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The centralizer of Morse shifts induced by arbitrary blocks

by

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Abstract. The centralizer $C(T_x)$ of the Morse shift given by a continuous Morse sequence $x = b^0 \times b^1 \times \ldots$ over Z_2 is described. Let $|b^t| = \lambda_t$, $t \geqslant 0$, and let G be the set of $\bar{\lambda}$-adic integers, $\bar{\lambda} = (\lambda_t)_0^\infty$. $C(T_x)$ can be identified with a set of pairs (g_0, φ), where $g_0 \in G_0 \subset G$ and φ is a measurable function from G to Z_2. The set G_0 satisfies the following properties:

(a) $m(G_0) = 0$, where m is the normalized Haar measure on G.

(b) Either $G_0 = \hat{Z}$ or G_0 is uncountable. T_x is rigid iff G_0 is uncountable.

(c) There exists a Morse sequence x such that T_x is rigid and $C(T_x) \neq \overline{\{T^i\}}$.

Introduction. There are some reasons to investigate the centralizers of Morse shifts. The Morse shift (Ω_x, T, μ_x) is metrically isomorphic to a Z_2-extension of the group of $\bar{\lambda}$-adic integers with the Haar measure m and the translation by $\hat{1}$. Newton's results [5] imply that each automorphism S commuting with T can be identified with (g_0, φ), where $g_0 \in G$ and φ is a measurable function from G to Z_2. The transformation S is an extension of the translation of G by g_0 to an automorphism of $C(T)$.

Lemańczyk [4] has proved that the centralizer of the Morse shift (Ω_x, T, μ_x) is countable but not trivial assuming that x is a regular Morse sequence and the sequence $\{|b^t|\}$ is bounded. In this case G_0 coincides with the set of all $\bar{\lambda}$-adic rational integers. On the other hand, Lemańczyk described some class of Morse shifts having the property to be rigid. The centralizer of such systems is uncountable. The question arose what are G_0 and φ for an arbitrary Morse shift. The next reason for investigating the commutant of Morse shifts is connected with Walter's question in [7]. He asked whether there is a rigid automorphism T with simple spectrum such that the commutant of T is not the closure of the powers of T in the weak topology. A supposition arose that an example could be found in the class of Morse shifts. In this paper we confirm that supposition.

§ 1. Centralizer of Morse shifts. We first give the necessary preliminaries. For a more complete treatment, the reader is referred to [1], [2], [4], [6].

Let $x = b^0 \times b^1 \times \ldots$ be a 0-1 Morse sequence. There exists an almost periodic two-sided sequence ω such that $\omega[k] = x[k]$, $k \geqslant 0$. Putting Ω_x

$=\overline{\{T^i\omega\colon i\in \boldsymbol{Z}\}}$ we obtain a strictly ergodic system (Ω_x, T) where T is the shift. The unique T-invariant ergodic measure is denoted by μ_x and the system $\theta(x)=(\Omega_x, T, \mu_x)$ is said to be a *Morse dynamical system* or *Morse shift*. Denote by σ the *mirror map* on Ω_x, i.e. $\sigma y=\tilde{y}$, $y\in\Omega_x$, where $\tilde{y}[i]=y[i]^{\sim}=1+y[i]$ in $\boldsymbol{Z}_2$, $i\in \boldsymbol{Z}$.

In this paper we shall assume that x is a continuous Morse sequence [1]. We use the following notation:

$$\lambda_t=|b^t| \quad \text{(the length)}, \qquad n_t=\lambda_0\ldots\lambda_t, \qquad c_t=b^0\times b^1\times\ldots\times b^t.$$

Given a block A, the successive places of A will be numbered from 0 to $n-1$ ($n=|A|$), i.e. $A=A[0]A[1]\ldots A[n-1]$. If $l=0$ or 1 then $A+l$ means A if $l=0$ and $\tilde{A}$ otherwise. For $0\leqslant i\leqslant k\leqslant n-1$, $A[i,k]$ denotes the block $A[i]A[i+1]\ldots A[k]$. If $B=B[0]\ldots B[m-1]$ is another block then AB is the block $A[0]\ldots A[n-1]B[0]\ldots B[m-1]$. We shall write $AB[i,k]$ instead of $(AB)[i,k]$. If $m\leqslant n$ then $\mathrm{fr}(B,A)$ denotes the number

$$\frac{1}{n}\cdot\operatorname{card}\{j\colon 0\leqslant j\leqslant n-m,\ A[j,j+m-1]=B\}.$$

Kwiatkowski [3] has shown that (Ω_x, T, μ_x) is a $\boldsymbol{Z}_2$-extension of the system $(G, m, T_{\hat{1}})$, where $G=\varprojlim \boldsymbol{Z}/n_t\boldsymbol{Z}$, m is the normalized Haar measure on G and $T_{\hat{1}}$ is the translation by $\hat{1}$. Here $\hat{1}=(1,1,\ldots)$. Every element $g\in G$ may be represented in the form $g=(j_t)_0^\infty$, $0\leqslant j_t\leqslant n_t-1$, $j_t\equiv j_{t+1} \pmod{n_t}$, $t\geqslant 0$. We will also use the representation of g in the form $g=\sum_{t=0}^\infty q_t n_{t-1}$, $0\leqslant q_t\leqslant \lambda_t-1$, $n_{-1}=1$. We will call it a *$\bar{\lambda}$-adic integer*, $\bar{\lambda}=(\lambda_t)_0^\infty$. In particular, if l is an integer then the symbol $\hat{l}$ will denote the $\bar{\lambda}$-adic rational integer corresponding to l.

In the sequel we will use the symbol "+" in different meanings, for addition in $\boldsymbol{Z}_2$, $\bmod n_t$, in G and in the usual sense.

(Ω_x, T, μ_x) is metrically isomorphic to the system $(G\times \boldsymbol{Z}_2, m\times\frac{1}{2}, T_x)$, where

$$(1)\qquad T_x(g,i)=(g+\hat{1},\ i+\varphi_x(g)),\qquad g\in G,\ i\in \boldsymbol{Z}_2,$$

$$(2)\qquad \varphi_x(g)=c_t[1+j_t]+c_t[j_t] \quad\text{in } \boldsymbol{Z}_2,\qquad g=(j_t)_0^\infty,$$

$$j_t\equiv j_{t+1} \pmod{n_t},\quad t\geqslant 0.$$

The above formula determines the function $\varphi_x\colon G\to \boldsymbol{Z}_2$ except for $g=(n_t-1)_0^\infty$, because

$$c_u[1+j_u]+c_u[j_u]=c_t[1+j_t]+c_t[j_t]$$

for all $u>t$, whenever $j_t<n_t-1$.

The map on $G\times \boldsymbol{Z}_2$ defined by $(g,i)\to(g,i+1)$ corresponds to the mirror σ on Ω_x. We will also denote it by σ.

In order to describe the centralizer $C(T_x)$ of the dynamical system $(G\times \boldsymbol{Z}_2, m\times\frac{1}{2}, T_x)$ we use the isomorphism theorem (see § 4 below). In the sequel we assume that the lengths λ_t of b^t satisfy the condition

$$\sum_{t=0}^{\infty} 1/\lambda_t<\infty.$$

Let (Ω_y, T, μ_y) be another continuous Morse shift,

$$y=\beta^0\times\beta^1\times\ldots,\qquad |\beta^t|=|b^t|=\lambda_t,\qquad t\geqslant 0.$$

Newton's results [5] allow to find the form of isomorphisms between the dynamical systems (Ω_x, T, μ_x) and (Ω_y, T, μ_y) regarded as $\boldsymbol{Z}_2$-extensions of G. Let $\varphi_y\colon G\to \boldsymbol{Z}_2$ be the function defined by

$$(3)\qquad \varphi_y(g)=d_t[1+j_t]+d_t[j_t],\qquad g=(j_t)_0^\infty,\ 0\leqslant j_t\leqslant n_t-1,$$

where $d_t=\beta^0\times\beta^1\times\ldots\times\beta^t$, $t\geqslant 0$, and $g\neq(n_t-1)_0^\infty$. The system (Ω_y, T, μ_y) is metrically isomorphic to $(G\times \boldsymbol{Z}_2, m\times\frac{1}{2}, T_y)$. Then each isomorphism W from $(G\times \boldsymbol{Z}_2, m\times\frac{1}{2}, T_y)$ to $(G\times \boldsymbol{Z}_2, m\times\frac{1}{2}, T_x)$ is of the form

$$W(g,i)=(g+g_0,\ i+\varphi(g)),$$

where g_0 is a $\bar{\lambda}$-adic integer and $\varphi\colon G\to \boldsymbol{Z}_2$ is a function such that

$$(4)\qquad \varphi_x(g+g_0)+\varphi(g)=\varphi(g+\hat{1})+\varphi_y(g)$$

for a.e. $g\in G$.

Applying the isomorphism theorem (see § 4, Th. 5) we will describe g_0 and the function φ. Put

$$g_0=\sum_{t=0}^{\infty}\bar{q}_t n_{t-1},\qquad l_t=\sum_{k=0}^{t}\bar{q}_k n_{k-1}$$

and for fixed t define

$$\varphi_t(g)=(c_t+r_t)(c_t+s_t)[l_t+j_t]+d_t[j_t],\qquad g=(j_t)_0^\infty.$$

Next let

$$H_t=\{g\in G\colon \varphi_{t+1}(g)=\varphi_t(g)\}.$$

We estimate $m(H_t)$. Let $g=\sum_{t=0}^\infty q_t n_{t-1}$. We have

$$\varphi_{t+1}(g)=\varphi_t(g)+(b^{t+1}+r_{t+1})(b^{t+1}+s_{t+1})[q_{t+1}+\bar{q}_{t+1}]+\beta^{t+1}[q_{t+1}]+r_t$$

whenever $q_t+\bar{q}_t\leqslant\lambda_t-2$ and $q_{t+1}+\bar{q}_{t+1}\neq\lambda_{t+1}-1$. If $q_t+\bar{q}_t\geqslant\lambda_t$ and $q_{t+1}+\bar{q}_{t+1}\neq\lambda_{t+1}-1$ and $q_{t+1}+\bar{q}_{t+1}\neq\lambda_{t+1}$ then

$$\varphi_{t+1}(g)=\varphi_t(g)+(b^{t+1}+r_{t+1})(b^{t+1}+s_{t+1})[1+q_{t+1}+\bar{q}_{t+1}]+\beta^{t+1}[q_{t+1}]+s_t.$$

The above equalities imply

$$m(H_t) \geqslant 1 - \frac{2}{\lambda_{t+1}} - \frac{2}{\lambda_t} - \left[\left(1 - \frac{\bar{q}_t}{\lambda_t}\right) D_{t+1} + \frac{\bar{q}_t}{\lambda_t} \bar{D}_{t+1}\right],$$

where D_t, $\bar{D}_t$ are defined in § 4.

Putting $E_t = \bigcap_{u \geqslant t} H_u$ we obtain

$$m(E_t) \geqslant 1 - 4 \sum_{u \geqslant t} \frac{1}{\lambda_u} - \sum_{u \geqslant t} \left[\left(1 - \frac{\bar{q}_u}{\lambda_u}\right) D_{u+1} + \frac{\bar{q}_u}{\lambda_u} \bar{D}_{u+1}\right].$$

Finally we define $E = \bigcup_{t=0}^{\infty} E_t$ and

(5) $$\varphi(g) = \varphi_t(g) \quad \text{for } g \in E_t.$$

We have $E_t \subset E_{t+1}$, $t \geqslant 0$, and from Theorem 5 we get $m(E) = 1$. It is not hard to see that the function (5) satisfies (4) with g_0 defined above. It follows from the proofs of [3, Th. 1] and [6, Th. 1] that every isomorphism W from (Ω_y, T, μ_y) to (Ω_x, T, μ_x) is of the form described above.

Now we are in a position to describe the centralizer $C(T_x)$ of the Morse shift $(G \times \mathbf{Z}_2, m \times \frac{1}{2}, T_x)$. Taking $\beta^t = b^t$, $t \geqslant 0$, and applying Theorem 5 we get the following

THEOREM 1. *Each $S \in C(T_x)$ has the form*

$$S(g, i) = (g + g_0, i + \varphi(g))$$

and $g_0 = (l_t)_0^\infty = \sum_{t=0}^{\infty} \bar{q}_t n_{t-1}$ satisfies the following condition: there exist $\bar{r}_t, \bar{s}_t \in \mathbf{Z}_2$, $t \geqslant 0$, such that

(6) $$\sum_{t=0}^{\infty} \left[\left(1 - \frac{l_t}{n_t}\right) D_{t+1} + \frac{l_t}{n_t} \bar{D}_{t+1}\right] < \infty$$

where

$$D_t = d\left((b^t + \bar{r}_t)(b^t + \bar{s}_t)[\bar{q}_t, \bar{q}_t + \lambda_t - 1], b^t + \bar{r}_{t-1}\right),$$

$$\bar{D}_t = d\left((b^t + \bar{r}_t)(b^t + \bar{s}_t)[1 + \bar{q}_t, \bar{q}_t + \lambda_t], b^t + \bar{s}_{t-1}\right)$$

(here for blocks A, B of length n, $d(A, B)$ denotes the number $n^{-1} \operatorname{card}\{0 \leqslant i \leqslant n-1: A[i] \neq B[i]\}$).

The function φ is the limit a.e. of the sequence of functions φ_t,

(7) $$\varphi_t(g) = (c_t + \bar{r}_t)(c_t + \bar{s}_t)[l_t + j_t] + c_t[j_t], \qquad g = (j_t)_0^\infty.$$

Moreover, φ satisfies the condition

(8) $$\varphi_x(g + g_0) + \varphi(g) = \varphi(g + \hat{1}) + \varphi_x(g).$$

Since $\sum_{t=0}^{\infty} 1/\lambda_t < \infty$ the condition (6) is equivalent to

(6′) $$\sum_{t=0}^{\infty} d\left((b^t + r_t)(b^t + s_t)[q_t, q_t + \lambda_t - 1], b^t\right) < \infty,$$
$$\sum_{t=0}^{\infty} \min\left(1 - \frac{q_t}{\lambda_t}, \frac{q_t}{\lambda_t}\right) \cdot [\operatorname{fr}(r_t \bar{s}_t, b^{t+1}) + \operatorname{fr}(\bar{r}_t s_t, b^{t+1})] < \infty.$$

Remark 1. The numbers q_t in (6′) and $r_t, s_t \in \mathbf{Z}_2$, $t \geqslant 0$, may be so chosen that $q_0 = \bar{q}_0$ and

$$q_t = \begin{cases} \bar{q}_t & \text{if } \bar{q}_{t-1} \leqslant \lambda_{t-1} - \bar{q}_{t-1} - 1, \\ \bar{q}_t + 1 \pmod{\lambda_t} & \text{otherwise,} \end{cases}$$

$$r_t + s_t = \bar{r}_t + \bar{s}_t \quad \text{in } \mathbf{Z}_2.$$

The $\bar{\lambda}$-adic integer g_0 is defined by $g_0 = \sum_{t=0}^{\infty} \bar{q}_t n_{t-1}$.

Thus each $S \in C(T_x)$ is an extension of the translation T_{g_0}, $T_{g_0}(g) = g_0 + g$, of G. Note that T_{g_0} is an element of the centralizer of $T_{\hat{1}}$. We will prove that extension of T_{g_0} to an element of $C(T_x)$ is possible only if g_0 runs through a subset of G of Haar measure zero. In the sequel let G_0 be the set of all $g_0 \in G$ satisfying (6).

§ 2. **Properties of** $C(T_x)$. For $S \in C(T_x)$ we will write $S = (g_0, \varphi)$ if $g_0 \in G_0$ and φ is a function satisfying (8).

I. *For every $g_0 \in G_0$ there exist exactly two functions φ, ψ satisfying* (8).

In fact, if φ and ψ are such functions then $\varphi + \psi$ is a $T_{\hat{1}}$-invariant function so either $\varphi \equiv \psi$ or $\varphi \equiv \psi + 1$. On the other hand, if φ satisfies (8) then so does $\varphi + 1$. Note that if φ is determined by g_0 and the sequences $\{r_t, s_t\}$ which satisfy (6) then $\varphi + 1$ is determined by g_0 and $\{r_t', s_t'\}$, $r_t' = r_t + 1$, $s_t' = s_t + 1$, $t \geqslant 0$.

II. *The set G_0 is a $T_{\hat{1}}$-invariant subgroup of G.*

First we show that G_0 is measurable. To this end we use the condition (6) in the following form:

$$\sum_{t=0}^{\infty} \left[\left(1 - \frac{\bar{q}_t}{\lambda_t}\right) D_{t+1} + \frac{\bar{q}_t}{\lambda_t} \bar{D}_{t+1}\right] < \infty.$$

Put

$$A_{nkp} = \left\{g_0 = \sum_{t=0}^{\infty} \bar{q}_t n_{t-1}: \text{there exist } \bar{r}_t, \bar{s}_t \in \mathbf{Z}_2,\ n \leqslant t \leqslant n+k, \right.$$
$$\left. \text{such that } \sum_{t=n}^{n+k} \left[\left(1 - \frac{\bar{q}_t}{\lambda_t}\right) D_{t+1} + \frac{\bar{q}_t}{\lambda_t} \bar{D}_{t+1}\right] < \frac{1}{p}\right\},$$

where n, k, p are positive integers. Then it is clear that

$$G_0 = \bigcap_{p=1}^{\infty} \bigcup_{n=0}^{\infty} \bigcap_{k=1}^{\infty} A_{nkp}.$$

Since A_{nkp} is a finite union of cylinder sets, G_0 is measurable.

G_0 is a subgroup of G, because if $S = (g_0, \varphi)$, $\bar{S} = (\bar{g}_0, \bar{\varphi})$ then

$$S \circ \bar{S} = (g_0 + \bar{g}_0, \psi_1), \quad \psi_1(g) = \bar{\varphi}(g) + \varphi(\bar{g}_0 + g),$$
$$\bar{S} \circ S = (g_0 + \bar{g}_0, \psi_2), \quad \psi_2(g) = \varphi(g) + \bar{\varphi}(g_0 + g).$$

Therefore $g_0 + \bar{g}_0 \in G_0$. Since $T_x = (\hat{1}, \varphi_x)$ we have $g_0 + \hat{1} \in G_0$ whenever $g_0 \in G_0$. In particular,

$$T_x^m = \Big(\hat{m}, \sum_{i=0}^{m-1} \varphi_x(g + \hat{i})\Big), \quad m = \pm 1, \pm 2, \ldots,$$

$$I = T_x^0 = (0, 1_G), \quad \sigma = (0, 1_G + 1), \quad S \circ \sigma = (g_0, \varphi + 1).$$

Properties I and II are valid for every ergodic $\boldsymbol{Z}_2$-extension of G.

III. *The set $\hat{Z}$ of $\bar{\lambda}$-adic rational integers is contained in G_0. Either $G_0 = \hat{Z}$ or G_0 is an uncountable subset of G.*

The first statement is evident. Suppose that $g_0 \in G_0 \setminus \hat{Z}$. Then there exist numbers q_t and $r_t, s_t \in \boldsymbol{Z}_2$ that satisfy (6′) and determine g_0 as in Remark 1. The set Z_0 of all t such that $q_t > 0$ is infinite. Take any infinite subset I of Z_0. We construct a $\bar{\lambda}$-adic integer g_I and r_t', s_t', $t \geqslant 0$, satisfying (6′). Namely, we put

$$q_t' = q_t, \ r_t' = r_t, \ s_t' = s_t \quad \text{for } t \in I,$$
$$q_t' = 0, \ r_t' = s_t' = 0 \quad \text{for } t \in Z_0 \setminus I.$$

It is easy to see that q_t' and r_t', s_t', $t \geqslant 0$, satisfy (6′) so they determine a $\bar{\lambda}$-adic integer $g_I \notin \hat{Z}$. Taking different subsets I of Z_0 we obtain different $\bar{\lambda}$-adic integers g_I. Thus G_0 is uncountable.

Observe that $G_0 = \hat{Z}$ iff $C(T_x) = \{T^i \sigma^j\}$, $i = 0, \pm 1, \ldots$, $j \in \boldsymbol{Z}_2$. The above considerations allow us to generalize Lemańczyk's result [4, Th. 1]. If $x = b^0 \times b^1 \times \ldots$ is a regular Morse sequence then the conditions (6′) can be written as

$$(6'') \qquad \begin{aligned} &\sum_{t=0}^{\infty} d\big((b^t + r_t)(b^t + s_t)[q_t, q_t + \lambda_t - 1], b^t\big) < \infty, \\ &\sum_{t=0}^{\infty} \min(l_t/n_t, 1 - l_t/n_t) < \infty, \end{aligned}$$

where $(l_t)_0^\infty \in G$ and the l_t are defined in Remark 1.

In this case the assumption $\sum_{t=0}^{\infty} 1/\lambda_t < \infty$ is not necessary [3]. In addition, if the sequence λ_t is bounded then (6) implies $G_0 = \hat{Z}$.

For convenience we use the following notation:

$$\mathrm{fr}(00 \vee 11; A) = \mathrm{fr}(00, A) + \mathrm{fr}(11, A),$$
$$\mathrm{fr}(01 \vee 10; A) = \mathrm{fr}(01, A) + \mathrm{fr}(10, A).$$

EXAMPLE 1. Let $x = b^0 \times b^1 \times \ldots$ be a continuous nonregular Morse sequence. Then G_0 is uncountable. In fact, if x is not regular then we can represent it in such a way that

$$\sum_{t=0}^{\infty} \min\big(\mathrm{fr}(00 \vee 11; b^t), \mathrm{fr}(01 \vee 10; b^t)\big) < \infty$$

(see [3]). Put $q_t = 1$ and

$$r_t = 1, \ s_t = 0 \quad \text{if } \mathrm{fr}(00 \vee 11; b^t) < \mathrm{fr}(01 \vee 10; b^t),$$
$$r_t = s_t = 0 \quad \text{otherwise}.$$

Then $g_0 = \sum_{t=0}^{\infty} q_t n_{t-1} \in G_0$ and $g_0 \notin \hat{Z}$. Thus G_0 is uncountable.

THEOREM 2. *The set G_0 has Haar measure zero.*

Proof. Property II implies $m(G_0) = 0$ or 1. To prove that $m(G_0) = 0$ it suffices to show that $G \setminus G_0 \neq \emptyset$. If

$$(9) \qquad \sum_{t=0}^{\infty} \min\big(\mathrm{fr}(01 \vee 10; b^t), \mathrm{fr}(00 \vee 11; b^t)\big) = +\infty,$$

then the element $g = \sum_{t=0}^{\infty} q_t n_{t-1}$, where $q_t = [\lambda_t/2]$, $t \geqslant 0$, does not satisfy the second condition of (6′), so $g \in G \setminus G_0$. Therefore we may assume that the condition (9) is not satisfied. In this case we use the inequalities

$$\mathrm{fr}(00 \vee 11; A \times B) \leqslant \mathrm{fr}(00 \vee 11; A) + 1/|A|,$$
$$\mathrm{fr}(01 \vee 10; A \times B) \leqslant \mathrm{fr}(01 \vee 10; A) + 1/|A|,$$
$$\sum_{t=0}^{\infty} 1/\lambda_t < \infty,$$

and we group the blocks $\{b^t\}_0^\infty$ to find a new representation (also denoted by $x = b^0 \times b^1 \times \ldots$) such that $\sum_t \mathrm{fr}(01 \vee 10; b^t) < \infty$ or $\sum_t \mathrm{fr}(00 \vee 11; b^t) < \infty$. Moreover, we may assume that

$$\tfrac{1}{3} < \mathrm{fr}(0, b^t) < \tfrac{2}{3},$$

because $\mathrm{fr}(0, c_t) \to \frac{1}{2}$ as $t \to \infty$.

(A) Suppose first $\sum_t \mathrm{fr}(01 \vee 10; b^t) < \infty$. Let

$$d(q) = d(b^t b^t [q, q+\lambda_t - 1], b^t), \quad q = 0, 1, \ldots, \lambda_{t-1}.$$

It follows from the definition of $d(q)$ that

$$\lambda_t^{-1} \sum_{q=0}^{\lambda_t - 1} d(q) = 2\,\mathrm{fr}(0, b^t)\,\mathrm{fr}(1, b^t) > 2 \cdot \tfrac{1}{3} \cdot \tfrac{2}{3} = \tfrac{4}{9}.$$

Hence

$$\tfrac{9}{7} \lambda_t^{-1} \sum_{\lambda_t/9 < q \leqslant 8\lambda_t/9} d(q) > \tfrac{9}{7}(\tfrac{4}{9} - \tfrac{2}{9}) = \tfrac{2}{7}.$$

Since $d(0) = 0$ and

$$|d(q+1) - d(q)| \leqslant \mathrm{fr}(01 \vee 10; b^t) + 1/\lambda_t, \quad q = 0, \ldots, \lambda_t - 2,$$

we deduce that there exist q_t, $\frac{1}{9}\lambda_t < q_t \leqslant \frac{8}{9}\lambda_t$, such that $\frac{1}{7} < d(q_t) < \frac{2}{7}$ (for sufficiently large t). Note that the sequence $\{q_t\}$ does not satisfy (6'). Indeed, the inequality $\frac{1}{9}\lambda_t < q_t \leqslant \frac{8}{9}\lambda_t$ and (A) imply that the second condition of (6') is possible only if $r_t = s_t$ for sufficiently large t. Then the first condition of (6') is impossible, because $\frac{1}{7} \leqslant d(q_t) \leqslant \frac{2}{7}$ and

$$d\big((b^t)^\sim (b^t)^\sim [q_t, q_t + \lambda_t - 1], b^t\big) = 1 - d(q_t).$$

Thus $g_0 = \sum_{t=0}^{\infty} \bar{q}_t n_{t-1}$ does not belong to G_0, where $\bar{q}_t$ is defined as in Remark 1.

(B) Suppose now $\sum_t \mathrm{fr}(00 \vee 11; b^t) < \infty$. Let

$$\bar{d}(q) = d(b^t (b^t)^\sim [q, q+\lambda_t - 1], b^t), \quad q = 0, 1, \ldots, \lambda_t - 1.$$

If infinitely many of the integers λ_t are even, then by grouping the blocks $\{b^0, b^1, \ldots\}$ we may assume that the λ_t are even for $t \geqslant t_0$. It is easy to verify that

$$\bar{d}(q) = 1 - \bar{d}(\lambda_t - q), \quad q = 1, \ldots, \lambda_t - 1.$$

In particular, $\bar{d}(\lambda_t/2) = 1 - \bar{d}(\lambda_t/2)$ with implies $\bar{d}(\lambda_t/2) = \frac{1}{2}$. Put $q_t = 0$ for $t < t_0$ and $q_t = \lambda_t/2$ for $t \geqslant t_0$. Just as above, we check that the sequence $\{q_t\}$ does not satisfy (6').

Finally, suppose that the λ_t are odd for $t \geqslant t_0$. We reduce this case to (A). Since $x = b^0 \times b^1 \times \ldots$ is a continuous Morse sequence, we have

$$\sum_{t \geqslant t_0} \min\big(d(b^t, 010 \ldots 010), d(b^t, 101 \ldots 101)\big) = +\infty.$$

Now we use the following equality:

$$1 - 2d(A_1 \times \ldots \times A_u, B_1 \times \ldots \times B_u + l)$$
$$= \prod_{i=1}^{u-1} [1 - 2d(A_i, B_i + l_i)]\big(1 - 2d(A_u, B_u + l)\big),$$

where $l, l_1, \ldots, l_{u-1} \in Z_2$ and $l = l + l_1 + \ldots + l_{u-1}$, $|A_i| = |B_i|$, which can be easily proved by induction. Take $A_1 = b^{t+1}, \ldots, A_u = b^{t+u}$, $B_1 = 010 \ldots 010$ (with 010 repeated λ_{t+1} times), $\ldots$, $B_u = 010 \ldots 010$ (λ_{t+u}-fold repetition), and $l = 0$ if $d(b^{t+1} \times \ldots \times b^{t+u}, 010 \ldots 010) \leqslant \frac{1}{2}$ and $l = 1$ otherwise. In the same way we choose l_i, $i = 1, \ldots, u-1$. For $t \geqslant t_0$ we obtain

$$1 - 2\min\big(d(b^{t+1} \times \ldots \times b^{t+u}, 010 \ldots 010), d(b^{t+1} \times \ldots \times b^{t+u}, 101 \ldots 101)\big)$$
$$= \prod_{i=1}^{u} (1 - 2a_{t+i}),$$

where $a_t = \min\big(d(b^t, 010 \ldots 010), d(b^t, 101 \ldots 101)\big)$.

Since $\sum_{t=0}^{\infty} a_t = \infty$, we have $\prod_{i=1}^{\infty} (1 - 2a_{t+i}) = 0$ for every $t \geqslant t_0$. Thus $\min\big(d(b^{t+1} \times \ldots \times b^{t+u}, 010 \ldots 010), d(b^{t+1} \times \ldots \times b^{t+u}, 101 \ldots 101)\big) \xrightarrow[u]{} \frac{1}{2}$.

By grouping the blocks $\{b^t\}$ we can find a representation of x, $x = b^0 \times b^1 \times \ldots$, such that

$$\tfrac{1}{3} < d(b^t, 010 \ldots 010) < \tfrac{2}{3} \quad \text{for } t \geqslant 1.$$

Putting $E_t = b^t + 010 \ldots 010$ (λ_t-fold repetition) we have $\frac{1}{3} < \mathrm{fr}(0, E_t) < \frac{2}{3}$, $t \geqslant 1$. Since $\mathrm{fr}(01 \vee 10; E_t) = \mathrm{fr}(00 \vee 11; b^t)$, it follows that $y = b^0 \times E_1 \times E_2 \times \ldots$ is a continuous Morse sequence satisfying (A). Repeating the considerations in (A) we obtain a sequence $\{q_t\}$, $\frac{1}{9}\lambda_t \leqslant q_t \leqslant \frac{8}{9}\lambda_t$ and $\frac{1}{7} < d'(q_t) < \frac{2}{7}$, where $d'(q_t)$ is defined by y. By easy computations we establish that

$$\min\big(\bar{d}(q), 1 - \bar{d}(q)\big) = \min\big(d'(q), 1 - d'(q)\big), \quad 0 \leqslant q \leqslant \lambda_t - 1.$$

The last conditions imply $\frac{1}{9} \leqslant \bar{d}(q_t) \leqslant \frac{8}{9}$. As above, we show that the sequence $\{q_t\}$ does not satisfy (6'). This finishes the proof of the theorem.

§ 3. **Topology of $C(T_x)$.** We start with the following remark: if $S, S_n \in C(T_x)$, $S_n = (g_n, \varphi_n)$, $S = (g_0, \varphi)$, $g_n, g_0 \in G_0$ then $S_n \to S$ in the weak topology iff $g_n \to g$ in G and $\varphi_n \to \varphi$ in Haar measure m.

THEOREM 3. *$C(T_x)$ is the closure of $\{T^i \sigma^j\}$, $i = 0, \pm 1, \ldots, j \in Z_2$, in the weak topology. T_x is a rigid transformation iff G_0 is uncountable.*

Proof. First we show $C(T_x) = \overline{\{T^i \sigma^j\}}$.

Assume that G_0 is uncountable and let $S = (g_0, \varphi)$, $g_0 = (l_t)_0^\infty$, $g_0 \notin \hat{Z}$. Take $\bar{r}_t, \bar{s}_t \in Z_2$ satisfying (6). Define

$$i_t = \begin{cases} \bar{r}_t & \text{if } l_t \leqslant n_t - l_t - 1, \\ \bar{s}_t & \text{otherwise}, \end{cases} \qquad \bar{l}_t = \begin{cases} l_t & \text{if } l_t \leqslant n_t - l_t - 1, \\ -(n_t - l_t) & \text{otherwise}. \end{cases}$$

Now we show that $T_x^{\bar{l}_t} \circ \sigma^{i_t} \to S$ in the weak topology. We have $T_x^m = (\hat{m}, \varphi_m(g))$ and

$$\varphi_m(g) = c_t[j_t + m] + c_t[j_t] \tag{10}$$

for sufficiently large t (see § 2, II). Put $\varphi_{m,t}(g) = c_t[j_t+m]+c_t[j_t]$. Notice that the equality

(11) $$\varphi_m(g) = \varphi_{m,t}(g)$$

holds on a set of Haar measure greater than $1-m/n_t$. At the same time, $\varphi(t)$ is the limit of the sequence of the functions φ_t, defined by (7). From the preceding considerations, if $g \in E_t$ then

(12) $$\varphi(g) = \varphi_t(g) = \begin{cases} c_t[j_t+l_t]+c_t[j_t]+\bar{r}_t & \text{if } j_t+l_t \leqslant n_t-1, \\ c_t[j_t+l_t-n_t]+c_t[j_t]+\bar{s}_t & \text{if } j_t+l_t \geqslant n_t. \end{cases}$$

Suppose that $j_{t+1} = j_t+ln_t$, where $0 \leqslant l < \lambda_{t+1}-1$. It follows from (11) that

(13) $$\varphi_{l_t}(g) = \varphi_{l_t,t+1}(g) = \begin{cases} c_t[j_t+l_t]+c_t[j_t] & \text{if } j_t+l_t \leqslant n_t-1, \\ c_t[j_t+l_t-n_t]+c_t[j_t]+b^{t+1}[l]+b^{t+1}[l+1] & \text{otherwise} \end{cases}$$

except on a set of Haar measure $l_t/n_{t+1} \leqslant 1/\lambda_{t+1}$.

Assume that $l_t \leqslant n_t-l_t-1$. Then

(14) $$\varphi_t(g) = \varphi_{l_t,t+1}(g)+\bar{r}_t$$

whenever $j_t+l_t \geqslant n_t$ and $b^{t+1}[l]+b^{t+1}[l+1] = \bar{r}_t+\bar{s}_t$, except on a set of Haar measure $(l_t/n_t)\,\mathrm{fr}(\bar{r}_t\tilde{\bar{s}}_t \vee \tilde{\bar{r}}_t\bar{s}_t;\, b^{t+1})$.

By using (6′) with l_t instead of q_t, the equality $r_t+s_t = \bar{r}_t+\bar{s}_t$ (see Remark 1) gives

(15) $$\varphi_t(g) = \varphi_{l_t,t+1}(g)+\bar{r}_t$$

except on a set of measure $(l_t/n_t)\,\mathrm{fr}(r_t\tilde{s}_t \vee \tilde{r}_t s_t;\, b^{t+1})$.

In view of (12), (13), (15) we obtain

$$\varphi(g) = \varphi_{l_t}(g)+\bar{r}_t$$

on a set of measure $\geqslant 1-\varepsilon_t$, where $\varepsilon_t \to 0$.

If $l_t > n_t-l_t-1$ then by similar arguments we establish that

$$\varphi(g) = \varphi_{-(n_t-l_t)}(g)+\bar{s}_t$$

on a set of measure $\geqslant 1-\varepsilon_t'$, where $\varepsilon_t' \to 0$.

Since $T_x^{\bar{l}_t}\circ\sigma^{i_t} = (\bar{\bar{l}}_t, \varphi_{\bar{l}_t}+i_t)$, $\bar{l}_t \to g_0$ and $\varphi_{\bar{l}_t}+i_t \to \varphi$ in measure, we conclude that

$$T_x^{\bar{l}_t}\circ\sigma^{i_t} \to S.$$

In this way the equality $C(T_x) = \overline{\{T^i\sigma^j\}}$, $i \in \boldsymbol{Z}$, $j \in \boldsymbol{Z}_2$, is proved.

To show T_x is rigid it suffices to remark that there exists a subsequence of $\{\bar{l}_t\}$ (denote it by $\{\bar{l}_t\}$ again) satisfying $|\bar{l}_t| \to \infty$. Then $T_x^{2\bar{l}_t} \to S^2$ or $T_x^{-2\bar{l}_t} \to S^2$, hence $T_x^{2(\bar{l}_t-\bar{l}_{t-1})} \to I$ or $T_x^{-2(\bar{l}_{t-1}-\bar{l}_t)} \to I$, and so T_x is rigid.

THEOREM 4. *Let $x = b^0\times b^1\times\ldots$ be a continuous Morse sequence such that*

$$\min_{0\leqslant q\leqslant\lambda_t/2} d\big(b^t[0, \lambda_t-q-1], (b^t)^\sim[q, \lambda_t-1]\big) \geqslant \varrho > 0$$

for each $t \geqslant 0$. Then $\sigma \notin \overline{\{T_x^i\}}$, $i \in \boldsymbol{Z}$.

Proof. Suppose that $T_x^{m_t} \to \sigma$. Then $\hat{m}_t \to 0$ in G and $\varphi_{m_t} \to 1$ in measure m, where φ_{m_t} is defined by (10). We may assume $m_t \equiv 0 \pmod{n_t}$, $t \geqslant 0$, i.e. $m_t = \bar{m}_t n_t$, $\bar{m}_t \geqslant 1$. Write

$$\varepsilon_t = m\{g\in G\colon \varphi_{m_t}(g) = 0\}, \quad t \geqslant 0.$$

Then $\varepsilon_t \to 0$. Next let

$$A_{tu} = \{g = (j_v)_0^\infty;\ 0 \leqslant j_u \leqslant n_u-m_t-1\}, \quad u > t.$$

We have $m(A_{tu}) = 1-m_t/n_u$ and $\varphi_{m_t}(g) = c_u[j_u+m_t]+c_u[j_u]$ for $g \in A_{tu}$. Further,

(16) $$m\{g\in A_{tu}\colon \varphi_{m_t}(g) = 0\} \leqslant \varepsilon_t+m(A_{tu}^c) \leqslant \varepsilon_t+m_t/n_u.$$

We choose $u = u(t)$ such that $u > t$ and $m_t/n_u < \varepsilon_t$. Then (16) implies

$$m\{g\in A_{tu}\colon \varphi_{m_t}(g) = 0\} < 2\varepsilon_t \quad \text{for } u = u(t).$$

Write

$$n_t^{(u)} = \lambda_{t+1}\ldots\lambda_u, \quad c_t^{(u)} = b^{t+1}\times\ldots\times b^u, \quad u > t.$$

We have

$$m\{g\in A_{tu}\colon \varphi_{m_t}(g) = 0\} = \frac{1}{n_t^{(u)}}\operatorname{card}\{0 \leqslant l \leqslant n_t^{(u)}-\bar{m}_t-1\colon c_t^{(u)}[l+\bar{m}_t] = c_t^{(u)}[l]\}.$$

It follows that

$$\frac{1}{n_t^{(u)}}\operatorname{card}\{0 \leqslant l \leqslant n_t^{(u)}-1\colon c_t^{u}[l] = c_t^{(u)}(c_t^{(u)}+i_u)[l+\bar{m}_t]\} \leqslant 3\varepsilon_t,$$

because $\bar{m}_t/n_t^{(u)} = m_t/n_u < \varepsilon_t$. Here i_u is an arbitrary element of $\boldsymbol{Z}_2$.

In this way we obtain the following property:

(A) *For every t there exists $u = u(t) > t$ such that for every $u \geqslant u(t)$*

$$\frac{1}{n_t^{(u)}}\operatorname{card}\{0 \leqslant l \leqslant n_t^{(u)}-1\colon c_t^{(u)}(c_t^{(u)}+i_u)[l+\bar{m}_t] = c_t^{(u)}[l]\} < 3\varepsilon_t$$

and $\bar{m}_t \leqslant n_t^{(u)}/2$.

Now we show that (A) implies

(B) *For every t there exists $v = v(t) \geqslant t$ such that*

$$\min_{0 \leqslant q \leqslant \lambda_v/2} d\big(b^v[0, \lambda_u - q - 1], (b^v)^{\sim}[q, \lambda_u - 1]\big) < 2\sqrt{3\varepsilon_t}.$$

To do this we represent the number $\bar{m}_t$ as

$$\bar{m}_t = q' n_t^{(u-1)} + r', \qquad 0 \leqslant q' \leqslant \frac{\lambda_u}{2}, \ 0 \leqslant r' < n_t^{(n-1)}$$

and $c_t^{(u)} = c_t^{(u-1)} \times b^u$.

Suppose that $q' \geqslant 1$ and $r \leqslant n_t^{(u-1)}/2$. We analyse the distances of the successive fragments of $c_t^{(u)}$ and $c_t^{(u)}(c_t^{(u)} + i_u)$ having the lengths $n_t^{(u-1)}$. Using (A) we get the following possibilities:

(a) $d\big(b^u[0, \lambda_u - q' - 1], (b^u)^{\sim}[q', \lambda_u - 1]\big) < 2\sqrt{3\varepsilon_t}$,

(b) $d\big(c_t^{(u-1)}, (c_t^{(u-1)})^{\sim}(c_t^{(u-1)} + i_{u-1})[r, r + n_t^{(u-1)}]\big) < 3\varepsilon_t$,

for some $i_{u-1} \in \boldsymbol{Z}_2$. Case (a) implies (B). If (b) holds then we can repeat the above reasoning taking $u := u-1$ and $\bar{m}_t = r'$. If $r > n_t^{(u-1)}/2$ then we obtain (a) or (b) again with $q'+1$ instead of q' and $r := n_t^{(u-1)} - r$. If $q' = 0$ and $r \leqslant n_t^{(u-1)}/2$ then (b) holds. If $q' = 0$ and $r > n_t^{(u-1)}/2$ then we obtain (a) with $q' = 1$. Proceeding in this way either we choose $u \geqslant t+1$ satisfying (a) or (b) is satisfied for $u = t+1$. The last means (a) for $u = t$. We have shown (B) and consequently Theorem 4.

EXAMPLE 2. Take $b^t = 011011\ldots011$, where 011 is repeated 2^{t+1} times, $t \geqslant 0$. The sequence $x = b^0 \times b^1 \times \ldots$ is regular. It satisfies the assumption of Theorem 3 with $\varrho = \frac{1}{6}$. At the same time, G_0 is uncountable. Indeed, we have $d(b^t b^t[3, \lambda_t + 2], b^t) = 0$, $t \geqslant 0$. Taking $q_t = 3$, $t \geqslant 0$, we obtain

$$l_t = 3 \sum_{u=0}^{t} n_{u-1} < 6n_{t-1},$$

which implies $l_t/n_t < 1/2^t$. Thus $\min(l_t/n_t, 1 - l_t/n_t) < 1/2^t$ and (6″) is satisfied with $r_t = s_t = 0$.

Hence $C(T_x)$ is rigid. Theorem 3 implies $C(T_x) \supsetneqq \overline{\{T_x^i\}}$. In this way we obtain a negative answer to Walter's question [7], because T_x has simple spectrum.

§ **4. Isomorphism theorem.** To describe the centralizer of Morse shifts we have used a certain form of isomorphisms between such systems. Here we give a modification of isomorphism theorems from [3] and [6], omitting the details, which can be found in the above-mentioned papers.

THEOREM 5. *If $x = b^0 \times b^1 \times \ldots$, $y = \beta^0 \times \beta^1 \times \ldots$ are continuous Morse sequences such that $\lambda_t = |b^t| = |\beta^t|$, $t \geqslant 0$, and $\sum_{t=0}^{\infty} 1/\lambda_t < \infty$, then the Morse dynamical systems $\theta(x)$ and $\theta(y)$ are metrically isomorphic if and only if there are sequences of integers $\{\bar{r}_t\}$, $\{\bar{s}_t\}$, $\bar{r}_t, \bar{s}_t \in \boldsymbol{Z}_2$, $t \geqslant 0$, and an element $g_0 = (l_t)_0^{\infty} = \sum_{t=0}^{\infty} \bar{q}_t n_{t-1}$ such that*

$$\sum_t \left[\left(1 - \frac{l_t}{n_t}\right) D_{t+1} + \frac{l_t}{n_t} \bar{D}_{t+1}\right] < \infty, \qquad \text{where} \tag{17}$$

$$D_t = d\big((b^t + \bar{r}_t)(b^t + \bar{s}_t)[\bar{q}_t, \bar{q}_t + \lambda_t - 1], \beta^t + \bar{r}_{t-1}\big),$$

$$\bar{D}_t = d\big((b^t + \bar{r}_t)(b^t + \bar{s}_t)[1 + \bar{q}_t, \bar{q}_t + \lambda_t], \beta^t + \bar{s}_{t-1}\big).$$

Proof. It suffices to show the necessity of the theorem. Indeed, the condition (17) enables us to define a function $\varphi\colon G \to \boldsymbol{Z}_2$ by (5) (see § 1). Then φ satisfies the condition (4).

Necessity. Let us group the blocks $b^0, b^1, b^2, \ldots$ and $\beta^0, \beta^1, \beta^2, \ldots$ to obtain new representations $x = \bar{b}^0 \times \bar{b}^1 \times \ldots$, $y = \bar{\beta}^0 \times \bar{\beta}^1 \times \ldots$ such that

$$\bar{b}^0 = b^0, \quad \bar{b}^t = b^{2t-1} \times b^{2t}, \quad t \geqslant 1,$$

$$\bar{\beta}^t = \beta^{2t} \times \beta^{2t+1}, \quad t \geqslant 0.$$

It follows from [6, Lemma 5] that there exist sequences of blocks $\{a_t\}$, $\{\bar{a}_t\}$, $t \geqslant 0$, $|a_t| = \lambda_{2t}$, $|\bar{a}_t| = \lambda_{2t+1}$, $t \geqslant 0$, and sequences of integers $\{w_t\}$, $\{p_t\}$, $0 \leqslant w_t \leqslant \lambda_{2t-1}\lambda_{2t} - 1$, $t \geqslant 1$, $0 \leqslant p_t \leqslant \lambda_{2t}\lambda_{2t+1} - 1$, $t \geqslant 0$, such that

$$\sum_t d(\bar{b}^t \bar{b}^t[w_t, w_t + \lambda_{2t-1}\lambda_{2t} - 1], \bar{a}_{t-1} \times a_t) < \infty,$$

$$\sum_t d(\bar{\beta}^t \bar{\beta}^t[p_t, p_t + \lambda_{2t}\lambda_{2t+1} - 1], a_t \times \bar{a}_t) < \infty.$$

The proof of Lemma 5 shows that p_t, w_t may be chosen in such a way that if $w_t \neq 0$ then $w_t \geqslant \lambda_{2t-1}\lambda_{2t} - \lambda_{2t-1} + 1$, and if $p_t \neq 0$ then $p_t \geqslant \lambda_{2t}\lambda_{2t+1} - \lambda_{2t}$.

Now we are in a position to define numbers $\bar{r}_t, \bar{s}_t \in \boldsymbol{Z}_2$. Write

$$\eta_t = \max\big(d(\bar{b}^t \bar{b}^t[w_t, w_t + \lambda_{2t-1}\lambda_{2t} - 1], \bar{a}_{t-1} \times a_t),$$
$$d(\bar{\beta}^t \bar{\beta}^t[p_t, p_t + \lambda_{2t}\lambda_{2t+1} - 1], a_t \times \bar{a}_t)\big), \quad t \geqslant 1.$$

Put $k_t = \lambda_{2t-1}\lambda_{2t} - w_t$, $t \geqslant 1$, $m_t = \lambda_{2t}\lambda_{2t+1} - p_t$, $t \geqslant 0$. Then $1 \leqslant k_t \leqslant \lambda_{2t-1} - 1$ (if $w_t \neq 0$) and $1 \leqslant m_t \leqslant \lambda_{2t}$ (if $p_t \neq 0$). Suppose that $w_t \neq 0$. It is easy to verify that

$$d(\bar{b}^t, \bar{a}_{t-1} \times a_t'[k_t, k_t + \lambda_{2t-1}\lambda_{2t} - 1]) \tag{18}$$
$$\leqslant d(\bar{b}^t \bar{b}^t[w_t, w_t + \lambda_{2t-1}\lambda_{2t} - 1], \bar{a}_{t-1} \times a_t) + \frac{k_t}{\lambda_{2t-1}\lambda_{2t}} \leqslant \eta_t + \frac{1}{\lambda_{2t}},$$

where $a_t' = a_t a_t[0]$. Notice that

(19) $$d(\bar b^t, \bar a_{t-1}\times a_t'[k_t, k_t+\lambda_{2t-1}\lambda_{2t}-1])$$
$$= \left(1-\frac{k_t}{\lambda_{2t-1}}\right) d(b^{2t-1}[0, \lambda_{2t-1}-k_t-1]\times b^{2t}, \bar a_{t-1}[k_t, \lambda_{2t-1}-1]\times a_t)$$
$$+\frac{k_t}{\lambda_{2t-1}} d(b^{2t-1}[\lambda_{2t-1}-k_t, \lambda_{2t-1}-1]\times b^{2t}, \bar a_{t-1}[0, k_t-1]\times a_t a_t[1, \lambda_{2t}]).$$

Let $\bar r_{2t-1}, \bar s_{2t-1}\in \boldsymbol{Z}_2$ be integers such that
$$d((b^{2t-1}+\bar r_{2t-1})(b^{2t-1}+\bar s_{2t-1})[\lambda_{2t-1}-k_t, 2\lambda_{2t-1}-k_t-1], \bar a_{t-1})$$
is smallest possible. If $k_t = 0$ then we define $\bar r_{2t-1}, \bar s_{2t-1}$ in such a way that $\bar r_{2t-1} = \bar s_{2t-1}$.

To define $\bar r_t, \bar s_t$ for t even we use the following inequalities:
$$d(A_1, B_1+l)\leqslant d(A_1\times A_2, B_1\times B_2), \qquad d(A_2, B_2+l)\leqslant 3d(A_1\times A_2, B_1\times B_2).$$
Here A_1, A_2, B_1, B_2 are blocks, $|A_1| = |B_1|$, $|A_2| = |B_2|$ and $l\in \boldsymbol{Z}_2$ is such that $d(A_1, B_1+l)\leqslant \frac{1}{2}$. The above inequalities are simple properties of the distance d. Applying them to (19) and using (18) we obtain
$$d((b^{2t-1}+\bar r_{2t-1})(b^{2t-1}+\bar s_{2t-1})[\lambda_{2t-1}-k_t, 2\lambda_{2t-1}-k_t-1], \bar a_{t-1})\leqslant \eta_t+\frac{1}{\lambda_{2t}},$$

(20) $$\frac{k_t}{\lambda_{2t-1}} d(b^{2t}+\bar r_{2t-1}, a_t a_t[1, \lambda_{2t}])$$
$$+\left(1-\frac{k_t}{\lambda_{2t-1}}\right) d(b^{2t}+\bar s_{2t-1}, a_t)\leqslant 3\left(\eta_t+\frac{1}{\lambda_{2t}}\right).$$

Notice that the last inequalities are also valid for $w_t = 0$ if we put $k_t = 0$.

Suppose that $p_t\neq 0$. Let $\bar r_{2t}, \bar s_{2t}\in \boldsymbol{Z}_2$ be integers such that
$$d((\beta^{2t}+\bar s_{2t})(\beta^{2t}+\bar r_{2t})[\lambda_{2t}-m_t, 2\lambda_{2t}-m_t-1], a_t)$$
is smallest possible. If $m_t = 0$ then we can define $\bar r_{2t}, \bar s_{2t}$ in such a way that $\bar r_{2t} = \bar s_{2t}$. As above, we establish the inequalities
$$d((\beta^{2t}+\bar s_{2t})(\beta^{2t}+\bar r_{2t})[\lambda_{2t}-m_t, 2\lambda_{2t}-m_t-1], a_t)\leqslant \eta_t+\frac{1}{\lambda_{2t+1}},$$

(21) $$\left(1-\frac{m_t}{\lambda_{2t}}\right) d(\beta^{2t+1}+\bar r_{2t}, \bar a_t)$$
$$+\frac{m_t}{\lambda_{2t}} d(\beta^{2t+1}+\bar s_{2t}, \bar a_t \bar a_t[1, \lambda_{2t+1}])\leqslant 3\left(\eta_t+\frac{1}{\lambda_{2t+1}}\right).$$

These inequalities are also true for $p_t = 0$ if we put $m_t = 0$.

Now we can define numbers $\bar q_t$, $t\geqslant 0$. Namely we set $\bar q_0 = 0$, and for $t\geqslant 1$
$$\bar q_{2t-1} = \begin{cases} \lambda_{2t-1}-k_t \pmod{\lambda_{2t-1}} & \text{if } m_{t-1}\neq 0, \text{ or } m_{t-1} = 0 \text{ and } \bar q_{2t-2} = 0,\\ \lambda_{2t-1}-k_t-1 & \text{if } m_{t-1} = 0 \text{ and } \bar q_{2t-2} = \lambda_{2t-2}-1,\end{cases}$$
$$\bar q_{2t} = \begin{cases} m_t-1 \pmod{\lambda_{2t}} & \text{if } k_t\neq 0, \text{ or } k_t = 0 \text{ and } \bar q_{2t-1} = \lambda_{2t-1}-1,\\ m_t & \text{if } k_t = 0 \text{ and } \bar q_{2t-1} = 0.\end{cases}$$

To check that the numbers $\bar q_t$ and $\bar r_t, \bar s_t\in \boldsymbol{Z}_2$ satisfy (17) we use the following, easily verified, equalities:
$$d((\beta^{2t}+\bar s_{2t})(\beta^{2t}+\bar r_{2t})[\lambda_{2t}-m_t, 2\lambda_{2t}-m_t-1], a_t)$$
$$= d(\beta^{2t}, (a_t+\bar r_{2t})(a_t+\bar s_{2t})[m_t, m_t+\lambda_{2t}-1]+\bar r_{2t-1}),$$
$$d((b^{2t}+\bar r_{2t})(b^{2t}+\bar s_{2t})[m_t-1, m_t+\lambda_{2t}-2],$$
$$(a_t+\bar r_{2t})(a_t+\bar s_{2t})[m_t, m_t+\lambda_{2t}-1]+\bar r_{2t-1})$$
$$= d(b^{2t}+\bar r_{2t-1}, a_t a_t[1, \lambda_{2t}]),$$
$$d((b^{2t}+\bar r_{2t})(b^{2t}+\bar s_{2t})[m_t, m_t+\lambda_{2t}-1],$$
$$(a_t+\bar r_{2t})(a_t+\bar s_{2t})[m_t, m_t+\lambda_{2t}-1]+\bar s_{2t-1})$$
$$= d(b^{2t}+\bar s_{2t-1}, a_t).$$
The above, (20) and (21) imply
$$\left(1-\frac{m_t}{\lambda_{2t}}\right) D_{2t+1}+\frac{m_t}{\lambda_{2t}}\bar D_{2t+1}\leqslant 2\left(\eta_t+\frac{1}{\lambda_{2t}}\right)+\frac{1}{\lambda_{2t+1}}+3\left(\eta_t+\frac{1}{\lambda_{2t+1}}\right)$$
if $m_t\neq 0$ or $m_t = 0$ and $\bar q_{2t} = 0$,
$$\bar D_{2t+1}\leqslant 2\left(\eta_t+\frac{1}{\lambda_{2t}}\right)+\frac{1}{\lambda_{2t+1}}+3\left(\eta_t+\frac{1}{\lambda_{2t+1}}\right) \quad \text{if } m_t = 0 \text{ and } \bar q_{2t} = \lambda_{2t}-1,$$
and
$$\frac{k_t}{\lambda_{2t-1}} D_{2t}+\left(1-\frac{k_t}{\lambda_{2t-1}}\right)\bar D_{2t}\leqslant 2\left(\eta_t+\frac{1}{\lambda_{2t+1}}\right)+3\left(\eta_t+\frac{1}{\lambda_{2t+1}}\right)$$
if $k_t\neq 0$, or $k_t = 0$ and $\bar q_{2t-1} = \lambda_{2t-1}-1$,
$$D_{2t}\leqslant 2\left(\eta_t+\frac{1}{\lambda_{2t+1}}\right)+3\left(\eta_t+\frac{1}{\lambda_{2t+1}}\right) \quad \text{if } k_t = 0 \text{ and } \bar q_{2t-1} = 0.$$
To obtain (17) we observe that if $m_t\neq 0$ and $k_t\neq 0$ then
$$\left|\frac{m_t}{\lambda_{2t}}-\frac{\bar q_{2t}}{\lambda_{2t}}\right|\leqslant \frac{1}{\lambda_{2t}}, \qquad \left|\frac{k_t}{\lambda_{2t-1}}-\left(1-\frac{\bar q_{2t-1}}{\lambda_{2t-1}}\right)\right|\leqslant \frac{1}{\lambda_{2t-1}},$$
and the series $\sum_{t=0}^{\infty} 1/\lambda_t$, $\sum_{t=0}^{\infty}\eta_t$ are convergent.

This finishes the proof of the theorem.

Remark 2. One can show that the assumption $\sum_{t=0}^{\infty} 1/\lambda_t < \infty$ may be omitted. This can be proved by using Theorem 5 and not difficult, but laborious computations.

We finish this paper with another version of Theorem 5 we have used in § 1.

THEOREM 5′. *Let $x = b^0 \times b^1 \times \ldots$, $y = \beta^0 \times \beta^1 \times \ldots$ be continuous Morse sequences, $|b^t| = |\beta^t| = \lambda_t$, $t \geq 0$ and $\sum_{t=0}^{\infty} 1/\lambda_t < \infty$. Then $\theta(x)$ and $\theta(y)$ are metrically isomorphic iff there exist numbers q_t, $0 \leq q_t \leq \lambda_t - 1$, and $r_t, s_t \in Z_2$ such that*

$$\sum_{t=0}^{\infty} d\big((b^t + r_t)(b^t + s_t)[q_t, q_t + \lambda_t - 1], \beta^t\big) < \infty,$$

$$\sum_{t=0}^{\infty} \min(1 - q_t/\lambda_t, q_t/\lambda_t)\, \mathrm{fr}(r_t \tilde{s}_t \vee \tilde{r}_t s_t; b^{t+1}) < \infty.$$

The proof of this theorem follows immediately from Theorem 5. Namely, let

$$r_0 = \bar{r}_0, \qquad r_{t+1} = \begin{cases} \bar{r}_{t+1} + \bar{r}_t & \text{if } \bar{q}_t \leq \lambda_t - \bar{q}_t - 1, \\ \bar{r}_{t+1} + \bar{s}_t & \text{otherwise}, \end{cases}$$

$$s_0 = \bar{s}_0, \qquad s_{t+1} = \begin{cases} \bar{s}_{t+1} + \bar{r}_t & \text{if } \bar{q}_t \leq \lambda_t - \bar{q}_t - 1, \\ \bar{s}_{t+1} + \bar{s}_t & \text{otherwise}, \end{cases}$$

$$q_0 = \bar{q}_0, \qquad q_t = \begin{cases} \bar{q}_t & \text{if } \bar{q}_{t-1} \leq \lambda_{t-1} - \bar{q}_{t-1} - 1, \\ \bar{q}_t + 1 \pmod{\lambda_t} & \text{otherwise}. \end{cases}$$

Then it is easy to verify that

$$\left(1 - \frac{q_t}{\lambda_t}\right) D_{t+1} + \frac{q_t}{\lambda_t} \bar{D}_{t+1} \leq d\big((b^t + r_t)(b^t + s_t)[q_t, q_t + \lambda_t - 1], \beta^t\big) + \min\left(\frac{q_t}{\lambda_t}, 1 - \frac{q_t}{\lambda_t}\right) \cdot |D_{t+1} - \bar{D}_{t+1}|,$$

$$\left(1 - \frac{q_t}{\lambda_t}\right) D_{t+1} + \frac{q_t}{\lambda_t} \bar{D}_{t+1} \geq \frac{1}{2}\left\{\frac{1}{2} d\big((b^t + r_t)(b^t + s_t)[q_t, q_t + \lambda_t - 1], \beta^t\big) + \min\left(1 - \frac{q_t}{\lambda_t}, \frac{q_t}{\lambda_t}\right) \cdot (D_{t+1} + \bar{D}_{t+1})\right\}.$$

It remains to observe that

$$D_{t+1} + \bar{D}_{t+1} \geq \mathrm{fr}(r_t \tilde{s}_t \vee \tilde{r}_t s_t; b^{t+1}) - 1/\lambda_{t+1},$$
$$|D_{t+1} - \bar{D}_{t+1}| \leq \mathrm{fr}(r_t \tilde{s}_t \vee \tilde{r}_t s_t; b^{t+1}) + 1/\lambda_{t+1}, \qquad t \geq 0.$$

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