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Vector-valued Calderón–Zygmund theory and Carleson measures on spaces of homogeneous nature

by

FRANCISCO J. RUIZ (Zaragoza) and JOSÉ L. TORREA (Madrid)

Abstract. We develop a method for the study of boundedness properties of operators defined by means of a kernel and mapping functions defined on X into functions defined on $X \times [0, \infty)$, where X is a space of homogeneous nature. This study allows us to obtain in this context the Fefferman–Stein theorem concerning the vector-valued inequalities for the Hardy–Littlewood maximal operator. Carleson measures and tent spaces fall under the scope of this method. Also, almost everywhere convergence results for parabolic differential operators are obtained.

0. Introduction. The main objective of this paper is to study Carleson measures and related topics in the context of spaces of homogeneous nature. Some work in this direction was done in [1]. Also, we want to study the boundedness properties of some related maximal operators.

In order to study these properties we introduce a vector-valued singular integral technique. This technique allows us to obtain vector-valued inequalities of the following type:

$$\text{(I)} \qquad \Big\|\Big(\sum_j |Tf_j|^q\Big)^{1/q}\Big\|_{L^p(X\times[0,\infty),\alpha)} \leqslant C \Big\|\Big(\sum_j |f_j|^q\Big)^{1/q}\Big\|_{L^p(X,\mu)},$$

$$\text{(II)} \qquad \alpha\Big(\Big\{(x, t) \in X\times[0, \infty)\colon \sum_j |Tf_j(x, t)|^q > \lambda^q\Big\}\Big) \leqslant \frac{C}{\lambda} \int_X \Big(\sum_j |f_j(x)|^q\Big)^{1/q} d\mu(x),$$

where X is a space of homogeneous nature with a doubling (in general not translation invariant) measure μ and α is a Carleson measure on $X \times [0, \infty)$ (i.e. $\alpha(B(x, r)\times[0, r]) \leqslant C\mu(B(x, r))$ for any ball $B(x, r)$).

T will be an operator with some a priori boundedness properties and such that T can be represented by a kernel in a certain way (see Section 3, Theorems 2–4).

Different substitutes of the classical dyadic Calderón–Zygmund decomposition can be tried in this context. It appears that Whitney type decompositions (see [6]) are not enough for the weighted theory (see Sections 2 and 3) and we use some ideas developed in the case of topological groups with a translation invariant measure (see Koranyi and Vagi [13]).

In the particular case of $d\alpha(x, t) = d\mu(x)\otimes\delta_0(t)$ we apply the theory to parabolic singular integrals and obtain some results about their almost everywhere convergence.

The organization of the paper is as follows.

In Section 1 we introduce the notation and preliminaries.

Section 2 is devoted to stating and proving the lemmas which will be the main tool in the Calderón–Zygmund decomposition of functions. Also, we find weighted norm inequalities for maximal operators related with the Hardy–Littlewood maximal function.

Section 3 contains the main theorems concerning the boundedness properties of operators (defined by means of a kernel) mapping E-valued functions on X into F-valued ones on $X\times[0,\infty)$ (where E, F are Banach spaces).

Some applications to maximal functions are considered in Section 4, where we obtain the Fefferman–Stein theorem (see [10]) for spaces of homogeneous type with a translation invariant pseudometric. In this section, some vector-valued inequalities are proved for the functional A_∞ which appears in the theory of tent spaces (see [5]).

Finally, in Section 5 the theory is applied in the context of parabolic singular integrals.

1. Preliminaries and basic facts. Let X be a set. A *pseudometric* on X is a map ϱ: $X\times X\to[0,\infty)$ such that:

(i) $\varrho(x,y)>0$ if and only if $x\neq y$.

(ii) $\varrho(x,y)=\varrho(y,x)$ for all $x, y\in X$.

(iii) There exists a constant $k\geqslant 1$ such that

$$\varrho(x,z)\leqslant k(\varrho(x,y)+\varrho(y,z)),\qquad x, y, z\in X.$$

By a "ball with center x and radius r" we mean the set $B(x,r) =\{y\in X:\ \varrho(x,y)<r\}$. Sometimes, we shall write $\lambda B(x,r)$ for the set $B(x,\lambda r)$ with $\lambda>0$.

In the case $k=1$, the pseudometric (in fact metric) will be denoted by $d(x,y)$.

Definition 1 (see [6]). A *space of homogeneous nature* (or, simply, a *homogeneous space*) is a topological space X endowed with a pseudometric ϱ such that:

(a) The family $\{B(x,r):\ x\in X,\ r>0\}$ is a basis of the topology of X.

(b) There exists a natural number N such that for any $x\in X$ and $r>0$ the ball $B(x,r)$ contains at most N points x_i with $\varrho(x_i,x_j)>r/2$.

The constants k (in (iii)) and N (in (b)) are called the *space constants*.

All along this paper we shall consider a topological space X with a pseudometric satisfying (a), (b) and with a *doubling Borel measure* μ, i.e. there exists $C>0$ such that

(1) $$\mu(B(x,2r))\leqslant C\mu(B(x,r)),\qquad x\in X,\ r>0.$$

It is known (see [6]) that then X is a homogeneous space.

Observe that the condition (1) implies the following stronger one:

(2) If $\eta>1$ then there exists a constant $C(\eta)$ such that

$$\mu(B(x,\eta r))\leqslant C(\eta)\,\mu(B(x,r)),\qquad x\in X,\ r>0.$$

Covering Lemma (see [6]). *Let E be a bounded subset of X (i.e. contained in a ball) and $\{B(x,r(x))\}_{x\in E}$ a covering of E. Then there exists a sequence of disjoint balls $\{B(x_i,r(x_i))\}_{i=1}^{\infty}$ such that $\{KB(x_i,r(x_i))\}_{i=1}^{\infty}$ is a covering of E, where K is a constant depending only on the space constants.*

As usual, given a Banach space A, we shall denote by $L^p_A(X,\mu)$, $1\leqslant p\leqslant\infty$, the Bochner–Lebesgue space of A-valued strongly measurable functions f such that $\int_X\|f(x)\|_A^p\,d\mu(x)<+\infty$. If $A=C$ we shall write simply $L^p(X,\mu)$.

The following definitions include in our context the analogues to several well-known operators in Euclidean harmonic analysis.

Definition 2. Let f: $X\to C$ be a locally integrable function (i.e. integrable over balls). For $0<\gamma\leqslant 1$, we define:

(3) $$M_\gamma f(x)=\sup\{\mu(B(x,r))^{-\gamma}\int_{B(x,r)}|f(y)|\,d\mu(y):\ r>0\},\qquad x\in X.$$

(4) $$\mathfrak{M}_\gamma f(x,t)=\sup\{\mu(B(x,r))^{-\gamma}\int_{B(x,r)}|f(y)|\,d\mu(y):\ r\geqslant t\},\qquad x\in X,\ t\geqslant 0.$$

Remark 1. The balls taken in these definitions are centered at x. It is possible to define similar operators buth with the supremum taken over all balls containing x; let us call them $\tilde{M}_\gamma$, $\tilde{\mathfrak{M}}_\gamma$. An easy consequence of the inequality (2) is the existence of a constant C_γ depending on the space constants and γ such that:

(5) $$\begin{aligned} M_\gamma f(x)\leqslant \tilde{M}_\gamma f(x)\leqslant C_\gamma M_\gamma f(x),&\qquad x\in X,\\ \mathfrak{M}_\gamma f(x,t)\leqslant \tilde{\mathfrak{M}}_\gamma f(x,t)\leqslant C_\gamma\mathfrak{M}_\gamma f(x,t),&\qquad x\in X,\ t>0.\end{aligned}$$

Note also that

(6) $$\mathfrak{M}_\gamma f(x,0)=M_\gamma f(x),\qquad x\in X.$$

Remark 2. If $\gamma=1$ the operator $M_1 f=Mf$ is the Hardy–Littlewood maximal operator. For a homogeneous space (X,ϱ,μ) with μ doubling it is well known (see [6]) that M is bounded from $L^p(X,\mu)$ into $L^p(X,\mu)$, $1<p\leqslant\infty$, and from $L^1(X,\mu)$ into weak-$L^1(X,\mu)$. Moreover, if the boundedly supported continuous functions are dense in $L^p(X,\mu)$ then the classical Lebesgue differentation theorem holds.

Note. In all the rest of this paper we shall suppose that the continuous functions with bounded support are dense in $L^p(X,\mu)$, $1\leqslant p<\infty$.

Hereafter, the space $X \times [0, \infty) = \{(x, t) \colon x \in X, t \geq 0\}$ will be denoted by X^+.

The following question arises: Let (X, ϱ, μ) be a homogeneous space and let α, β be Borel measures on X and X^+ respectively; what is the exact condition on the pair (α, β) for $\mathfrak{M}_\gamma$ to be a bounded operator from $L^p(X, \alpha)$ into weak-$L^q(X^+, \beta)$?

In other words, the problem is to find conditions on (α, β) in order that

$$\beta(\{(x, t) \in X^+ \colon \mathfrak{M}_\gamma f(x, t) > \lambda\}) \leq C\lambda^{-q} \|f\|^q_{L^p(X,\alpha)}, \quad \lambda > 0. \tag{7}$$

It is clear that the singular part of α with respect to μ has no contribution in the inequality (7). Thus α can be taken as $d\alpha(x) = \omega(x)\, d\mu(x)$, where ω is a weight (positive measurable function). So the above question can be reduced to characterizing the pairs (ω, β) such that $\mathfrak{M}_\gamma$ maps $L^p(X, \omega\, d\mu)$ into weak-$L^q(X^+, \beta)$.

The question will be solved in the next section. Moreover, owing to (6) we shall be able to answer similar questions for M_γ. On the other hand, the study of these operators will give us a key to a "good" Calderón–Zygmund decomposition.

2. Boundedness of $\mathfrak{M}_\gamma$ and technical lemmas. In order to simplify the notation in the rest of this paper, the measure μ of a set E will be denoted by $|E|$, and the integral with respect to μ by $\int_E f$. Also, B will always mean a ball in X, and $\tilde{B}$ the cylinder

$$\{(x, t) \in X^+ \colon x \in B,\ 0 \leq t \leq \text{radius of } B\}.$$

DEFINITION 3. Suppose ω is a weight on X and β a positive Borel measure on X^+. We say that $(\omega, \beta) \in C_{p,q,\gamma}$, $0 < \gamma \leq 1$, if

$$\frac{\beta(\tilde{B})^{1/q}}{|B|^\gamma} \Big(\int_B \omega^{-p'/p}\Big)^{1/p'} \leq C, \quad 1 < p < \infty,\ 1 \leq q < \infty, \tag{8}$$

where the supremum is taken over all balls B in X.

In case $p = 1$, $C_{1,q,\gamma}$ will denote the set of pairs (ω, β) such that

$$\sup_{x \in B} \frac{\beta(\tilde{B})^{1/q}}{|B|^\gamma} \leq C\omega(x) \quad \text{a.e.}, \quad 1 \leq q < \infty. \tag{9}$$

The infimum of the constants appearing in (8) and (9) will be called $C_{p,q,\gamma}(\omega, \beta)$ and $C_{1,q,\gamma}(\omega, \beta)$ respectively. If $q = p$ and $\gamma = 1$ we shall write simply $C_{p,p,1} = C_p$.

LEMMA 1. *Let $1 \leq p, q < \infty$, $0 < \gamma \leq 1$ and $(\omega, \beta) \in C_{p,q,\gamma}$. Suppose f is a positive function with bounded support and such that $f \in L^p(X, \omega\, d\mu)$. Then for*

$$\lambda > \frac{C_{p,q,\gamma}(\omega, \beta)}{\beta(X^+)^{1/q}} \|f\|_{L^p(X,\omega d\mu)} \quad (\lambda > 0 \text{ if } \beta(X^+) = +\infty) \tag{10}$$

there exists a sequence of disjoint balls $\{B_i\}_{i=1}^\infty$ in X such that:

(i) $$\lambda \leq |B_i|^{-\gamma} \int_{B_i} f \leq C\lambda, \quad i = 1, 2, \ldots$$

(ii) *There exists a constant K such that $\{KB_i\}_{i=1}^\infty$ is a covering of $E_\lambda^\gamma = \{x \in X \colon M_\gamma f(x) > \lambda\}$.*

(iii) *$\{(KB_i)^\sim\}_{i=1}^\infty$ is a covering of the set*

$$A^\gamma_{C_\gamma \lambda} = \{(x, t) \in X^+ \colon \mathfrak{M}_\gamma f(x, t) > C_\gamma \lambda\}$$

where K is the constant which appears in the covering lemma of Section 1 and C_γ is the constant in (5).

Proof. Only the case $1 < p < \infty$ will be considered since the modifications needed to deal with the case $p = 1$ are fairly straightforward.

Given $x \in E_\lambda^\gamma$, consider the family of radii r such that

$$|B(x, r)|^{-\gamma} \int_{B(x,r)} f > \lambda. \tag{11}$$

This family has a finite supremum, say $r_\lambda(x)$, since if B is a ball satisfying (11) then

$$\beta(\tilde{B}) < \frac{\beta(\tilde{B})}{\lambda^q} \Big(|B|^{-\gamma} \int_B f\Big)^q \leq \frac{\beta(\tilde{B})}{\lambda^q |B|^{\gamma q}} \Big(\int_B f^p \omega\Big)^{q/p} \Big(\int_B \omega^{-p'/p}\Big)^{q/p'}$$
$$\leq \frac{C_{p,q,\gamma}(\omega, \beta)^q}{\lambda^q} \|f\|^q_{L^p(X,\omega d\mu)} < \beta(X^+)$$

where (10) and Hölder's inequality (applied to $(f\omega^{1/p})\omega^{-1/p}$) have been used.

On the other hand, E_λ^γ is bounded. This can be proved as follows. Suppose that $X \neq B(x, r)$ for any $x \in X$ and $r > 0$ (otherwise there is nothing to prove) and that $\operatorname{supp} f \subset B(x_0, r_0)$. We shall show that for n large enough $E_\lambda^\gamma \subset B(x_0, nr_0)$.

We choose n such that $n/2 + k < n$ (i.e. $k < n/2$), where k is the space constant. It is easy to see that if $y \notin B(x_0, nr_0)$ then $B(y, (n/2k) r_0) \cap B(x_0, r_0) = \emptyset$ and therefore

$$M_\gamma f(y) = \sup_{s > (n/2k) r_0} |B(y, s)|^{-\gamma} \int_{B(y,s)} f$$
$$\leq \sup_{s > (n/2k) r_0} |B(y, s)|^{-\gamma} \Big(\int_{B(y,s)} f^p \omega\Big)^{1/p} \Big(\int_{B(y,s)} \omega^{-p'/p}\Big)^{1/p'}$$
$$\leq C_{p,q,\gamma}(\omega, \beta) \|f\|_{L^p(X,\omega d\mu)} \sup_{s > (n/2k) r_0} \beta(\tilde{B}(y, s))^{-1/q}.$$

Now, the condition imposed on λ allows us to choose n large enough to make the last expression smaller than λ. So we have the bounded set E_λ^γ covered by the family $\{B(x, r_\lambda(x)) \colon x \in E_\lambda^\gamma\}$. Then by the covering lemma in Section 1 we get a sequence $\{B(x_i, r(x_i))\}_{i=1}^\infty$ of disjoint balls satisfying (ii). The definition of $r_\lambda(x)$ and the properties of doubling measures imply (i).

Finally, we do some more work to prove (iii). Given $(x, t) \in A_{C_\gamma\lambda}^\gamma$ it is clear that $x \in E_{C_\gamma\lambda}^\gamma \subset E_\lambda^\gamma$ and so there exists a ball $B(x_i, Kr(x_i))$ in the above covering such that $x \in B(x_i, Kr(x_i))$. Suppose $t > Kr(x_i)$; then $x_i \in B(x, s)$ for any $s \geqslant t$ and therefore

$$C_\gamma \lambda < \mathfrak{M}_\gamma f(x, t) \leqslant \widetilde{\mathfrak{M}}_\gamma f(x_i, t) \leqslant C_\gamma \mathfrak{M}_\gamma f(x_i, t).$$

In particular, $\mathfrak{M}_\gamma f(x_i, t) > \lambda$ and $r(x_i) < Kr(x_i) < t$ which gives us a contradiction with the definition of $r(x_i)$. In other words, $t \leqslant Kr(x_i)$ and then $(x, t) \in \tilde{B}(x_i, Kr(x_i))$. This concludes the proof of the lemma.

Lemma 2. *Let $1 \leqslant p < \infty$ and $(\omega, \beta) \in C_p$. Suppose f is a positive function with bounded support and such that $f \in L^p(X, \omega\, d\mu)$. Then for*

$$\lambda^p > \frac{C_p(\omega, \beta)}{\beta(X^+)} \|f\|_{L^p(X, \omega d\mu)}^p \quad (\lambda > 0 \textit{ if } \beta(X^+) = +\infty)$$

there exists a sequence $\{Q_i\}_{i=1}^\infty$ of disjoint subsets of X such that:

(i) *$f(x) \leqslant \lambda$ for a.e. $x \notin \bigcup_{i=1}^\infty Q_i$.*

(ii) *$c\lambda \leqslant |Q_i|^{-1} \int_{Q_i} f \leqslant C\lambda$, $i = 1, 2, \ldots$*

(iii) *There exists a sequence $\{x_i, r(x_i)\}_{i=1}^\infty$ in X^+ such that*

$$B(x_i, r(x_i)) \subset Q_i \subset B(x_i, Kr(x_i)).$$

Here C and c are constants depending on the space.

Proof. Let $\{B(x_i, r(x_i))\}_{i=1}^\infty$ be the family obtained in Lemma 1. We define Q_i by recurrence as follows:

$$Q_1 = B(x_1, Kr(x_1)) \setminus \bigcup_{j>1} B(x_j, r(x_j)),$$

$$Q_n = B(x_n, Kr(x_n)) \setminus \bigcup_{j<n} Q_j \setminus \bigcup_{j>n} B(x_j, r(x_j)), \qquad n > 1.$$

Now, conditions (i), (ii) and (iii) are easily checked by using the properties of $B(x_i, r(x_i))$ and Remark 2 in Section 1.

Remark 3. In the literature, various lemmas related with Lemma 2 can be found. One way of trying to obtain part (i) and the right-hand side inequality in (ii) is using a lemma of Whitney's type for the set E_λ^γ (see [4], [6]). With this method the radius of the ball with center x is essentially $\varrho(x, (E_\lambda^\gamma)^c)$. The balls obtained in this way do not in general satisfy the left-hand side inequality of (ii) as can be easily seen in the case $X = \{2^n \colon n \in N\}$ with $\varrho(2^n, 2^m) = |2^n - 2^m|$ and $\mu(2^n) = 2^{n+1}$.

Some of the ideas developed in Lemmas 1 and 2 can be found in [13] for the particular case in which X is a locally compact Hausdorff topological group with a left-invariant Haar measure μ and a translation invariant pseudometric ϱ.

Theorem 1. *Let $1 \leqslant p, q < \infty$ and $0 < \gamma \leqslant 1$. Then the inequality*

(12)
$$\beta(\{x, t) \in X^+ \colon \mathfrak{M}_\gamma f(x, t) > \lambda\}) \leqslant \frac{C}{\lambda^q} \|f\|_{L^p(X, \omega d\mu)}^q, \quad f \in L^p(X, \omega\, d\mu),\ \lambda > 0,$$

holds if and only if $(\omega, \beta) \in C_{p,q,\gamma}$.

Remark 4. In the particular case of $d\beta(x, t) = v(x)\, d\mu(x) \otimes \delta_0(t)$ with v a weight on X and δ_0 the Dirac delta at $t = 0$, we get the exact condition on the pair (v, ω) for M_γ to be a bounded operator from $L^p(X, \omega\, d\mu)$ into weak-$L^q(X, v\, d\mu)$. If $\omega \equiv v$, $\gamma = 1$ and $p = q$, it was already obtained by A. P. Calderón (see [3]). The proof there consists in proving the theorem for maximal functions of the type

$$M^R f(x) = \{|B|^{-1} \int_B |f| \colon x \in B, \text{ radius of } B \leqslant R\},$$

and then the fact that $M^R f(x) \nearrow Mf(x)$ gives the result.

This technique was also used in the case $X = \boldsymbol{R}^n$ with Euclidean norm (see [18]). In the present case, an easy modification of the covering lemma suffices to apply the method. However, since Lemma 1 is at our disposal we give here a short proof based on it.

Proof of Theorem 1. For any ball B it is clear that

$$\tilde{B} \subset \{(x, t) \in X^+ \colon \mathfrak{M}_\gamma f(x, t) \geqslant |B|^{-\gamma} \int_B f\}.$$

Then, by taking $f = \chi_B \omega^{-p'/p}$ in the case $p > 1$, the inequality (12) gives $(\omega, \beta) \in C_{p,q,\gamma}$.

In the case $p = 1$ take $f = \chi_{B'} \omega^{-1}$ where B' is any ball contained in B. The hypothesis says that

$$\beta(\tilde{B})^{1/q} \leqslant C|B|^\gamma \Big(\int_{B'} \omega^{-1}\Big)^{-1} \Big(\int_{B'} \omega^{-1} \omega\Big) = C|B|^\gamma \Big(\int_B \omega^{-1}\Big)^{-1} |B'|.$$

Now on letting B' tend to x, Remark 2 in Section 1 gives $(\omega, \beta) \in C_{1,q,\gamma}$.

Conversely, we assume $(\omega, \beta) \in C_{p,q,\gamma}$, $p > 1$ (the case $p = 1$ is completely similar) and we have to prove (12). It is sufficient to consider $f \in L^p(X, \omega\, d\mu)$

with bounded support. If

$$\lambda \leqslant \frac{C_{p,q,\gamma}(\omega, \beta)}{\beta(X^+)^{1/q}} \|f\|_{L^p(X,\omega d\mu)}$$

then inequality (12) is obviously satisfied.

Otherwise we apply Lemma 1 to the function $|f|$ and we have

$$\begin{aligned}\beta(A^{\gamma}_{C_\gamma\lambda}) &\leqslant \beta\Big(\bigcup_{i=1}^{\infty} (KB_i)^{\sim}\Big) \leqslant \sum_{i=1}^{\infty} \frac{\beta\big((KB_i)^{\sim}\big)}{\lambda^q} \Big(|B_i|^{-\gamma} \int_{B_i} |f|\Big)^q \\ &\leqslant \frac{C}{\lambda^q} \sum_{i=1}^{\infty} \frac{\beta\big((KB_i)^{\sim}\big)}{|KB_i|^{\gamma q}} \Big(\int_{B_i} |f|^p \omega\Big)^{q/p} \Big(\int_{B_i} \omega^{-p'/p}\Big)^{q/p'} \\ &\leqslant C \frac{C_{p,q,\gamma}(\omega, \beta)}{\lambda^q} \|f\|^q_{L^p(X,\omega d\mu)}\end{aligned}$$

and this concludes the proof of Theorem 1.

Note. In the first part of the proof we have assumed that $\omega^{-p'/p}$ and ω^{-1} are integrable over balls. This can be obtained from (12) as follows:

If $\int_B \omega^{-p'/p} = \int_B \omega^{1-p'} = \int_B \omega^{-p'} \omega = \infty$ then there exists $f \in L^p(B, \omega d\mu) \subset L^p(X, \omega d\mu)$ such that $\int_B f\omega^{-1} \omega = \infty$ and then $\mathfrak{M}_\gamma f \equiv \infty$.

If $\int_B \omega^{-1} = \infty$ then $\mathfrak{M}_\gamma (\chi_B \omega^{-1}) \equiv \infty$ with $\chi_B \omega^{-1} \in L^1(X, \omega d\mu)$.

Both cases contradict (12).

Some results concerning the boundedness of the operator $\mathfrak{M}_1 = \mathfrak{M}$ in the case $X = \boldsymbol{R}^n$ can be found in [16] and [17].

3. Main results. Given E, F Banach spaces, $\mathscr{L}(E, F)$ will be the set of bounded linear operators from E into F.

THEOREM 2. *Let E, F be Banach spaces. Suppose that*:

(a) *T is a bounded linear operator from $L^\infty_E(X, \omega d\mu)$ into $L^\infty_F(X^+, \beta)$ with $(\omega, \beta) \in C_1$.*

(b) *There exists an $\mathscr{L}(E, F)$-valued function K on $X \times X \times [0, \infty) \setminus \{(x, x, t)\colon x \in X,\ t \geqslant 0\}$ such that*:

(b1) *For any $(x, t) \in X^+$ and any ball B such that $(x, t) \notin 2\tilde{B}$ we have $\int_B \|K(x, y, t)\| d\mu(y) < +\infty$. Moreover, for f in $L^\infty_E(X, \mu)$ with support contained in a ball B the following representation formula holds*:

$$Tf(x, t) = \int_X K(x, y, t) f(y) d\mu(y) \quad \textit{for } (x, t) \notin 2\tilde{B}.$$

(b2) $$\|K(x, y, t) - K(x, y', t)\|_{\mathscr{L}(E,F)} \leqslant C \frac{\varrho(y, y')}{\big(\varrho(x, y') + t\big)\,|B(y', \varrho(x, y') + t)|}$$

$$\textit{for } \varrho(x, y') + t > 2\varrho(y, y').$$

Then:

(i) *T maps $L^p_E(X, \omega d\mu)$ into $L^p_F(X^+, \beta)$ for $1 < p < \infty$.*

(ii) *T maps $L^1_E(X, \omega d\mu)$ into weak-$L^1_F(X^+, \beta)$.*

In the case $\omega(x) \equiv 1$ or, in other words, when the condition C_1 means that we have a Carleson measure on X^+ (i.e. $\beta(\tilde{B}) \leqslant C|B|$ for every ball B in X) the following result holds:

THEOREM 3. *Suppose that T satisfies hypothesis* (b) *of Theorem* 2 *and also*

(a) *There exists a p_0, $1 < p_0 \leqslant \infty$, such that T is a bounded linear operator from $L^{p_0}_E(X, \mu)$ into $L^{p_0}_F(X^+, \beta)$ where β is a Carleson measure.*

Then:

(i) *T maps $L^p_E(X, \mu)$ into $L^p_F(X^+, \beta)$ for $1 < p \leqslant p_0$.*

(ii) *T maps $L^1_E(X, \mu)$ into weak-$L^1_F(X^+, \beta)$.*

THEOREM 4. *Suppose that T satisfies hypotheses* (a) *and* (b) *of Theorem* 2 *for any pair $(\omega, \beta) \in C_1$.*

Then the following vector-valued inequalities hold for any Carleson measure α on X^+:

(i) *For $1 < p, q < \infty$*

$$\Big\|\Big(\sum_{j=1}^{\infty} \|Tf_j\|^q_F\Big)^{1/q}\Big\|_{L^p(X^+,\alpha)} \leqslant C \Big\|\Big(\sum_{j=1}^{\infty} \|f_j\|^q_E\Big)^{1/q}\Big\|_{L^p(X,\mu)}.$$

(ii) *For $1 < q < \infty$*

$$\alpha\Big(\Big\{(x, t) \in X^+\colon \sum_{j=1}^{\infty} \|Tf_j(x, t)\|^q_F > \lambda^q\Big\}\Big) \leqslant \frac{C}{\lambda} \int_X \Big(\sum_{j=1}^{\infty} \|f_j(x)\|^q_E\Big)^{1/q} d\mu(x).$$

The above theorems have already been shown in the case $X = \boldsymbol{R}^n$ with Euclidean norm (see [18]). As will be seen below a crucial step in the proof of Theorem 2 is to obtain a Calderón–Zygmund decomposition where the considered disjoint sets Q_i satisfy

$$c\lambda \leqslant |Q_i|^{-1} \int_{Q_i} \|f\|_E.$$

This will be possible thanks to Lemma 2. Once it is done the proofs of Theorems 2 to 4 follow, except for more or less complicated technicalities, along the same lines as in [18].

Nevertheless, for clarity we give here the complete proofs.

Proof of Theorem 2. It is enough to prove (ii) and then apply Marcinkiewicz's interpolation theorem.

For a fixed $\lambda > 0$, we want to prove

$$\beta\big(\{(x, t) \in X^+\colon \|Tf(x, t)\|_F > \lambda\}\big) \leqslant \frac{C}{\lambda} \int_X \|f(x)\|_E\, \omega(x)\, d\mu(x). \tag{13}$$

Take a function f in $L_E^\infty(X, \omega\,d\mu) \cap L_E^1(X, \omega\,d\mu)$ with bounded support and apply Lemma 2 to the function $f \to \|f(x)\|_E$ and λ (observe that if λ does not satisfy the hypothesis of Lemma 2 then (13) is obvious).

Now, we decompose f into the functions g and b given by

$$g(x) = f(x) \quad \text{for } x \notin \bigcup Q_i,$$

$$g(x) = |Q_i|^{-1} \int_{Q_i} f \quad \text{for } x \in Q_i,\ i = 1, 2, \ldots,$$

$$b(x) = f(x) - g(x) = \sum_i \Big(f(x) - |Q_i|^{-1} \int_{Q_i} f\Big)\chi_{Q_i}(x) = \sum_i b_i(x).$$

In the rest of this proof $B_i = B(y_i, r(y_i))$ is the ball associated with Q_i by the relation $B_i \subset Q_i \subset KB_i$ (see Lemma 2). We shall also consider the set $D_\lambda = \sum_{i=1}^\infty (2KB_i)^\sim$.

Using the properties of Q_i it is easy to see that $g(x) \leqslant C\lambda$ for a.e. x and $\|g\|_{L_E^1(X,\omega d\mu)} \leqslant \|f\|_{L_E^1(X,\omega d\mu)}$. By hypothesis (a) we have

$$\|Tg\|_{L_F^\infty(X^+,\beta)} \leqslant C\|g\|_{L_E^\infty(X,\omega d\mu)} \leqslant C\lambda$$

and therefore

$$\beta(\{(x, t) \in X^+ : \|Tf(x, t)\|_F > 2C\lambda\}) \leqslant \beta(\{(x, t) \in X^+ : \|Tb(x, t)\|_F > C\lambda\}),$$

and so it is enough to show that

$$\beta(\{(x, t) \in X^+ : \|Tb(x, t)\|_F > \lambda\}) \leqslant \frac{C}{\lambda} \int_X \|f(x)\|_E\, \omega(x)\, d\mu(x). \tag{14}$$

We have

$$\beta(\{(x, t) \in X^+ : \|Tb(x, t)\|_F > \lambda\}) \leqslant \beta(D_\lambda) + \beta(\{(x, t) \notin D_\gamma : \|Tb(x, t)\|_F > \lambda\}).$$

Let us estimate these two measures:

$$\beta(D_\lambda) = \sum_{i=1}^\infty \frac{\beta((2KB_i)^\sim)}{\lambda} |B_i|^{-1} \int_{B_i} \|f(x)\|_E\, d\mu(x)$$

$$\leqslant \frac{C}{\lambda} \sum_{i=1}^\infty \int_{B_i} \|f(x)\|_E\, \omega(x)\, d\mu(x) \leqslant \frac{C}{\lambda} \int_X \|f(x)\|_E\, \omega(x)\, d\mu(x)$$

where the constant appearing in the second inequality depends on the C_1 constant and on the space constants.

On the other hand, we have

$$\beta(\{(x, t) \notin D_\lambda : \|Tb(x, t)\|_F > \lambda\}) \leqslant \lambda^{-1} \int_{D_\lambda^c} \|Tb(x, t)\|_F\, d\beta(x, t)$$

$$\leqslant \lambda^{-1} \sum_i \int_{D_\lambda^c} \|Tb_i(x, t)\|_F\, d\beta(x, t).$$

By hypothesis (b1) this is smaller than

$$\lambda^{-1} \sum_i \int_{D_\lambda^c} \int_{Q_i} \|K(x, y, t) - K(x, y_i, t)\|\, \|b_i(y)\|_E\, d\mu(y)\, d\beta(x, t).$$

In this case either $t > 2Kr(y_i) > 2\varrho(y, y_i)$ or $\varrho(x, y_i) > 2Kr(y_i) > 2\varrho(y, y_i)$. Then hypothesis (b2) applies and we deduce that the last expression is less than

$$\frac{C}{\lambda} \sum_i \Big(\int_{Q_i} \|b_i(y)\|_E\, d\mu(y)\Big) M_i \leqslant \frac{2C}{\lambda} \sum_i \Big(\int_{Q_i} \|f(y)\|_E\, d\mu(y)\Big) M_i$$

where

$$M_i = \operatorname*{ess\,sup}_{y \in Q_i} \int_{D_\lambda^c} \frac{\varrho(y, y_i)}{(\varrho(x, y_i) + t)|B(y_i, \varrho(x, y_i) + t)|}\, d\beta(x, t).$$

But if we put

$$A_n^i = \{(x, t) \in X^+ : 2^{n+1} Kr(y_i) > \varrho(x, y_i) + t > 2^n Kr(y_i)\}, \quad n = 1, 2, \ldots,$$

then

$$\int_{D_\lambda^c} \frac{\varrho(y, y_i)}{(\varrho(x, y_i) + t)|B(y_i, \varrho(x, y_i) + t)|}\, d\beta(x, t)$$

$$\leqslant Kr(y_i) \sum_{n=1}^\infty \int_{A_n^i} \frac{d\beta(x, t)}{(\varrho(x, y_i) + t)|B(y_i, \varrho(x, y_i) + t)|}$$

$$\leqslant Kr(y_i) \sum_{n=1}^\infty \frac{1}{2^n Kr(y_i)} \cdot \frac{\beta(\tilde{B}(y_i, 2^{n+1} Kr(y_i)))}{|B(y_i, 2^n Kr(y_i))|}$$

since if $\varrho(x, y_i) + t = 2^{n+1} Kr(y_i)$ then $(x, t) \in \tilde{B}(y_i, 2^{n+1} Kr(y_i))$. Now, since $(\omega, \beta) \in C_1$ we get $M_i \leqslant C\omega(y)$ for a.e. $y \in Q_i$. In particular,

$$\Big(\int_{Q_i} \|f(y)\|_E\, d\mu(y)\Big) M_i \leqslant \int_{Q_i} \|f(y)\|_E\, \omega(y)\, d\mu(y)$$

and therefore the disjointness of Q_i gives the result.

Proof of Theorem 3. The same scheme as in the above proof works except for the following computation:

$$\beta(\{(x, t) \in X^+ : \|Tg(x, t)\|_F > \lambda\})$$

$$\leqslant \lambda^{-p_0} \int_{X^+} \|Tg(x, t)\|_F^{p_0}\, d\beta(x, t) \leqslant \frac{C}{\lambda^{p_0}} \int_X \|g(x)\|_E^{p_0}\, d\mu(x)$$

$$\leqslant \frac{C}{\lambda} \int_X \|g(x)\|_E\, d\mu(x) \leqslant \frac{C}{\lambda} \int_X \|f(x)\|\, d\mu(x)$$

and the fact that for β a Carleson measure, $M_i \leqslant C$.

Proof of Theorem 4. Given an operator as in Theorem 2, we can define a new operator $\tilde{T}$ mapping $l^s(E)$-valued functions to $l^s(F)$-valued ones (where s is fixed, $1<s<\infty$) by

$$\tilde{T}(f_1, f_2, \ldots, f_j, \ldots) = (Tf_1, Tf_2, \ldots, Tf_j, \ldots).$$

By Theorem 3, T maps $L^q_E(X, \mu)$ into $L^q_F(X^+, \alpha)$ $(1<q<\infty)$, and so it is clear that $\tilde{T}$ is bounded from $L^q_{l^q(E)}(X, \mu)$ into $L^q_{l^q(F)}(X^+, \alpha)$.

Moreover, $\tilde{T}$ is an operator like T but with associated kernel $\tilde{K}(x, y, t)$ such that

$$\tilde{K}(x, y, t)\big((\alpha_j)_{j=1}^{\infty}\big) = \big(K(x, y, t)\alpha_j\big)_{j=1}^{\infty}, \quad (\alpha_j) \subset E.$$

Then

$$\|\tilde{K}(x, y, t)\|_{\mathscr{L}(l^q(E), l^q(F))} = \|K(x, y, t)\|_{\mathscr{L}(E,F)}.$$

Now, by Theorem 3 taking $l^q(E)$ and $l^q(F)$ as the Banach spaces and $p_0 = q$ we obtain part (ii) of Theorem 4 and also part (i) in the range $1 < p \leqslant q < \infty$.

To prove part (i) in the case $1<q<p<\infty$ we shall need the following:

Lemma. *Let u be a function in $L^r(X^+, \alpha)$, $1<r\leqslant\infty$, where α is a Carleson measure. Consider the maximal function*

$$u^*(x) = \sup |B|^{-1} \int_{\tilde{B}} |u(x, t)|\, d\alpha(x, t)$$

where the supremum is taken over the balls in X which contain x. Then

$$\|u^*\|_{L^r(X,\mu)} \leqslant C\|u\|_{L^r(X^+,\alpha)}.$$

Before proving the lemma we shall finish the proof of Theorem 4.

Let $r = p/q$ and

(15) $$\int_{X^+} \big(\sum_j \|Tf_j\|_F^q\big)^{p/q} d\alpha = \big(\int_{X^+} \sum_j \|Tf_j\|_F^q\, u\, d\alpha\big)^r$$

where $u\geqslant 0$, $u\in L^{r'}(X^+, \alpha)$ and $\|u\|_{L^{r'}(X^+,\alpha)} \leqslant 1$. It is obvious that the pair $(u\,d\alpha, u^*\,d\mu)\in C_1$ and thus by Theorem 2 the right-hand side of (15) is smaller than

$$C\big(\int_X \sum_j \|f_j\|_E^q u^*\, d\mu\big)^r \leqslant C\|u^*\|^r_{L^{r'}(X,\mu)} \big(\int_X \big(\sum_j \|f_j\|_E^q\big)^r d\mu\big) \leqslant C\int_X \big(\sum_j \|f_j\|_E^q\big)^{p/q} d\mu.$$

Therefore Theorem 4 is proved.

Proof of the lemma. It is clear that

$$\|u^*\|_{L^\infty(X,\mu)} \leqslant C\|u\|_{L^\infty(X^+,\alpha)}.$$

Therefore it is enough to prove

(16) $$|\{x\in X:\ u^*(x) > \lambda\}| \leqslant \frac{C}{\lambda}\int_{X^+} |u(x, t)|\, d\alpha(x, t), \quad \lambda > 0,$$

and apply Marcinkiewicz's interpolation theorem.

But (16) can be shown by an argument parallel to the proof of Lemma 1 and Theorem 1. We omit the details.

Remark 5. If $0<a<\infty$ and

$$B^a = \{(x, t)\in X^+:\ x\in B,\ 0\leqslant t^a \leqslant \text{radius of } B\}$$

then all the theorems in this section remain valid if we make the following changes:

1. $\tilde{B}$ is replaced by B^a (even in the definition of Carleson measures and conditions $C_{p,q,\gamma}$).

2. Condition (b2) of Theorem 2 is substituted by

$$\|K(x, y, t) - K(x, y', t)\| \leqslant C\frac{\varrho(y, y')}{\big(\varrho(x, y') + t^a\big)\big|B\big(y', \varrho(x, y') + t^a\big)\big|}$$

for $\varrho(x, y') + t^a > 2\varrho(y, y')$.

Remark 6. In Theorems 2–4, condition (b2) on K can be replaced by the following:

(b2)′
$$\int_{\varrho(x,y')+t>2\varrho(y,y')} \|K(x, y, t) - K(x, y', t)\|\, d\beta(x, t) \leqslant C\omega(y) \text{ for a.e. } y, y',$$

and the proofs go along the same lines.

Remark 7. With obvious changes in Theorems 2–4, we can get similar results for operators mapping E-valued functions on X into F-valued ones on X. For example, we have:

Proposition. *Suppose that:*

(a) *S is a bounded linear operator from $L^\infty_E(X, \mu)$ into $L^\infty_F(X, \mu)$.*

(b) *There exists an $\mathscr{L}(E, F)$-valued function K on $X\times X\setminus\{(x, x):\ x\in X\}$ such that:*

(b1) $$Sf(x) = \int_X K(x, y) f(y)\, d\mu(y), \quad x\notin B,\ \operatorname{supp} f\subset B.$$

(b2) $$\|K(x, y) - K(x, y')\|_{\mathscr{L}(E,F)} \leqslant \frac{C\varrho(y, y')}{\varrho(x, y')\big|B\big(y', \varrho(x, y')\big)\big|}$$

for $\varrho(x, y') > 2\varrho(y, y')$.

Then S satisfies vector-valued inequalities analogous to those for T in Theorem 4 (with (X^+, α) replaced by (X, μ)).

4. **Applications.** The first application is the following:

THEOREM 5. *Suppose that* (X, ϱ, μ) *is a homogeneous space which satisfies one of the following conditions*:

(a) ϱ *is a metric.*

(b) X *is an abelian (additive) group and* ϱ *is translation invariant.*

Then, if α *is a Carleson measure on* X^+ *and* $\mathfrak{M}_1 = \mathfrak{M}$ *is the operator defined in Section* 1, *the following vector-valued inequalities hold*:

(i) *For* $1 < p, q < \infty$

$$\Big\|\Big(\sum_{j=1}^{\infty} (\mathfrak{M}f_j)^q\Big)^{1/q}\Big\|_{L^p(X^+,\alpha)} \leqslant C\Big\|\Big(\sum_{j=1}^{\infty} |f_j|^q\Big)^{1/q}\Big\|_{L^p(X,\mu)}.$$

(ii) *For* $1 < q < \infty$

$$\alpha\Big(\Big\{(x, t)\in X^+:\ \sum_{j=1}^{\infty} (\mathfrak{M}f_j(x, t))^q > \lambda^q\Big\}\Big) \leqslant \frac{C}{\lambda} \int_X \Big(\sum_{j=1}^{\infty} |f_j(x)|^q\Big)^{1/q} d\mu(x).$$

The proof will be given after the following construction. Suppose that the homogeneous structure is given by a metric d. Consider a function φ: $\boldsymbol{R}^2 \to \boldsymbol{R}$ such that:

(A) $$\chi_{[0,1]\times[0,1]} \leqslant \varphi \leqslant \chi_{[-1,2]\times[-1,2]},$$

(B) $$\|\nabla\varphi(u, v)\| \leqslant \frac{C}{|u|+|v|}, \qquad (u, v)\in \boldsymbol{R}^2\setminus\{0\}.$$

Condition (A) says that the operator

$$\Phi f(x, t) = \sup_{r>0} |B(x, r)|^{-1} \int_X \varphi\big(d(x, y)/r,\ t/r\big) f(y)\, d\mu(y), \qquad (x, t)\in X^+,$$

is well defined for $f\in L^\infty(X, \mu)$.

We now state the following:

THEOREM 6. Φ *satisfies all the conclusions of Theorems* 2 *and* 4.

Proof of Theorem 6. We put

$$Tf(x, t) = \Big\{|B(x, r)|^{-1} \int_X \varphi\big(d(x, y)/r,\ t/r\big) f(y)\, d\mu(y)\Big\}_{r>0}.$$

Since $\|Tf(x, t)\|_{l^\infty} = \Phi f(x, t)$ the boundedness properties of T from L^p into $L^p_{l^\infty}$ are equivalent to those of Φ from L^p into L^p. Thus in order to finish the proof les us check that T satisfies the hypotheses of Theorems 2 and 4.

Condition (A) on the function φ implies easily that T is bounded from $L^\infty(X, \omega\, d\mu)$ into $L^\infty(X^+, \beta)$ for any (ω, β) in C_1.

On the other hand, the mean value property and condition (B) imply that

(17) $$\left|\varphi\left(\frac{d(x, y)}{r}, \frac{t}{r}\right) - \varphi\left(\frac{d(x, y')}{r}, \frac{t}{r}\right)\right| \leqslant C\frac{d(y, y')}{\xi+t}$$

where $\xi = d(x, y)+\theta\big(d(x, y')-d(x, y)\big)$ with $0\leqslant\theta\leqslant 1$. Now, if $2d(y, y')\leqslant d(x, y')+t$ we have

$$\begin{aligned}\xi &= (1-\theta)\,d(x, y)+\theta d(x, y') \geqslant (1-\theta)\big(d(x, y')-d(y', y)\big)+\theta d(x, y')\\ &\geqslant d(x, y')-\tfrac{1}{2}(1-\theta)\,d(x, y')-\tfrac{1}{2}(1-\theta)\,t \geqslant \tfrac{1}{2}d(x, y')-\tfrac{1}{2}t\end{aligned}$$

and thus the right-hand side of (17) is smaller than or equal to

$$C\frac{d(y, y')}{d(x, y')+t}.$$

Suppose that we prove the following:

(18) *If* $2d(y, y')\leqslant d(x, y')+t$ *and*

$$(*) = \varphi\left(\frac{d(x, y)}{r}, \frac{t}{r}\right) - \varphi\left(\frac{d(x, y')}{r}, \frac{t}{r}\right) \neq 0$$

then $B\big(y', d(x, y')+t\big) \subset B(x, 14r)$.

Then since μ is doubling we get

$$\left\|\left\{|B(x, r)|^{-1}\left(\varphi\left(\frac{d(x, y)}{r}, \frac{t}{r}\right) - \varphi\left(\frac{d(x, y')}{r}, \frac{t}{r}\right)\right)\right\}_{r>0}\right\|_{l^\infty}$$

$$\leqslant C\frac{d(y, y')}{\big(d(x, y')+t\big)\big|B\big(y', d(x, y')+t\big)\big|} \qquad \text{for } 2d(y, y')\leqslant d(x, y')+t,$$

and Theorem 6 follows.

Proof of (18). If $z\in B\big(y', d(x, y')+t\big)$ then

$$d(z, x)\leqslant d(z, y')+d(y', x)\leqslant 2d(x, y')+t.$$

Now, $t\leqslant 2r$ because otherwise $(*)$ in zero. Analogously, $d(x, y')\leqslant 6r$ because otherwise

$$d(x, y)\geqslant d(x, y')-d(y, y')\geqslant d(x, y')-\tfrac{1}{2}d(x, y')-\tfrac{1}{2}t\geqslant 2r$$

and then $(*)$ is zero again.

In particular, $d(z, x)$ must be less than $14r$.

Proof of Theorem 5. If ϱ is a metric it is clear that $\mathfrak{M}f(x, t)\leqslant \Phi f(x, t)$ and then Theorem 6 gives the result.

If ϱ is a translation invariant pseudometric it is well known (see [2]) that there exists a metric d, an exponent $0 < a \leqslant 1$ and a positive constant M such that

$$d(x, y) \leqslant \varrho(x, y)^a \leqslant Md(x, y), \qquad x, y \in X.$$

If we write $B_\varrho(x, r) = \{y: \varrho(x, y) < r\}$ and $B_d(x, r) = \{y: d(x, y) < r\}$ then we have

$$(19) \qquad B_d(x, r^a/M) \subset B_\varrho(x, r) \subset B_d(x, r^a), \qquad (x, r) \in X^+.$$

This says that

$$(20) \qquad \mathfrak{M}^\varrho f(x, t) \leqslant C\mathfrak{M}^d f(x, t^a), \qquad (x, t) \in X^+$$

(where $\mathfrak{M}^\varrho$, $\mathfrak{M}^d$ are the maximal functions $\mathfrak{M}$ associated with ϱ and d respectively).

It is clear that if

$$\Psi f(x, t) = \sup_{r>0} |B_d(x, r)|^{-1} \int_X \varphi\big(d(x, y)/r, t^a/r\big) f(y)\, d\mu(y)$$

then

$$(21) \qquad \mathfrak{M}^d f(x, t^a) \leqslant \Psi f(x, t),$$

and the argument in the proof of Theorem 6 can be used to prove that Ψ is in the situation described in Remark 5 of Section 3; then it can be concluded that Ψ satisfies inequalities (i) and (ii) of Theorem 5 for any α such that

$$\alpha\big(B_d(x, r^a) \times [0, r]\big) \leqslant C|B_d(x, r^a)|.$$

But, by (19), this is equivalent to

$$\alpha\big(B_\varrho(x, r) \times [0, r]\big) \leqslant C|B_\varrho(x, r)|$$

(i.e. α is a Carleson measure). Now, (20) and (21) give the result.

Remark 8. In the case $X = \boldsymbol{R}^n$ and $d\alpha(x, t) = dx \otimes \delta_0(t)$, Theorem 5 is due to Fefferman and Stein [10]. Their proof is based strongly on the existence of dyadic cubes in $\boldsymbol{R}^n$. The proof by the Calderón–Zygmund theory is given in [15], where we give a systematic discussion of vector-valued singular integrals in $\boldsymbol{R}^n$.

For $X = \boldsymbol{R}^n$ with Euclidean norm and Lebesgue measure, Theorem 5 can be found in [18]. The translation invariance of the measure avoids some technicalities.

The second application we shall give needs some notation connected with tent spaces.

Now, we shall suppose that X is an abelian additive group with a translation invariant pseudometric ϱ and a doubling measure μ.

Given $x \in X$ we shall denote by $\Gamma(x)$ the set

$$\{(y, t) \in X^+: \varrho(x, y) < t\}.$$

Following [5] we define the functional mapping functions on X^+ into functions on X given by

$$A_\infty(F)(x) = \sup_{(y,t)\in\Gamma(x)} \|F(y, t)\|_E.$$

THEOREM 7. *Suppose that:*

(a) *T is an operator mapping E-valued functions on X into F-valued ones on X^+ such that*

$$\|A_\infty(Tf)\|_{L^\infty(X,\mu)} \leqslant C\|f\|_{L^\infty_E(X,\mu)}.$$

(b) *There exists an $\mathscr{L}(E, F)$-valued kernel K on $X \times X \times [0, \infty) \setminus \{(x, x, t): x \in X,\ t \geqslant 0\}$ satisfying* (b1) *and* (b2) *of Theorem* 2.

Then:

(i) *For* $1 < p, q < \infty$

$$\Big\|\Big(\sum_j A_\infty(Tf_j)^q\Big)^{1/q}\Big\|_{L^p(X,\mu)} \leqslant C\Big\|\Big(\sum_j \|f_j\|_E^q\Big)^{1/q}\Big\|_{L^p(X,\mu)}.$$

(ii) *For* $1 < q < \infty$

$$\Big|\Big\{x \in X: \sum_j |A_\infty(Tf_j)(x)|^q > \lambda^q\Big\}\Big| \leqslant \frac{C}{\lambda}\int_X \Big(\sum_j \|f_j(x)\|_E^q\Big)^{1/q} d\mu(x).$$

In other words, T satisfies vector-valued inequalities from $L^p(X, \mu)$ into the tent spaces $T^p_\infty(X^+, \mu)$ (see [5]).

Proof. By the translation invariance of ϱ we have $(y, t) \in \Gamma(x) \Leftrightarrow (x-y, t) \in \Gamma(0)$ and therefore

$$(22) \qquad A_\infty(Tf)(x) = \sup_{(u,t)\in\Gamma(0)} \|Tf(x-u, t)\|_F = \sup_{(u,t)\in X^+} \|Tf(x-u, t)\chi_{B(0,t)}(u)\|_F.$$

This gives us an opportunity of considering the following operator: let ν be the measure on X^+ given by $d\nu(x, t) = d\mu(x) \otimes dt$ (where dt is the Lebesgue measure on $[0, \infty)$) and let $G = L_F(X^+, \nu)$. We define

$$Sf(x)[(u, t)] = Tf(x-u, t)\,\chi_{B(0,t)}(u).$$

Hypothesis (b1) for T implies that S is an operator mapping E-valued functions on X into G-valued ones on X given by the $\mathscr{L}(E, G)$-valued kernel (in the sense of Remark 7)

$$\tilde{K}(x, y)(e)[(u, t)] = K(x-u, y, t)(e)\,\chi_{B(0,t)}(u), \qquad e \in E,\ (u, t) \in X^+.$$

In order to apply Remark 7 observe the following two facts:

1. Hypothesis (a) and (22) say that S maps $L^\infty_E(X, \mu)$ into $L^\infty_G(X, \mu)$.
2. $$\|\tilde{K}(x, y)-\tilde{K}(x, y')\|_{\mathscr{L}(E,G)} \leqslant \sup_{(u,t)\in\Gamma(0)} \|K(x-u, y, t)-K(x-u, y', t)\|_{\mathscr{L}(E,F)},$$

and so from hypothesis (b2) it is not hard to prove that

$$\|\tilde{K}(x, y)-\tilde{K}(x, y')\|_{\mathscr{L}(E,G)} \leqslant C\frac{\varrho(y, y')}{\varrho(x, y')\,|B(y', \varrho(x, y'))|} \quad \text{for } \varrho(x, y') > 2\varrho(y, y').$$

Now, Remark 7 gives the vector-valued inequalities for S and the result follows by observing that $A_\infty(Tf)(x) = \|Sf(x)\|_G$.

Note. Some examples of operators as in Theorem 7 are provided by the Poisson kernel in different settings. For instance, the Poisson integral $Pf(x, t)$ (where $X = \mathbf{R}^n$, $d\mu = dx$, $\varrho(x, y) = |x-y|$ and $K(x, y, t) = c_n t(|x-y|^2+t^2)^{-(n+1)/2}$) or the heat equation solution $Wf(x, t)$ (where $X = \mathbf{R}^n$, $d\mu = dx$, $\varrho(x-y) = |x-y|^2$ and $K(x, y, t) = (4\pi t)^{-n/2}\exp(-|x-y|^2/4t)$).

In particular, Theorem 7 implies (by the standard arguments) the well known results about "nontangential" (i.e. along the corresponding domains $\Gamma(x)$) convergence for these operators.

5. Applications to parabolic singular integrals. B. F. Jones Jr. introduced in [11] a class of convolution singular integrals of the form

$$\lim_{\varepsilon\to 0} \sum_{0}^{t-\varepsilon} \int_{\mathbf{R}^n} K(x-y, t-s) f(y, s)\, dy\, ds, \qquad 0 < t < \infty, \tag{23}$$

where $f(y, s)$ belongs to $L^p(\mathbf{R}^n\times(0, \infty), dx\,dt)$, $1 < p < \infty$, and $K(x, t)$ satisfies:

(i) $K(x, t) = 0$, $t < 0$.

(ii) $K(\lambda x, \lambda^m t) = \lambda^{-n-m}K(x, t)$, where λ is any positive number and m is a fixed number greater than 1.

(iii) If $\Omega(x) = K(x, 1)$ then:

$$\int_{\mathbf{R}^n} (1+|x|)|\Omega(x)|\, dx \leqslant C, \qquad \int_{\mathbf{R}^n} \Omega(x)\, dx = 0,$$

$$\int_{\mathbf{R}^n} |\Omega(x-y)-\Omega(x)|\, dx \leqslant C|y|, \qquad y\in\mathbf{R}^n,$$

$$\int_{\mathbf{R}^n} |\Omega((1+\delta)x)-\Omega(x)|\, dx \leqslant C\delta \qquad \text{for } \delta \leqslant 1,$$

$$\int_{|x|>a} |\Omega(x)|\, dx \leqslant Ca^{-n}, \qquad a > 0.$$

Jones shows the existence of the above limit in the L^p sense. In conclusion he raises two questions:

(a) Under what conditions does the limit in (23) exist pointwise almost everywhere in $\mathbf{R}^n\times(0, \infty)$?

(b) If

$$f^*(x, t) = \sup_{\varepsilon>0}\Big|\int_0^{t-\varepsilon}\int_{\mathbf{R}^n} K(x-y, t-s) f(y, s)\, dy\, ds\Big|,$$

under what conditions $\|f^*\|_p \leqslant C_p\|f\|_p$, $1 < p < \infty$?

Later Fabes and Sadosky [9] showed that (i), (ii) above, $\int_{\mathbf{R}^n} \Omega(x)\, dx = 0$ and

$$\text{(iv)} \qquad |K(x, 1)| + \Big|\frac{\partial}{\partial x_i}K(x, 1)\Big| \leqslant \frac{C}{1+|x|^{n+2}}, \qquad i = 1, \ldots, n$$

(with C an absolute constant) are the desired conditions for both (a) and (b).

Using the techniques developed in Section 3 we can prove the following.

Theorem 8. *With the above notation, if K satisfies* (i), (ii), $\int_{\mathbf{R}^n} \Omega(x)\, dx = 0$ *and* (iv) *then*:

(a) $$|\{(x, t)\colon f^*(x, t) > \lambda\}| \leqslant \frac{C}{\lambda}\int_0^\infty\int_{\mathbf{R}^n} |f(x, t)|\, dx\, dt, \qquad \lambda > 0.$$

(b) *For $f\in L^1(\mathbf{R}^n\times(0, \infty), dx\,dt)$ the limit in* (23) *exists pointwise for almost every $(x, t)\in\mathbf{R}^n\times(0, \infty)$.*

One of the main ingredients in the proof of this theorem will be the following:

Theorem 9. *Given two Banach spaces E, F and a homogeneous space (X, d, μ) with d a metric, suppose that*:

(a) *T is a linear operator from A-valued simple functions into B-valued measurable ones.*

(b) *There exists an $\mathscr{L}(E, F)$-valued function K in $X\times X\setminus\{(x, x)\colon x\in X\}$ such that*:

(b1) $$Tf(x) = \int_X K(x, y) f(y)\, d\mu(y), \qquad \operatorname{supp} f \subset B,\ x\notin B.$$

(b2) $$\int_{d(x,y')>2d(y,y')} \|K(x, y)-K(x, y')\|\, d\mu(x) = C, \qquad y, y'\in X.$$

(b3) $$\int_{a<d(x,y)<2a} \|K(x, y)\|\, d\mu(x) + \int_{a<d(x,y)<2a} \|K(x, y)\|\, d\mu(y) \leqslant C, \qquad x, y\in X,\ a > 0.$$

(c) *The sublinear operator*

$$T^* f(x) = \sup_{\varepsilon>0} \Big\| \int_{d(x,y)>\varepsilon} K(x, y) f(y)\, d\mu(y) \Big\|_F$$

is bounded from $L^p_E(X, \mu)$ *into* $L^p(X, \mu)$ *for some* p, $1 < p < \infty$.

Then:

(i) T^* *maps* $L^q_E(X, \mu)$ *into* $L^q(X, \mu)$ *for* $1 < q \leqslant p$.

(ii) T^* *maps* $L^1_E(X, \mu)$ *into* weak-$L^1(X, \mu)$.

Proof of Theorem 9. Let $\varphi\colon \boldsymbol{R} \to \boldsymbol{R}$ be any derivable function such that $\chi_{[2,\infty)} \leqslant \varphi \leqslant \chi_{[1,\infty)}$ and $|\varphi'(t)| \leqslant C/t$.

Consider the operator

$$Sf(x) = \{S_\varepsilon f(x)\}_{\varepsilon>0} = \{\int_X K(x, y)\, \varphi(d(x, y)/\varepsilon) f(y)\, d\mu(y)\}_{\varepsilon>0}.$$

It is to be thought of as an operator with kernel $J\colon X \times X \to \mathscr{L}(E, l^\infty(F))$ given by

$$J(x, y) = \{J_\varepsilon(x, y)\}_{\varepsilon>0} = \{K(x, y)\, \varphi(d(x, y)/\varepsilon)\}_{\varepsilon>0}.$$

Observe that

$$\|J_\varepsilon(x, y) - J_\varepsilon(x, y')\|$$
$$\leqslant \|K(x, y) - K(x, y')\| + \|K(x, y')\| \left|\varphi\left(\frac{d(x, y)}{\varepsilon}\right) - \varphi\left(\frac{d(x, y')}{\varepsilon}\right)\right|.$$

In particular, by applying the mean value property and the fact that $|\varphi'(t)| \leqslant C/t$ we have

$$\int_{2d(y,y')<d(x,y')} \|J(x, y) - J(x, y')\|\, d\mu(x)$$
$$= \int_{2d(y,y')<d(x,y')} \sup_{\varepsilon>0} \|J_\varepsilon(x, y) - J_\varepsilon(x, y')\|\, d\mu(x)$$
$$\leqslant \int_{2d(y,y')<d(x,y')} \|K(x, y) - K(x, y')\|\, d\mu(x)$$
$$+ C \int_{2d(y,y')<d(x,y')} \|K(x, y')\| \frac{d(y, y')}{d(x, y')}\, d\mu(x).$$

Now, we decompose the set $\{2d(y, y') < d(x, y')\}$ into the pieces $\{2^j d(y, y') < d(x, y') < 2^{j+1} d(y, y')\}$ $(j = 1, 2, \ldots)$ and we apply hypotheses (b2) and (b3). We thus get

$$\int_{2d(y,y')<d(x,y')} \|J(x, y) - J(x, y')\|\, d\mu(x) \leqslant C. \tag{24}$$

On the other hand, the operator

$$\tilde{T}f(x) = \{\int_{d(x,y)>\varepsilon} K(x, y) f(y)\, d\mu(y)\}_{\varepsilon>0}$$

is bounded from $L^p_E(X, \mu)$ into $L^p_{l^\infty(F)}(X, \mu)$ by hypothesis (c). The difference operator $S - \tilde{T}$ has kernel

$$\{K(x, y)\, \varphi(d(x, y)/\varepsilon)\, \chi_{\{\varepsilon<d(x,y)<2\varepsilon\}}\}_{\varepsilon>0}. \tag{25}$$

Hypothesis (b3) says that the kernel (25) is in $L^1_{\mathscr{L}(E,l^\infty(F))}(X, \mu)$ (considered as a function of the variable y). This means that the boundedness properties of $\tilde{T}$ are the same as those of S.

Now, (24), Remark 6 and Theorem 3 (taken in the spirit of Remark 7) say that S is bounded from $L^q_E(X, \mu)$ into $L^q_{l^\infty(F)}(X, \mu)$ for $1 < q \leqslant p$ and from $L^1_E(X, \mu)$ into weak-$L^1_{l^\infty(F)}(X, \mu)$, and this gives the result for T.

Note. This class of results was considered by Rivière [14] using different methods.

Proof of Theorem 8. Take $X = \boldsymbol{R}^n \times (0, \infty)$ and $d\mu(x, t) = dx \otimes dt$. We shall denote by $\bar{x}$, $\bar{y}$ the elements (x, t), (y, s) in X. (X, μ) becomes a homogeneous space with the metric d defined by $\|\bar{x}\| = |x| + t^{1/m}$ (i.e. $d(\bar{x}, \bar{y}) = \|\bar{x} - \bar{y}\|$).

In this context, f^* can be considered as an operator with $\mathscr{L}(C, l^\infty) \equiv l^\infty$-valued kernel given by

$$K(\bar{x}, \bar{y}) = \{K(\bar{x} - \bar{y})\, \chi_{\{|t|>\varepsilon\}}(t - s)\}_{\varepsilon>0}.$$

Fabes and Rivière showed in [8] that the L^1-norm of the function

$$K(\bar{x})\, \{\chi_{\{|t|>\varepsilon\}}(t) - \chi_{\{\|\bar{x}\|>\varepsilon\}}(\bar{x})\}$$

is bounded independently of ε. From this fact it is easily deducible that the boundedness properties of f^* are equivalent to those of the operator T^* as in Theorem 9 given by the kernel

$$J(\bar{x}, \bar{y}) = \{K(\bar{x} - \bar{y})\, \chi_{\{\|\bar{x}\|>\varepsilon\}}(\bar{x} - \bar{y})\}_{\varepsilon>0}.$$

So that we shall verify conditions (b2), (b3) and (c) in Theorem 9 for the operator T^*.

Condition (b2) is essentially proved in [8]. In order to prove (b3), by the translation invariance of μ it is enough to verify that

$$\int_{a<\|\bar{x}\|<2a} \|J(\bar{x}, 0)\|\, dx\, dt \leqslant \int_{a<\|\bar{x}\|<2a} |K(x, t)|\, dx\, dt \leqslant C.$$

But

$$\int_{a<\|\bar{x}\|<2a} |K(x, t)|\, dx\, dt = \int_{a<\|\bar{x}\|<2a} t^{-n/m-1} |\Omega(x/t^{1/m})|\, dx\, dt$$
$$\leqslant \int_{\mathbf{R}^n} |\Omega(u)| \Big(\int_{a(1+|u|)^{-1}<t^{1/m}<2a(1+|u|)^{-1}} dt/t \Big) du$$

and then it suffices to apply hypothesis (iv).

Finally, the result of Fabes and Sadosky mentioned above gives hypothesis (c) of Theorem 9.

Therefore, we have proved part (a). The almost everywhere convergence is a standard consequence of the boundedness properties of f^* together with the existence of a nice dense set where this pointwise convergence holds.

Remark 9. The parabolic singular integral operators appeared in connexion with the heat equation and more generally in connexion with inhomogeneous parabolic differential equations with constant coefficients (see [7], [11]).

Spatial derivatives of solutions of these equations are typical examples of parabolic singular integrals.

In dimension 1, the standard example is the operator T in $\mathbf{R}\times(0, \infty)$ given as convolution with

$$K(x, t) = \frac{1}{t^{3/2}} \pi^{-1/2} \left(\frac{x^2}{8t} - \frac{1}{4}\right) e^{-x^2/(4t)}.$$

It is a straightforward calculation to see that if $\|(x, t)\| = |x| + t^{1/2}$ and $2\|(y, s)\| < \|(x, t)\|$ then

$$|K(x-y, t-s) - K(x, t)| \leqslant C \frac{\|(y, s)\|}{\|(x-y, t-s)\|^4}.$$

On the other hand, the operator T is bounded on $L^2(\mathbf{R}\times(0, \infty), dx\,dt)$ (see [7]).

The general theory of singular integrals can be applied in this homogeneous space and in particular (see [12]) Cotlar's inequality in this context says that

$$f^*(x, t) \leqslant C_1 M(Tf)(x, t) + C_2 Mf(x, t)$$

where M is the Hardy–Littlewood maximal operator in $\mathbf{R}\times(0, \infty)$.

Then the result of Fabes and Sadosky about the boundedness of f^* on L^p follows from Cotlar's inequality. In other words, hypothesis (c) in Theorem 9 can be omitted in this case.

References

[1] E. Amar et A. Bonami, *Mesures de Carleson d'ordre α et solutions au bord de l'équation $\bar{\partial}$*, Bull. Soc. Math. France 107 (1979), 23–48.

[2] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer, 1976.

[3] A. P. Calderón, *Inequalities for the maximal function relative to a metric*, Studia Math. 57 (1976), 297–306.

[4] R. Coifman and M. de Guzmán, *Singular integrals and multipliers on homogeneous spaces*, Rev. Un. Mat. Argentina 25 (1970), 137–143.

[5] R. Coifman, Y. Meyer et E. M. Stein, *Un nouvel espace fonctionnel adapté à l'étude des opérateurs définis par des intégrales singulières*, in: Lecture Notes in Math. 992, Springer, 1983, 1–15.

[6] R. Coifman et G. Weiss, *Analyse Harmonique Non-Commutative sur Certains Espaces Homogènes*, Lecture Notes in Math. 242, Springer, 1971.

[7] E. Fabes, *Singular integrals and partial differential equations of parabolic type*, Studia Math. 28 (1966), 81–131.

[8] E. Fabes and N. Rivière, *Singular integrals with mixed homogeneity*, ibid. 27 (1966), 19–38.

[9] E. Fabes and C. Sadosky, *Pointwise convergence for parabolic singular integrals*, ibid. 26 (1965), 15–22.

[10] C. Fefferman and E. M. Stein, *Some maximal inequalities*, Amer. J. Math. 93 (1971), 107–115.

[11] B. F. Jones, *A class of singular integrals*, ibid. 86 (1964), 441–462.

[12] J. L. Journé, *Calderón–Zygmund Operators, Pseudodifferential Operators and the Cauchy Integral of Calderón*, Lecture Notes in Math. 994, Springer, 1983.

[13] A. Koranyi and S. Vagi, *Singular integrals on homogeneous spaces and some problems of classical analysis*, Ann. Scuola Norm. Sup. Pisa 25 (1971), 575–647.

[14] N. Rivière, *Singular integrals and multiplier operators*, Ark. Math. 9 (1971), 243–278.

[15] J. L. Rubio de Francia, F. J. Ruiz and J. L. Torrea, *Calderón–Zygmund theory for operator-valued kernels*, Adv. in Math. 62 (1986), 7–48.

[16] F. J. Ruiz, *A unified approach to Carleson measures and A_p weights*, Pacific J. Math. 117 (1985), 397–404.

[17] F. J. Ruiz and J. L. Torrea, *A unified approach to Carleson measures and A_p weights II*, ibid. 120 (1985), 189–197.

[18] —, —, *Weighted and vector-valued inequalities for potential operators*, Trans. Amer. Math. Soc. 295 (1986), 213–232.

FACULTAD DE CIENCIAS
DEPARTAMENTO TEORÍA DE FUNCIONES
UNIVERSIDAD DE ZARAGOZA
50009 Zaragoza, Spain

and

DIVISIÓN DE MATEMÁTICAS
UNIVERSIDAD AUTÓNOMA DE MADRID
28049 Madrid, Spain

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