)$ and $\delta < \alpha$. These three inequalities yield, with (a), the relations $2^{|I|} \leqslant \alpha$, $\alpha^{|J|} \leqslant \alpha$ (see [13], p. 49), and $2^\delta \leqslant \alpha$. From 2.2(b) we then have

$$\alpha \leqslant \varkappa(X) \leqslant \prod_{i \in J} \varkappa(X_i) = \prod_{i \in J} \varkappa(X_i) \cdot \prod_{i \in I \setminus J} \varkappa(X_i)$$
$$\leqslant \alpha^{|J|} \cdot \delta^{|I \setminus J|} \leqslant \alpha^{|J|} \cdot \delta^{|I|} \leqslant \alpha^{|J|} \cdot 2^\delta \cdot 2^{|I|} \leqslant \alpha \cdot \alpha \cdot \alpha = \alpha,$$

so that $\varkappa(X) = \prod_{i \in I} \varkappa(X_i) = \alpha$.

(b) $\Rightarrow$ (c). This is obvious.

(c) $\Rightarrow$ (d). It follows from (c), replacing X by $\langle\alpha^+\rangle$ and β by α, that

$$\varkappa(\langle\alpha^+\rangle^\alpha) = (\varkappa(\langle\alpha^+\rangle))^\alpha = (\alpha^+)^\alpha = 2^\alpha.$$

(d) $\Rightarrow$ (a). From (d) and 2.6 we have

$$2^\alpha = \varkappa(\langle\alpha^+\rangle^\alpha) = \alpha^+$$

for every infinite cardinal α, as required.

2.8. THEOREM. *The following statements are equivalent, and* GCH *implies each of them.*

(i) *If $\varkappa(X_i) \leqslant \varkappa(Y_i)$ for each $i \in I$, then $\varkappa(\prod_{i \in I} X_i) \leqslant \varkappa(\prod_{i \in I} Y_i)$.*

(ii) *If $\varkappa(X_i) = \varkappa(Y_i)$ for each $i \in I$, then $\varkappa(\prod_{i \in I} X_i) = \varkappa(\prod_{i \in I} Y_i)$.*

Proof. That (i) ⇒ (ii) is clear. Now given X_i, Y_i as in (i), let Z_i be the "disjoint union" space $X_i \dot\cup Y_i$ and note that $\varkappa(Z_i) = \varkappa(Y_i)$. Since X_i is closed in Z_i the space $\prod_{i\in I} X_i$ is closed in $\prod_{i\in I} Z_i$ and from (ii) and 2.1(b) we have

$$\varkappa(\prod_{i\in I} X_i) \leqslant \varkappa(\prod_{i\in I} Z_i) = \varkappa(\prod_{i\in I} Y_i).$$

That GCH implies (ii) is immediate from the implication (a) ⇒ (b) of 2.7.

2.9. Remark. We did not determine whether or not conditions (i) and (ii) of 2.8 imply GCH.

§ 3. The relation $\simeq$.

For every infinite cardinal α we write

$$\Lambda(\alpha) = \{X\colon \varkappa(X) = \alpha\},$$

and for spaces X and Y we write $X \simeq Y$ if $\varkappa(X^\beta) = \varkappa(Y^\beta)$ for all cardinals β. It is clear that the equivalence relation $\simeq$ respects the classes $\Lambda(\alpha)$ in the sense that if $X \simeq Y$ then $\varkappa(X) = \varkappa(Y)$.

To fix ideas we summarize some of the results of Sections 1 and 2, using the notation and terminology of the preceding paragraph.

3.1. Theorem. (a) *Assume* GCH. *Then for every infinite cardinal α the class $\Lambda(\alpha)$ contains exactly one $\simeq$-equivalence class.*

(b) *Assume* MA + ¬ CH. *Then for every regular cardinal α such that $\omega < \alpha < 2^\omega$ the class $\Lambda(\alpha)$ contains at least two $\simeq$-equivalence classes.*

Proof. (a) is the implication (a) ⇒ (c) of 2.7.

(b) Since $\alpha \in \Lambda(\alpha)$ and $\langle\alpha\rangle \in \Lambda(\alpha)$, it is enough to show $\varkappa(\alpha^\omega) \neq \varkappa(\langle\alpha\rangle^\omega)$. Using 2.6 and 2.1(b) and arguing as in 1.1, we have

$$\varkappa(\langle\alpha\rangle^\omega) = \alpha < 2^\omega = \varkappa(\omega^\omega) \leqslant \varkappa(\alpha^\omega).$$

In the next lemma we obtain an inequality that strengthens 2.2(a) and yields the main result (3.4) of this section.

3.2. Lemma. *Let $\{X_i\colon i\in I\}$ be a set of spaces, and set $X = \prod_{i\in I} X_i$ and $P = \prod_{i\in I} \varkappa(X_i)$. Then $\varkappa(X) \leqslant \varkappa(P)$.*

Proof. Let $\{K_\xi\colon \xi<\alpha\}$ be a compact cover of P with $\alpha = \varkappa(P)$ and for $i\in I$ let $\{L_{i,\eta}\colon \eta < \varkappa(X_i)\}$ be a compact cover of X_i. For $\xi<\alpha$ the projection π_i from P onto $\varkappa(X_i)$ takes K_ξ to a compact (hence finite) subset of $\varkappa(X_i)$, so the set $M_{i,\xi}$ defined by

$$M_{i,\xi} = \bigcup \{L_{i,\eta}\colon \eta \in \pi_i[K_\xi]\}$$

is a compact subset of X_i. For $\xi<\alpha$ we set

$$M_\xi = \prod_{i\in I} M_{i,\xi};$$

it is enough to show that $\{M_\xi\colon \xi<\alpha\}$ is a cover of X. Given $x = \langle x_i\colon i\in I\rangle \in X$, for $i\in I$ there is $\eta_i < \varkappa(X_i)$ such that $x_i \in L_{i,\eta_i}$. This defines a point

$$p = \langle \eta_i\colon i\in I\rangle \in P.$$

There is $\xi<\alpha$ such that $p\in K_\xi$, and for $i\in I$ we have

$$\eta_i = \pi_i(p) \in \pi_i[K_\xi].$$

It follows that

$$x_i \in L_{i,\eta_i} \subset M_{i,\xi},$$

so that $x\in M_\xi$, as required.

3.3. Remark. It is immediate from 3.2 that for every $\alpha \geqslant \omega$ the discrete space α is "maximal" in $\Lambda(\alpha)$ in the following sense: for every $X\in\Lambda(\alpha)$ and cardinal β, we have $\varkappa(X^\beta) \leqslant \varkappa(\alpha^\beta)$.

3.4. Theorem. *If X contains a closed, discrete subspace of cardinality $\varkappa(X)$, then $X \simeq \varkappa(X)$.*

Proof. With $\alpha = \varkappa(X)$ we have of course $X\in\Lambda(\alpha)$ and $\varkappa(X)\in\Lambda(\alpha)$. For every cardinal β the inequality $\varkappa(X^\beta) \leqslant \varkappa(\alpha^\beta)$ is given by 3.3, while the reverse inequality follows from 2.1(b).

3.5. Corollary. *If X is a non-compact, σ-compact, T_1 space, then $X\simeq\omega$.*

Proof. The space X is not countably compact, hence X contains a closed copy of ω ([1], p. 20).

3.6. Remark. We do not know whether in 3.5 the hypothesis that X is a T_1 space is redundant.

The next result (first pointed out to us by David Feldman) gives a sufficient condition for the relation $X\simeq Y$ and simplifies the proofs of the subsequent corollaries.

3.7. Theorem. *Let X and Y be spaces. If there is a continuous function f from X onto Y such that $f^{-1}(K)$ is compact (in X) whenever K is compact (in Y), then $X\simeq Y$.*

Proof. Let β be a cardinal. The inequality $\varkappa(Y^\beta)\leqslant\varkappa(X^\beta)$ follows from 2.1(a). The reverse inequality follows from the observation that if $\mathscr{K}$ is a compact cover of Y^β then

$$\{\prod_{\xi<\beta} f^{-1}(\pi_\xi[K])\colon K\in\mathscr{K}\}$$

is a compact cover of X^β.

3.8. Corollary. *Let X be a space containing no infinite compact subset. Then $X\simeq\varkappa(X)$.*

Proof. If $|X|<\omega$ this is clear. If $|X|\geqslant\omega$, then $|X| = \varkappa(X)$, and any one-to-one function from $\varkappa(X)$ onto X satisfies the conditions (on f) in 3.7.

3.9. COROLLARY. *If there is a perfect map f from X onto Y, then $X \simeq Y$.*

Proof. If K is a compact subset of Y, then $f^{-1}(K)$ is compact in X. (See [9], p. 236.)

3.10. Remark. It has been pointed out by Johannes Vermeer that the invariance of compact-covering numbers under perfect maps can provide an alternative proof of 3.2, similar to his proof of the inequality $L(\prod_{i\in I} X_i) \leqslant \omega \cdot \sum_{i\in I} \varkappa(X_i)$ [5].

§ 4. Concerning hemicompact, chain-compact spaces. We first define and briefly discuss the spaces we investigate here.

4.1. DEFINITION. A space X is *chain-compact* if it is possible to write $X = \bigcup_{\eta<\alpha} K_\eta$ with each K_η compact in such a way that $K_{\eta_1} \subsetneqq K_{\eta_2}$ whenever $\eta_1 < \eta_2 < \alpha$. (We write $X = \bigcup_{\eta<\alpha} K_\eta \uparrow$.)

Of course, every σ-compact space is chain-compact. We remark in passing that every chain-compact space which is hereditarily separable or paracompact is σ-compact; the first of these statements is part of a more general result and can be derived as in [15], p. 16 or [4]; the latter is a consequence of the fact that every paracompact, chain-compact space is Lindelöf [20] and of the following theorem.

4.2. THEOREM. *Let X be a non-compact, chain-compact space. Then*

(a) *$\varkappa(X)$ is a regular cardinal;*

(b) (Galvin) *$\varkappa(X) = L(X)$; and*

(c) *X may be written in the form $X = \bigcup_{\eta<\alpha} K_\eta \uparrow$ with each K_η compact and with $\alpha = \varkappa(X)$.*

Proof. We may assume that $X = \bigcup_{\eta<\alpha} K_\eta \uparrow$ with α a regular cardinal. Since $L(X) \leqslant \varkappa(X)$, it suffices to show that if $X = \bigcup_{\eta<\alpha} K_\eta \uparrow$ with each K_η compact and α regular then $\alpha \leqslant L(X)$; arguing by contradiction, we do this by proving that if $X = \bigcup_{\eta<\alpha} K_\eta \uparrow$ with regular $\alpha > L(X)$, then X is compact.

If $X = \bigcup_{\eta<\alpha} K_\eta \uparrow$ with (regular) $\alpha > L(X)$, then — as has been pointed out by van Douwen — X must be $L(X)$-bounded (i.e. every subset of X of cardinality $\leqslant L(X)$ is contained in a compact subset of X); it follows then (see [21], p. 611) that X is initially $L(X)$-compact (i.e. every open cover of X of cardinality $\leqslant L(X)$ has a finite subcover), hence X is compact.

Alternatively, set $X = \bigcup_{\eta<\alpha} K_\eta \uparrow$ with (regular) $\alpha > L(X)$ and let $\mathscr{U}$ be an open cover of X; we may assume that $\omega \leqslant |\mathscr{U}| \leqslant L(X)$. For each $\eta < \alpha$, choose finite $\mathscr{U}_{(\eta)} \subset \mathscr{U}$ such that $K_\eta \subset \bigcup \mathscr{U}_{(\eta)}$. Since α is regular and $|\mathscr{U}| \leqslant L(X) < \alpha$ there are $A \subset \alpha$ and finite $\mathscr{V} \subset \mathscr{U}$ such that $|A| = \alpha$ and $\mathscr{U}_{(\eta)} = \mathscr{V}$ for all $\eta \in A$; clearly $X = \bigcup \mathscr{V}$, as required.

The following definition generalizes Arens' definition [2] of hemicompactness (equivalent to $\mathrm{kw}(X) = \omega$).

4.3. DEFINITION. A space X is *hemicompact* if it is possible to write $X = \bigcup_{\eta<\varkappa(X)} K_\eta$ with each K_η compact and in such a way that for every compact $K \subset X$ there is $\delta < \varkappa(X)$ such that $K \subset \bigcup_{\eta<\delta} K_\eta$.

It is clear from 4.3 and the definition of $\mathrm{kw}(X)$ that X is hemicompact whenever $\mathrm{kw}(X) = \varkappa(X)$. The equality $\mathrm{kw}(X) = \varkappa(X)$ occurs frequenty (see for example [7] and [16]); an important instance is provided by the following theorem.

4.4. THEOREM [16]. *Let X be a locally compact space. Then $kw(X) = \varkappa(X)$ (and hence X is hemicompact).*

Proof. It is enough to assume $\varkappa(X) \geqslant \omega$ and to show $\mathrm{kw}(X) \leqslant \varkappa(X)$. Let $\mathscr{K}$ be a compact cover of X such that $|\mathscr{K}| = \varkappa(X)$, for every $x \in X$ choose a compact set $F(x)$ such that $x \in \operatorname{int} F(x)$ and for $K \in \mathscr{K}$ choose finite $A(K) \subset K$ such that

$$K \subset \bigcup_{x\in A(K)} \operatorname{int} F(x).$$

It is then clear, writing

$$S = \bigcup \{A(K) \colon K \in \mathscr{K}\} \quad \text{and} \quad \mathscr{C} = \{\bigcup_{x\in B} F(x) \colon B \in [S]^{<\omega}\},$$

that $\mathscr{C}$ is a k-base for X satisfying

$$|\mathscr{C}| = |[S]^{<\omega}| = |S| \leqslant \omega \cdot |\mathscr{K}| = \varkappa(X).$$

We are ready for the principal result of this section.

4.5. THEOREM. Let *$\{X_i \colon i \in I\}$ be a set of spaces, and set $X = \prod_{i\in I} X_i$ and $P = \prod_{i\in I} \langle \varkappa(X_i) \rangle$.*

(a) *If each X_i is hemicompact then $\varkappa(P) \leqslant \varkappa(X)$.*

(b) *If each X_i is chain-compact then $\varkappa(X) \leqslant \varkappa(P)$.*

Proof. (a) Let $\{K_\xi \colon \xi < \alpha\}$ be a compact cover of X with $\alpha = \varkappa(X)$, and for $i \in I$ let $\{L_{i,\eta} \colon \eta < \varkappa(X_i)\}$ be a compact cover of X_i with the property that for every compact $K \subset X_i$ there is $\delta < \varkappa(X_i)$ such that $K \subset \bigcup_{\eta<\delta} L_{i,\eta}$. For $\xi < \alpha$ and $\pi_i \colon X \to X_i$ the natural projection, the set $\pi_i[K_\xi]$ is a compact subset of X_i, so there is $\delta_{i,\xi} < \varkappa(X_i)$ such that

$$\pi_i[K_\xi] \subset \bigcup_{\eta<\delta_{i,\xi}} L_{i,\eta}.$$

The set $M_{i,\xi} = \langle \delta_{i,\xi} + 1 \rangle$ is a compact subset of $\langle \varkappa(X_i) \rangle$. For $\xi < \alpha$ we set

$$M_\xi = \prod_{i\in I} M_{i,\xi}$$

and we show that $\{M_\xi \colon \xi < \alpha\}$ is a cover of P, establishing $\varkappa(P) \leqslant \varkappa(X)$.

Let $p = \langle \xi_i\colon i \in I\rangle \in P$ and notice that (by the definition of $\varkappa(X_i)$) there exists

$$x_i \in X_i \setminus \bigcup_{\eta<\xi_i} L_{i,\eta}.$$

The point $x = \langle x_i\colon i\in I\rangle \in X$ so defined satisfies $x \in K_\xi$ for some $\xi<\alpha$, and it is then clear for this ξ that $p \in M_\xi$. Indeed for $i\in I$ we have

$$x_i \in \pi_i[K_\xi] \subset \bigcup_{\eta<\delta_{i,\xi}} L_{i,\eta},$$

so that $\xi_i < \delta_{i,\xi} \subset M_{i,\xi}$ and hence $p \in M_\xi$.

(b) Let $\{K_\xi\colon \xi<\alpha\}$ be a compact cover of P with $\alpha = \varkappa(P)$ and for $i\in I$ use 4.2(c) to write $X_i = \bigcup_{\eta<\varkappa(X_i)} L_{i,\eta}\uparrow$ with each $L_{i,\eta}$ compact. For $\xi<\alpha$ the projection π_i form P onto $\langle\varkappa(X_i)\rangle$ takes K_ξ to a compact subset $\pi_i[K_\xi]$ of $\langle\varkappa(X_i)\rangle$. We denote by $\mu_{i,\xi}$ the largest element of $\pi_i[K_\xi]$ and we define

$$M_\xi = \prod_{i\in I} L_{i,\mu_{i,\xi}}.$$

It is enough to show that $\{M_\xi\colon \xi<\alpha\}$, a family of compact subsets of X, is a cover of X. Given $x = \langle x_i\colon i\in I\rangle \in X$, for $i\in I$ there is $\eta_i < \varkappa(X_i)$ such that $x_i \in L_{i,\eta_i}$. This defines a point

$$p = \langle \eta_i\colon i\in I\rangle \in P.$$

There is $\xi<\alpha$ such that $p\in K_\xi$, and for $i\in I$ we have

$$\eta_i \in \pi_i[K_\xi];$$

it follows that $\eta_i \leqslant \mu_{i,\xi}$, and hence

$$x_i \in L_{i,\eta_i} \subset L_{i,\mu_{i,\xi}};$$

thus $x\in M_\xi$, as required.

4.6. Corollary. *Let X and Y be hemicompact, chain-compact spaces such that $\varkappa(X) = \varkappa(Y)$. Then $X \simeq Y$.*

Proof. With $\varkappa(X) = \varkappa(Y) = \alpha$, for every cardinal β we have

$$\varkappa(X^\beta) = \varkappa(\langle\alpha\rangle^\beta) = \varkappa(Y^\beta).$$

4.7. Remark. Corollary 4.6 suggests the following question. Suppose that X is both chain-compact and hemicompact; does it follow that one can write $X = \bigcup_{\xi<\varkappa(X)} K_\xi\uparrow$ with $\{K_\xi\colon \xi<\varkappa(X)\}$ a k-base for X? We are grateful to Alan Dow for providing a (locally compact) example showing that the answer to this question is negative [8]. The problem of finding such a locally compact space X is equivalent to the problem of finding a compact space K with a point p such that one may write $\{p\} = \bigcap_{\xi<\chi(p)} U_\xi\downarrow$ with each U_ξ open, but no local base at p is linearly ordered. (Given K take $X = K\setminus\{p\}$; given X let $K = X\cup\{p\}$ be the one-point compactification of X and notice (referring to [1], p. 65) that $\psi(p,K) = \chi(p,K) = \mathrm{kw}(X)$.) Dow's compact space K is a suitably defined quotient of the Stone–Čech compactification $\beta(\omega\times Y)$ with compact Y chosen so that $Y = \bigcup_{\xi<\omega_1} Y_\xi\uparrow$ with each Y_ξ nowhere dense in Y (e.g., $Y = \langle\omega_1+1\rangle^\omega$ with $Y_\xi = \langle\xi\rangle^\omega$).

4.8. Question. Does the conclusion of 4.5(a) hold for all (not necessarily hemicompact) spaces?

4.9. Question. Can the "hemicompact hypothesis" be omitted from 4.6?

It follows from 2.7 that a negative answer to 4.8 can be given only in models of ZFC where GCH fails. On the other hand, an affirmative answer to 4.8 would provide an analogue of 3.3 concerning the existence of "minimal" elements in $\Lambda(\alpha)$ (for regular $\alpha>\omega$): for every $X\in\Lambda(\alpha)$ and cardinal β, we would have

$$\varkappa(\langle\alpha\rangle^\beta) \leqslant \varkappa(X^\beta).$$

We note also that an affirmative answer to 4.9 would provide a natural generalization of 3.5.

§ 5. The numbers $\varkappa(\langle\alpha\rangle^\beta)$ and $\varkappa(\alpha^\beta)$. In this final section we take a closer look at spaces of the form α^β and $\langle\alpha\rangle^\beta$, which, as we indicated in the previous sections, play an important role in our work.

Concerning the numbers $\varkappa(\langle\alpha\rangle^\beta)$, we already know that $\varkappa(\langle\alpha\rangle^\beta) = \varkappa(\langle\mathrm{cf}(\alpha)\rangle^\beta)$ for all α, $\beta\geqslant\omega$ (this follows from 4.6), and that $\varkappa(\langle\alpha\rangle^\beta) = \mathrm{cf}(\alpha)$ when $0<\beta<\mathrm{cf}(\alpha)$ (2.6). In order to say a bit more, we first need the following terminology and definitions which generalize concepts from [11].

Given an ordinal θ and an infinite cardinal α, for $f, g\in {}^\theta\alpha$ we write $f<g$ if $f(\xi)<g(\xi)$ for all $\xi<\theta$, and $f<^* g$ if there is $\eta<\theta$ such that $f(\xi)<g(\xi)$ for $\eta<\xi<\theta$.

5.1. Definition. Given an ordinal θ and an infinite cardinal α, $D({}^\theta\alpha)$ $(E({}^\theta\alpha))$ is the least cardinality of a subset of ${}^\theta\alpha$ which is cofinal with respect to $<$ (with respect to $<^*$).

5.2. Theorem. *Let α, β be infinite cardinals. Then $\varkappa(\langle\alpha\rangle^\beta) = D({}^\beta\alpha) = E({}^\beta\alpha)$.*

Proof. The equality $\varkappa(\langle\alpha\rangle^\beta) = D({}^\beta\alpha)$ can be proved as in [11]. Indeed, if $\{g_\eta\colon \eta<D({}^\beta\alpha)\}$ is $<$-cofinal in ${}^\beta\alpha$, set $K_\eta = \prod_{\xi<\beta}\langle g_\eta(\xi)+1\rangle$ and notice that $\{K_\eta\colon \eta<D({}^\beta\alpha)\}$ is a compact cover of $\langle\alpha\rangle^\beta$; this establishes $\varkappa(\langle\alpha\rangle^\beta)\leqslant D({}^\beta\alpha)$. The reverse inequality follows from the observation that if $\{K_\eta\colon \eta<\varkappa(\langle\alpha\rangle^\beta)\}$ is a compact cover of $\langle\alpha\rangle^\beta$, then $\{g_\eta\colon \eta<\varkappa(\langle\alpha\rangle^\beta)\}$ — where g_η is defined by

$$g_\eta(\xi) = \max(\pi_\xi[K_\eta])+1$$

for all $\xi < \beta$ — is a $<$-cofinal family in ${}^{\beta}\alpha$. (Indeed, if $f \in K_\eta$ with some $\eta < \varkappa(\langle\alpha\rangle^\beta)$, then $f < g_\eta$.)

The equality $D({}^{\beta}\alpha) = E({}^{\beta}\alpha)$ is folklore among set-theorists. A printed proof — based on the observation that $D({}^{\beta}\alpha) \leqslant E({}^{\beta}\alpha)$ for all $0 < \beta$ — can be found in [4].

5.3. Remark. For $\alpha = \beta = \omega$, Theorem 5.2 was proved by Hechler [11], who also investigated $\boldsymbol{d} = E({}^{\omega}\omega)$ [12]. On the other hand, 5.2 shows that the investigation of $\varkappa(\langle\alpha\rangle^\beta)$ (when $\alpha\cdot\beta > \omega$) is a difficult problem. For example, the consistency of $E({}^{\omega_1}\omega) < 2^{\omega_1}$ (and hence of $\varkappa(\omega^{\omega_1}) < 2^{\omega_1}$) with ZFC is an open problem, closely related to questions in combinatorial set theory and in the theory of large cardinals [14].

Concerning the numbers $\varkappa(\alpha^\beta)$, we do not have a set-theoretic characterization analogous to 5.2. The following theorem, however, provides some information on $\varkappa(\alpha^\beta)$ and a connection between discrete spaces (α) and ordinal spaces $(\langle\alpha\rangle)$.

5.4. Theorem. *Let α and β be infinite cardinals. Then*

$$\varkappa((\alpha^+)^\beta) = \varkappa(\langle\alpha^+\rangle^\beta)\cdot\varkappa(\alpha^\beta).$$

Proof. The inequality $\geqslant$ is immediate from 3.2 and 2.1(b), so we prove For notational simplicity set $\varkappa(\langle\alpha^+\rangle^\beta) = \gamma$ and let $\{K_\eta\colon \eta < \gamma\}$ be a compact of $\langle\alpha^+\rangle^\beta$; we assume without loss of generality that each K_η satisfies $K_\eta = \prod_{\xi<\beta} \pi$

For $\xi < \beta$ and $\eta < \gamma$ the set $\pi_\xi[K_\eta]$ is compact in $\langle\alpha^+\rangle$ and hence bounded. Topologized discretely, $\pi_\xi[K_\eta]$ is therefore homeomorphic to a (closed) subspace of the discrete space α, so $K_\eta = \prod_{\xi<\beta}\pi_\xi[K_\eta]$ can be viewed as a closed subspace of α^β and hence

$$\varkappa(K_\eta) \leqslant \varkappa(\alpha^\beta).$$

If $\mathscr{K}_\eta$ is a cover of K_η by sets compact in the topology which K_η inherits from $(\alpha^+)^\beta$, with $|\mathscr{K}_\eta| \leqslant \varkappa(\alpha^\beta)$, then $\mathscr{K} = \bigcup_{\eta<\gamma}\mathscr{K}_\eta$ is a compact cover of $(\alpha^+)^\beta$ with $|\mathscr{K}| \leqslant \gamma\cdot\varkappa(\alpha^\beta)$, as required.

5.5. Corollary. *Let $0 \leqslant n < \omega$. Then $\varkappa(\omega_n^\omega) = \omega_n\cdot\varkappa(\omega^\omega)$. We note in particular that $\varkappa(\omega_1{}^\omega) = \varkappa(\omega^\omega) = \boldsymbol{d}$.*

Proof. Using 5.4 and 2.6, we obtain $\varkappa(\omega_{n+1}^\omega) = \omega_{n+1}\cdot\varkappa(\omega_n^\omega)$; the result follows now by (finite) induction on n.

5.6. Theorem. *Let α and β be infinite cardinals with $\alpha \geqslant 2^\beta$. Then $\varkappa(\alpha^\beta) = \alpha^\beta$.*

Proof. Since each compact subset K of α^β projects onto a finite subset of each coordinate space, we have $|K| \leqslant \omega^\beta = 2^\beta$ and hence

$$\alpha^\beta \leqslant \varkappa(\alpha^\beta)\cdot 2^\beta.$$

If $\alpha > 2^\beta$ then the above inequality yields $\alpha^\beta \leqslant \varkappa(\alpha^\beta) \leqslant \alpha^\beta$, while if $\alpha = 2^\beta$ we have $\alpha \leqslant \varkappa(\alpha^\beta) \leqslant \alpha^\beta = \alpha$.

Concerning the search for additional equivalence classes in $\Lambda(\alpha)$ (for $\alpha > \omega$) and for spaces "strictly between" α and $\langle\alpha\rangle$, we propose the following problem.

5.7. Question. For infinite cardinals λ, μ, set

(a) $T_{\lambda,\mu} = \langle\omega_\lambda+1\rangle\times\langle\omega_\mu+1\rangle\setminus\{(\omega_\lambda, \omega_\mu)\}$ and

(b) $T_{\lambda,\bar{\mu}} = \langle\omega_\lambda+1\rangle\times(\omega_\mu+1)\setminus\{(\omega_\lambda, \omega_\mu)\}$;

are there models of ZFC where $T_{2,0}$ and/or $T_{2,1}$ are $\simeq$-equivalent to neither $\langle\omega_2\rangle$ no ω_2?

We leave it to the reader to formulate the analogous questions for $\alpha > \omega_2$ and to check (using 2.1(b), 3.2, 4.6, and 5.4 wherever appropriate) that $T_{\lambda+1,\bar{\lambda}} \simeq \omega_{\lambda+1}$ and $T_{\lambda,\lambda} \simeq \langle\mathrm{cf}(\omega_\lambda)\rangle$. One notices in particular that for the familiar Tychonoff plank $T_{1,0}$ we have $T_{1,0} \simeq \omega_1$; this relation, together with 3.8, suggests the following question.

5.8. Question. Is there a countably compact, first countable (hence sequentially pact) space X such that $X \simeq \omega_1$?

We hope that the questions raised in this paper will generate continuing activity th a set-theoretic direction (searching for models of ZFC with the appropriate inatorial properties) and a topological direction (achieving a better under- ing of the structure of topological spaces through their family of compact sc

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KANSAS
Lawrence, KS 66045

DEPARTMENT OF MATHEMATICS
WESLEYAN UNIVERSITY
Middletown, CT 06457

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