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# Comparison of joint spectra for certain classes of commuting operators

by

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**Abstract.** It is shown that if $T = (T_1, \ldots, T_m)$ is a commuting $m$-tuple of continuous linear operators in a complex Banach space such that each operator $T_j$, $1 \leqslant j \leqslant m$, has real spectrum, then the Taylor spectrum, the Harte spectrum, the commutant spectrum and the bicommutant spectrum of $T$ all coincide. This result remains valid for commuting $m$-tuples $T$ for which each operator $T_j$, $1 \leqslant j \leqslant m$, has a decomposition $U_j + iV_j$ where $U_j$ and $V_j$ have real spectrum, the $2m$-tuple $(U_1, \ldots, U_m, V_1, \ldots, V_m)$ is commuting and the commutants $\{T_j;\ 1 \leqslant j \leqslant m\}'$ and $\{U_j, V_j;\ 1 \leqslant j \leqslant m\}'$ are equal. It is shown that there are classes of operators, such as multiplication operators, spectral operators and regular generalized scalar operators, with the property that any commuting $m$-tuple $T$ of operators from such a class has a decomposition of the type above. A crucial role is played by a concept of spectral set $\gamma(T)$ introduced in [12, 13].

**Introduction.** In recent years there have been many definitions of joint spectrum of a commuting family of operators on a Banach space. In this note we compare some of the more important definitions of joint spectrum and show that they coincide for large classes of operators.

If $X$ is a complex Banach space, then $L(X)$ denotes the Banach algebra of all continuous linear operators of $X$ into itself, equipped with the uniform operator topology. The identity operator on $X$ is denoted by $I$. Throughout this paper $m$ denotes a positive integer and $T = (T_1, \ldots, T_m)$ a commuting $m$-tuple of elements $T_j \in L(X)$, $1 \leqslant j \leqslant m$, often abbreviated simply as $T = (T_j)$. The real number field and the complex number field are denoted by $\boldsymbol{R}$ and $\boldsymbol{C}$, respectively.

Let $T$ be an $m$-tuple in the centre of a closed unital Banach subalgebra $\mathscr{A}$ of $L(X)$. Then $\sigma(T, \mathscr{A})$ is the set of all $\lambda \in \boldsymbol{C}^m$ for which the equation $\sum_{j=1}^{m} A_j(T_j - \lambda_j) = I$ has no solution $A = (A_j)$ in $\mathscr{A}^m$. In the case when $\mathscr{A}$ is commutative this is equivalent to

$$\sigma(T, \mathscr{A}) = \{(\psi(T_1), \ldots, \psi(T_m));\ \psi \in \mathscr{M}(\mathscr{A})\}, \tag{1}$$

where $\mathscr{M}(\mathscr{A})$ is the maximal ideal space of $\mathscr{A}$. The *commutant spectrum*, $\sigma(T, \{T_j\}')$, and the *bicommutant spectrum*, $\sigma(T, \{T_j\}'')$, are denoted simply by

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$\sigma'(T)$ and $\sigma''(T)$, respectively, where $\{T_j\}'$ and $\{T_j\}''$ are the commutant and bicommutant of $\{T_j\}$ in $L(X)$, respectively.

The *Harte spectrum*, $\sigma_H(T)$, is the set of all $\lambda \in \boldsymbol{C}^m$ for which at least one of the equations $\sum_{j=1}^m (T_j - \lambda_j) A_j = I$ and $\sum_{j=1}^m A_j(T_j - \lambda_j) = I$ has no solution $A = (A_j)$ in $L(X)^m$; see [10].

The *Taylor spectrum*, $\mathrm{Sp}(T)$, is defined in terms of the Koszul complex of $T$. Its properties are systematically exposed in the penetrating studies of J. L. Taylor [16], [17].

Finally, the *polynomial spectrum*, $\sigma_P(T)$, is the set of all $\lambda \in \boldsymbol{C}^m$ such that $q(\lambda)$ belongs to the usual spectrum, $\sigma(q(T))$, of the (single) operator $q(T)$, for each polynomial $q$: $\boldsymbol{C}^m \to \boldsymbol{C}$; see [2], for example.

We remark that these notions of joint spectrum coincide in the case of a single operator, that is, when $m = 1$.

The following notion, introduced in [12] and [13], will play a crucial role in the sequel. Namely, for a commuting $m$-tuple $T = (T_j)$ in $L(X)^m$ define

$$\gamma(T) = \{\mu \in \boldsymbol{R}^m;\ \sum_{j=1}^m (T_j - \mu_j)^2 \text{ is not invertible in } L(X)\}. \tag{2}$$

If $m = 1$, then it is easily shown that $\gamma(T_1) = \sigma(T_1) \cap \boldsymbol{R}$, and hence it can happen that $\gamma(T) = \emptyset$.

The main purpose of this note is to establish the following two results and to deduce some consequences from them (cf. Sections 3 and 4).

THEOREM 1. *Let $T = (T_j)$ be a commuting m-tuple in $L(X)^m$. Then the following statements are equivalent*:

(i) $\sigma(T_j) \subseteq \boldsymbol{R}$ *for each* $j = 1, \ldots, m$.

(ii) *Any one of the five joint spectra* $\mathrm{Sp}(T)$, $\sigma'(T)$, $\sigma''(T)$, $\sigma_H(T)$, $\sigma_P(T)$ *is contained in* $\boldsymbol{R}^m$.

(iii) $\mathrm{Sp}(T) = \sigma_H(T) = \sigma'(T) = \sigma''(T) = \sigma_P(T) = \gamma(T)$.

Remark 1. For $m$-tuples satisfying the hypotheses of Theorem 1 it is clear that $\gamma(T) \neq \emptyset$.

An $m$-tuple $T = (T_j)$ of elements from $L(X)$ is called *strongly commuting* [13] if, for each $1 \leqslant j \leqslant m$, there exist operators $U_j$ and $V_j$, each with real spectrum, such that $T_j = U_j + iV_j$ and $\Pi(T) = (U_1, \ldots, U_m, V_1, \ldots, V_m)$ is a commuting $2m$-tuple in $L(X)^{2m}$. Such a commuting $2m$-tuple $\Pi(T)$, denoted briefly by $\Pi(T) = (U, V)$, is called a *partition* of $T$. If there exists a partition $\Pi(T) = (U, V)$ of $T$ such that $\{T_j\}' = \{U_j, V_j\}'$, then $T$ is said to be *extra strongly commuting*. We remark that $\{T_j\}'' = \{U_j, V_j\}''$ is an equivalent condition.

THEOREM 2. *Let $T$ be a strongly commuting m-tuple in $L(X)^m$. Then*

$$\mathrm{Sp}(T) = \sigma_H(T) = \sigma'(T). \tag{3}$$



*If $T$ is extra strongly commuting, then $\sigma''(T)$ is also equal to each of the spectra in* (3).

In practice, the spectra $\mathrm{Sp}(T)$, $\sigma_H(T)$, $\sigma'(T)$, $\sigma''(T)$ and $\sigma_P(T)$ are often difficult to compute. However, one of the appealing aspects of the spectral set $\gamma(T)$ is that it readily lends itself to explicit computation. This observation, together with Theorem 1, Theorem 2 and Lemma 2 in Section 1, provides an effective means for computing other joint spectra of $m$-tuples $T = (T_j)$ which are strongly or extra strongly commuting. In addition, the notion $\gamma(\cdot)$ often leads to more direct and natural proofs of statements otherwise proved by less transparent methods; see Example 1(A) and Proposition 7, for example.

The organization of this paper is as follows. In Section 1 we introduce some notation and establish various preliminary results needed in the sequel. Section 2 is devoted to proving the two theorems stated above and Section 3 is devoted to some applications and examples. In the final section it is shown that the spectra $\sigma_H(T)$, $\mathrm{Sp}(T)$, $\sigma'(T)$, $\sigma''(T)$ and $\sigma_P(T)$ all coincide with the joint point spectrum when the Banach space $X$ is finite-dimensional.

**1. Preliminaries.** Throughout this section $X$ is a (complex) Banach space and $T = (T_j)$ is a commuting $m$-tuple in $L(X)^m$. Then we have the inclusions

$$\mathrm{Sp}(T) \cup \sigma_H(T) \subseteq \sigma'(T) \subseteq \sigma''(T) \subseteq \sigma_P(T) \subseteq \prod_{j=1}^m \sigma(T_j). \tag{4}$$

The fact that $\mathrm{Sp}(T) \subseteq \sigma'(T) \subseteq \sigma''(T)$ is discussed in [16]. Since $\sigma''(T)$ is given by the right-hand side of (1) with $\mathscr{A} = \{T_j\}''$, it follows from Proposition 1.1.1 of [2] that $\sigma''(T) \subseteq \sigma_P(T)$. The inclusion $\sigma_P(T) \subseteq \prod_{j=1}^m \sigma(T_j)$ is clear; see [2; p. 275]. Finally, if $\lambda \notin \sigma'(T)$, then $\sum_{j=1}^m A_j(T_j - \lambda_j) = I$ for some $m$-tuple $A = (A_j)$ of elements from $\{T_j\}' \subseteq L(X)$ and hence $\lambda \notin \sigma_H(T)$. This shows that $\sigma_H(T) \subseteq \sigma'(T)$.

If $X$ is a Hilbert space, then it follows from Proposition 2.10 of [18] that $\sigma_H(T) \subseteq \mathrm{Sp}(T)$. We remark that this inclusion can be strict; see the example discussed in Remark 2 of [6]. Similarly, it is known that the inclusions $\mathrm{Sp}(T) \subseteq \sigma'(T) \subseteq \sigma''(T)$, valid in any Banach space, can also be strict [18; § 4].

LEMMA 1. *Let $T = (T_j)$ be a commuting m-tuple of elements from $L(X)$. Then $\sigma''(T) \cap \boldsymbol{R}^m$ is a subset of $\gamma(T)$.*

Proof. If $\lambda \in \boldsymbol{R}^m \setminus \gamma(T)$, then $W = \sum_{j=1}^m (T_j - \lambda_j)^2$ is invertible in $L(X)$. It is easily verified that $W^{-1}$ belongs to $\{T_j\}''$, and hence so does each of the operators $A_j = W^{-1}(T_j - \lambda_j)$, $1 \leqslant j \leqslant m$. Since $\sum_{j=1}^m A_j(T_j - \lambda_j) = I$ it follows that $\lambda \in \boldsymbol{R}^m \setminus \sigma''(T)$. ∎

Remark 2. It follows from (4) and Lemma 1 that $\mathrm{Sp}(T) \cap \boldsymbol{R}^m$, $\sigma_H(T) \cap \boldsymbol{R}^m$ and $\sigma'(T) \cap \boldsymbol{R}^m$ are also subsets of $\gamma(T)$.

If $\sigma^*(T)$ denotes either the Taylor, Harte or polynomial spectrum of $T$, then $\sigma^*(T)$ is a nonempty compact subset of $\prod_{j=1}^m \sigma(T_j)$ with the property that $q(\sigma^*(T)) = \sigma(q(T))$ for all polynomials $q\colon C^m \to C$; see [17], [10] and [2; Proposition 1.1.2], respectively. The Taylor spectrum actually satisfies this property for functions $q$ analytic in a neighbourhood of $\mathrm{Sp}(T)$. The following result is essentially Proposition 10.1 of [13].

PROPOSITION 1. *Let $T$ be a commuting $m$-tuple in $L(X)^m$ and let $\sigma^*(T)$ be a subset of $C^m$ with the property that $q(\sigma^*(T)) = \sigma(q(T))$ for all polynomials $q\colon C^m \to C$. Then $\sigma^*(T) \cap R^m \subseteq \gamma(T)$ with equality if $\sigma^*(T) \subseteq R^m$.*

Proof. For $\lambda \in R^m$ define a polynomial $q_\lambda\colon C^m \to C$ by the formula

$$q_\lambda(z) = \sum_{j=1}^m (z_j - \lambda_j)^2.$$

If $\nu \in \sigma^*(T) \cap R^m$, then $0 \in q_\nu(\sigma^*(T)) = \sigma(q_\nu(T))$ and hence $\nu \in \gamma(T)$; see (2). Conversely, if $\nu \in \gamma(T)$, then $0 \in q_\nu(\sigma^*(T))$ so that $q_\nu(z) = 0$ for some $z \in \sigma^*(T)$. However, if $\sigma^*(T) \subseteq R^m$, then this is possible only if $z = \nu$ and hence $\nu \in \sigma^*(T)$. ∎

The following result follows immediately from Lemma 1, Remark 2, the discussion prior to Proposition 1 and Proposition 1 itself.

COROLLARY 1.1. *Let $T$ be a commuting $m$-tuple in $L(X)^m$. If $\sigma^*(T)$ denotes either $\mathrm{Sp}(T)$, $\sigma_H(T)$, $\sigma'(T)$, $\sigma''(T)$ or $\sigma_P(T)$, then*

$$\sigma^*(T) \cap R^m \subseteq \gamma(T). \tag{5}$$

*Equality holds in (5) whenever $\sigma^*(T) \subseteq R^m$.*

Let $T$ be a commuting $m$-tuple in $L(X)^m$ and let $n$ be a positive integer. If $q_k\colon C^m \to C$ are polynomials, $1 \leqslant k \leqslant n$, and $q = (q_1, \ldots, q_n)$ denotes the obvious map from $C^m$ into $C^n$, then it is known that the Taylor spectrum and the Harte spectrum satisfy $q(\mathrm{Sp}(T)) = \mathrm{Sp}(q(T))$ and $q(\sigma_H(T)) = \sigma_H(q(T))$; see [17; Theorem 4.8] and [10, 15], respectively. This "multi-polynomial" version of the spectral mapping theorem is not satisfied by the commutant and bicommutant spectra [15].

LEMMA 2. *Let $T$ be a strongly commuting $m$-tuple in $L(X)^m$ with a partition $\Pi(T)$ and let $p\colon C^{2m} \to C^m$ be the polynomial given by*

$$p(z) = p(z_1, \ldots, z_{2m}) = (z_1 + iz_{m+1}, \ldots, z_m + iz_{2m}), \quad z \in C^{2m}. \tag{6}$$

*Then*

$$\mathrm{Sp}(T) = \sigma_H(T) = p(\gamma(\Pi(T))).$$

Proof. Let $\sigma^*(\cdot)$ denote either $\mathrm{Sp}(\cdot)$ or $\sigma_H(\cdot)$. Then it follows from Corollary 1.1 that $\sigma^*(\Pi(T)) = \gamma(\Pi(T))$. By the discussion following Corollary 1.1 we have

$$\sigma^*(T) = \sigma^*(p(\Pi(T))) = p(\sigma^*(\Pi(T))) = p(\gamma(\Pi(T))). \quad ∎$$



Two $m$-tuples $T$ and $S$ in $L(X)^m$ are said to be *mutually commuting* if $T$ and $S$ are commuting $m$-tuples such that $T_k S_j = S_j T_k$ for each $j$ and $k$. In this case the $m$-tuple $T - S = (T_j - S_j)$ is also commuting.

PROPOSITION 2. *Let $T$ and $S$ be mutually commuting $m$-tuples in $L(X)^m$. If each operator $Q_j = T_j - S_j$, $1 \leqslant j \leqslant m$, is quasinilpotent, then $\gamma(T) = \gamma(S)$ and $\sigma^*(T) = \sigma^*(S)$ where $\sigma^*(\cdot)$ denotes either the Taylor, Harte or polynomial spectrum.*

Proof. Since $T$ and $S$ are mutually commuting it follows that they are quasinilpotent equivalent in the sense of Definition 4.1 of [9]; see also [9; Remark 4.3]. Accordingly, the identities $\mathrm{Sp}(T) = \mathrm{Sp}(S)$ and $\sigma_H(T) = \sigma_H(S)$ follow from Theorem 4.1 and Corollary 4.1 of [9], respectively.

Let $q\colon C^m \to C$ be a polynomial. Then it is clear that $q(T) = q(S) + Q$ where $Q$ (being a finite linear combination of products of operators from the commuting family $\{S_j, Q_j;\ 1 \leqslant j \leqslant m\}$ with each product containing at least one element $Q_j$, for some $j$) commutes with $q(S)$ and is quasinilpotent [5; Ch. 4, Lemma 3.8]. It follows that $\sigma(q(T)) = \sigma(q(S))$ [8; XV, Lemma 4.4] and hence, for $z \in C^m$, we see that $q(z) \in \sigma(q(T))$ if and only if $q(z) \in \sigma(q(S))$. The definition of polynomial spectrum then implies that $\sigma_P(T) = \sigma_P(S)$.

Finally, if $\lambda \in R^m$, then it is easily seen that

$$Q_\lambda + \sum_{j=1}^m (S_j - \lambda_j)^2 = \sum_{j=1}^m (T_j - \lambda_j)^2,$$

where $Q_\lambda$ is a quasinilpotent operator commuting with the operator $\sum_{j=1}^m (S_j - \lambda_j)^2$ and hence

$$\sigma\Big(\sum_{j=1}^m (T_j - \lambda_j)^2\Big) = \sigma\Big(\sum_{j=1}^m (S_j - \lambda_j)^2\Big).$$

Since $\lambda \in \gamma(T)$ if and only if $0 \in \sigma\big(\sum_{j=1}^m (T_j - \lambda_j)^2\big)$ it follows that $\gamma(T) = \gamma(S)$. ∎

**2. Proofs of the main theorems.** We begin with the proof of Theorem 1. The implication (i) ⇒ (ii) is clear from (4).

Assume that (iii) holds. Then $\mathrm{Sp}(T) = \gamma(T) \subseteq R^m$. If $\pi_j\colon C^m \to C$ is the $j$th coordinate projection defined by $\pi_j(z) = z_j$, $z \in C^m$, for each $1 \leqslant j \leqslant m$, then it follows from Taylor's spectral mapping theorem that

$$\sigma(T_j) = \sigma(\pi_j(T)) = \mathrm{Sp}(\pi_j(T)) = \pi_j(\mathrm{Sp}(T)) \subseteq R,$$

for each $j = 1, \ldots, m$, which is (i).

It remains to establish (ii) ⇒ (iii). Suppose that $\mathrm{Sp}(T) \subseteq R^m$. Then we have just seen that $\sigma(T_j) \subseteq R$, $1 \leqslant j \leqslant m$, and hence (4) implies that

$$\sigma_H(T) \cup \mathrm{Sp}(T) \subseteq \sigma_P(T) \subseteq \prod_{j=1}^m \sigma(T_j) \subseteq R^m.$$

It follows from Corollary 1.1 that

(7) $$\mathrm{Sp}(T) = \sigma_{\mathrm{H}}(T) = \sigma'(T) = \sigma''(T) = \sigma_{\mathrm{P}}(T) = \gamma(T),$$

which is (iii).

If $\sigma_{\mathrm{H}}(T) \subseteq \boldsymbol{R}^m$, then the spectral mapping theorem for the Harte spectrum applied to the polynomials $\pi_j$, $1 \leqslant j \leqslant m$, defined above shows that $\sigma(T_j) \subseteq \boldsymbol{R}$, $j = 1, \ldots, m$. Then the same argument as above establishes (7) again.

Finally, suppose that one of the remaining spectra, namely $\sigma'(T)$, $\sigma''(T)$ or $\sigma_{\mathrm{P}}(T)$ is a subset of $\boldsymbol{R}^m$. Since $\mathrm{Sp}(T)$ is a subset of each of these it follows that $\mathrm{Sp}(T) \subseteq \boldsymbol{R}^m$ from which we have already seen that (iii) follows. ∎

We now prove Theorem 2. Suppose $T$ has the partition $\Pi(T) = (U, V)$. It suffices to prove the set inclusions

(8) $$\mathrm{Sp}(T) \cup \sigma_{\mathrm{H}}(T) \subseteq \sigma'(T) \subseteq \sigma(T, \{U_j, V_j\}'') \subseteq p(\gamma(\Pi(T))),$$

where $p\colon \boldsymbol{C}^{2m} \to \boldsymbol{C}^m$ is the polynomial (6). Indeed, on applying Lemma 2, we note that these sets are all equal and Theorem 2 follows immediately, after noting that $\sigma(T, \{U_j, V_j\}'') = \sigma''(T)$ in the case when $T$ is extra strongly commuting with respect to $\Pi(T)$.

The first inclusion in (8) was noted in (4). The next follows from

$$\{U_j, V_j\}'' \subseteq \{T_j\}'.$$

To prove the final inclusion in (8), choose $\lambda \in \boldsymbol{C}^m \setminus p(\gamma(\Pi(T))) = p(\boldsymbol{R}^{2m} \setminus \gamma(\Pi(T)))$. Then $\lambda = p(u, v) = u + iv$ where $(u, v) \in \boldsymbol{R}^{2m} \setminus \gamma(\Pi(T))$, and hence

$$W = \sum_{j=1}^{m} ((U_j - u_j)^2 + (V_j - v_j)^2)$$

is invertible in $L(X)$ with $W^{-1}$ clearly an element of $\{U_j, V_j\}''$. Now define

(9) $$A_j = W^{-1}((U_j - u_j) - i(V_j - v_j)), \quad j = 1, \ldots, m.$$

Then each $A_j \in \{U_j, V_j\}''$, $1 \leqslant j \leqslant m$. It follows from (9) that $\sum_{j=1}^m A_j(T_j - \lambda_j) = I$. Accordingly, $\lambda \in \boldsymbol{C}^m \setminus \sigma(T, \{U_j, V_j\}'')$ as required. ∎

We include here a result of K. Rudol which is related to Theorems 1 and 2.

A compact subset $K$ of $\boldsymbol{C}^m$ is called *rationally convex* if

$$K = \{z \in \boldsymbol{C}^m;\ q(z) \in q(K) \text{ for every polynomial } q\colon \boldsymbol{C}^m \to \boldsymbol{C}\}.$$

THEOREM 3. *Let $T$ be a commuting $m$-tuple in $L(X)^m$. Let $\tilde{\sigma}(\cdot)$ be an abstract spectrum for $L(X)$ in the sense of Żelazko [19]. Then $\sigma_{\mathrm{P}}(T) = \tilde{\sigma}(T)$ if and only if $\tilde{\sigma}(T)$ is rationally convex.*



Proof. Since $\tilde{\sigma}(\cdot)$ is an abstract spectrum the identities

$$q(\tilde{\sigma}(T)) = \tilde{\sigma}(q(T)) = \sigma(q(T))$$

are valid for every polynomial $q\colon \boldsymbol{C}^m \to \boldsymbol{C}$ [19]. It follows from the definition of $\sigma_{\mathrm{P}}(T)$ that

$$\sigma_{\mathrm{P}}(T) = \{z \in \boldsymbol{C}^m;\ q(z) \in q(\tilde{\sigma}(T)) \text{ for every polynomial } q\colon \boldsymbol{C}^m \to \boldsymbol{C}\}. \ ∎$$

Remark 3. The Taylor spectrum and Harte spectrum are abstract spectra [19]. Furthermore, it is not difficult to show that compact sets $K \subseteq \boldsymbol{C}^m$ which are contained in $\boldsymbol{R}^m$ are rationally convex (the polynomials $q_\lambda$ in the proof of Proposition 1 play a role here). Accordingly, (4) and Theorem 3 imply that

$$\mathrm{Sp}(T) = \sigma_{\mathrm{H}}(T) = \sigma'(T) = \sigma''(T) = \sigma_{\mathrm{P}}(T)$$

whenever $\sigma(T_j) \subseteq \boldsymbol{R}$ for each $j = 1, \ldots, m$ (cf. Theorem 1).

We end this section with a sufficient condition which ensures that a strongly commuting $m$-tuple is extra strongly commuting. First we need the following result from which Proposition 3 follows easily.

LEMMA 3. *Let $T \in L(X)$ and let $U$, $V$ be commuting operators such that $T = U + iV$ and the groups $t \to \exp(itU)$, $t \in \boldsymbol{R}$, and $s \to \exp(isV)$, $s \in \boldsymbol{R}$, are uniformly bounded in $L(X)$. Let $T^{\#} = U - iV$. If $A \in L(X)$ commutes with $T$, then $A$ also commutes with $T^{\#}$. In particular, $\{U, V\}' = \{T\}'$.*

Proof. The elegant proof, due to M. Rosenblum, of Fuglede's theorem for normal operators in a Hilbert space can easily be adapted to the present setting; see [7; Theorem 7.21], for example. ∎

Remark 4. The hypotheses of the lemma imply that $U$ and $V$ have real spectra and are generalized scalar operators [5; Ch. 5, Theorem 4.5]. If $X$ is a Hilbert space, then $U$ and $V$ are actually selfadjoint [7; Theorem 7.23]. However, in Banach spaces they need not be scalar-type spectral operators [7; p. 195] although, as already noted, they are generalized scalar operators.

PROPOSITION 3. *Let $T$ be a commuting $m$-tuple in $L(X)^m$. If there exists a partition $\Pi(T) = (U, V)$ of $T$ such that the groups $\lambda \to \exp(i\sum_{j=1}^m \lambda_j U_j)$, $\lambda \in \boldsymbol{R}^m$, and $\mu \to \exp(i\sum_{j=1}^m \mu_j V_j)$, $\mu \in \boldsymbol{R}^m$, are uniformly bounded in $L(X)$, then $\{U_j, V_j\}' = \{T_j\}'$ and hence $T$ is extra strongly commuting.*

**3. Examples.** In this section we exhibit some classes of operators with the property that any commuting tuple of operators from such a class is necessarily strongly or extra strongly commuting.

EXAMPLE 1. Let $T \in L(X)$ be a *generalized scalar operator* in the sense of Colojoară and Foiaş [5], that is, there exists a spectral distribution $\Phi\colon C^\infty(\boldsymbol{C}) = C^\infty(\boldsymbol{R}^2) \to L(X)$ such that $\Phi(\lambda) = T$, where $\lambda$ denotes the identity function on $\boldsymbol{C}$. Then $T$ is called *regular* if there exists a spectral distribution for $T$ which takes its values in the bicommutant $\{T\}''$. Such a spectral distribution for $T$ is said to be regular.

PROPOSITION 4. *Let $\boldsymbol{T}=(T_j)$ be a commuting m-tuple in $L(X)^m$ consisting of regular generalized scalar operators. Then $\boldsymbol{T}$ is extra strongly commuting and hence*

$$\mathrm{Sp}(\boldsymbol{T})=\sigma_{\mathrm{H}}(\boldsymbol{T})=\sigma'(\boldsymbol{T})=\sigma''(\boldsymbol{T}). \tag{10}$$

Proof. Let $\Phi_j$ be a regular spectral distribution for $T_j$, $1\leqslant j\leqslant m$. Then $T_j T_k = T_k T_j$ implies, by regularity, that $T_j\Phi_k(f)=\Phi_k(f)T_j$ for each $f\in C^\infty(\boldsymbol{C})$, which in turn implies (for each $f\in C^\infty(\boldsymbol{C})$) that $\Phi_j(g)\Phi_k(f)=\Phi_k(f)\Phi_j(g)$ whenever $g\in C^\infty(\boldsymbol{C})$. Accordingly, the spectral distributions $\Phi_j$, $1\leqslant j\leqslant m$, all commute with each other.

If $U_j=\Phi_j(\mathrm{Re}(\lambda))$ and $V_j=\Phi_j(\mathrm{Im}(\lambda))$, $1\leqslant j\leqslant m$, then it is clear that $(\boldsymbol{U},\boldsymbol{V})$ is a commuting $2m$-tuple in $L(X)^{2m}$ such that $T_j=U_j+iV_j$ for each $j=1,\ldots,m$. Since each element of $\{U_j, V_j\}$ has real spectrum (cf. proof of Lemma 6.1 in Chapter 4 of [5]) it follows that $\Pi(\boldsymbol{T})=(\boldsymbol{U},\boldsymbol{V})$ is a partition for $\boldsymbol{T}$. It is a simple consequence of the regularity of each $\Phi_j$, $1\leqslant j\leqslant m$, that $\{T_j\}'=\{U_j, V_j\}'$. Accordingly, $\boldsymbol{T}$ is extra strongly commuting and hence (10) follows from Theorem 2. ■

Remark 4. Still using the notation of Proposition 4 and its proof it follows that the tensor product

$$\Phi=\Phi_1\otimes\ldots\otimes\Phi_m\colon\ C^\infty(\boldsymbol{C}^m)\to L(X) \tag{11}$$

of the $\Phi_j$, $1\leqslant j\leqslant m$, exists, is again a spectral distribution [5; Ch. 4, Proposition 3.1] and satisfies $\Phi(\pi_j)=T_j$, $1\leqslant j\leqslant m$, where $\pi_j\colon \boldsymbol{C}^m\to\boldsymbol{C}$ is the $j$th coordinate projection. Hence $\boldsymbol{T}$ is a $C^\infty(\boldsymbol{C}^m)$-*scalar system* in the sense of [1] with $\Phi\colon C^\infty(\boldsymbol{C}^m)\to L(X)$ being a $C^\infty(\boldsymbol{C}^m)$-functional calculus for $\boldsymbol{T}$. Furthermore, $\Phi$ assumes its values in $\{T_j\}''$ by definition of the tensor product and the regularity of each $\Phi_j$, $1\leqslant j\leqslant m$. Accordingly, an alternative proof of (10), for $\boldsymbol{T}$ as specified in Proposition 4, follows from Theorem 6 of [1]. Actually, using this result (10) can be slightly strengthened to

$$\mathrm{Sp}(\boldsymbol{T})=\sigma_{\mathrm{H}}(\boldsymbol{T})=\sigma'(\boldsymbol{T})=\sigma''(\boldsymbol{T})=\mathrm{Supp}(\Phi), \tag{12}$$

where $\mathrm{Supp}(\Phi)$ denotes the smallest closed set $K\subseteq\boldsymbol{C}^m$ such that $\Phi(f)=0$ whenever $f\in C^\infty(\boldsymbol{C}^m)$ has support disjoint from $K$.

We now present some important examples of classes of operators which are regular generalized scalar operators.

(A) *Scalar-type spectral operators.* Let $\mathscr{B}(\boldsymbol{C})$ denote the $\sigma$-algebra of Borel subsets of $\boldsymbol{C}$. An operator $S\in L(X)$ is a *scalar-type spectral operator* if there exists a spectral measure $P\colon \mathscr{B}(\boldsymbol{C})\to L(X)$, supported in $\sigma(S)$, such that

$$S=\int_{\boldsymbol{C}} z\,dP(z)=\int_{\sigma(S)} z\,dP(z),$$



where the integral exists in the usual sense of integration with respect to a countably additive vector measure [8; p. 1938]. The spectral measure $P$, necessarily unique [8; XV, Corollary 3.8], is called the *resolution of the identity of* $S$. To say that $P$ is a spectral measure means that $E\to P(E)x$, $E\in\mathscr{B}(\boldsymbol{C})$, is an $X$-valued, $\sigma$-additive measure for each $x\in X$, $P(\boldsymbol{C})=I$ and $P(E\cap F)=P(E)P(F)$ for each $E$ and $F$ in $\mathscr{B}(\boldsymbol{C})$.

Define $\Phi_S\colon C^\infty(\boldsymbol{C})\to L(X)$ by

$$\Phi_S(f)=\int_{\sigma(S)} f(z)\,dP(z),\quad f\in C^\infty(\boldsymbol{C}). \tag{13}$$

That $\Phi_S$ is a spectral distribution follows from [8; XVII, Theorem 2.10] and that $\Phi_S$ is regular is a simple consequence of [8; XV, Corollary 3.7]. Accordingly, if $\boldsymbol{S}$ is a commuting $m$-tuple of scalar-type spectral operators in $L(X)$, then it follows from Proposition 4, Remark 4 and (12) that

$$\mathrm{Sp}(\boldsymbol{S})=\sigma_{\mathrm{H}}(\boldsymbol{S})=\sigma'(\boldsymbol{S})=\sigma''(\boldsymbol{S})=\mathrm{Supp}(\Phi_{\boldsymbol{S}}), \tag{14}$$

where the spectral distribution $\Phi_{\boldsymbol{S}}\colon C^\infty(\boldsymbol{C}^m)\to L(X)$ of $\boldsymbol{S}$ (cf. (11)) is constructed from the $\Phi_j=\Phi_{S_j}$, $1\leqslant j\leqslant m$, as specified by (13).

Remark 5. By (13), each spectral distribution $\Phi_{S_j}$ can be identified with a measure, namely the resolution of the identity $P_j\colon \mathscr{B}(\boldsymbol{C})\to L(X)$ of $S_j$, $1\leqslant j\leqslant m$. However, the tensor product $\Phi_{\boldsymbol{S}}$ as given by (11), although always existing as a spectral distribution in $\boldsymbol{C}^m$ (cf. Remark 4), need not correspond to any spectral measure $\boldsymbol{P}\colon \mathscr{B}(\boldsymbol{C}^m)\to L(X)$ via the formula

$$\Phi_{\boldsymbol{S}}(f)=\int_{\boldsymbol{C}^m} f(\lambda)\,d\boldsymbol{P}(\lambda),\quad f\in C^\infty(\boldsymbol{C}^m). \tag{15}$$

Indeed, this will only be the case if the *product measure* $\boldsymbol{P}$ (cf. [11] for the definition) of the $P_j$, $1\leqslant j\leqslant m$, exists. Although the product measure $\boldsymbol{P}$ need not exist in general there are classes of Banach spaces $X$ in which it always exists. For example, this is the case if $X$ is a Hilbert space, an $L^p$-space, $1\leqslant p\leqslant\infty$, an injective Banach space or $H^\infty(\boldsymbol{D})$ [8, 14]. For such spaces $X$ the identity (15) is valid and hence $\mathrm{Supp}(\Phi_{\boldsymbol{S}})$ is then precisely the support, $\mathrm{Supp}(\boldsymbol{P})$, of the product measure $\boldsymbol{P}\colon \mathscr{B}(\boldsymbol{C}^m)\to L(X)$ (called the *joint resolution of the identity of* $\boldsymbol{S}$), that is, the complement of the largest open set $U\subseteq\boldsymbol{C}^m$ such that $\boldsymbol{P}(U)=0$. So, if the joint resolution of the identity $\boldsymbol{P}$ exists, then (14) becomes

$$\mathrm{Sp}(\boldsymbol{S})=\sigma_{\mathrm{H}}(\boldsymbol{S})=\sigma'(\boldsymbol{S})=\sigma''(\boldsymbol{S})=\mathrm{Supp}(\boldsymbol{P}). \tag{16}$$

It is worth while to indicate a more direct proof of (16) based on the notion $\gamma(\cdot)$, assuming that the joint resolution of the identity $\boldsymbol{P}$ of the $P_j$, $1\leqslant j\leqslant m$, exists. If we define $U_j=\int\mathrm{Re}(\lambda)\,dP_j(\lambda)$ and $V_j=\int\mathrm{Im}(\lambda)\,dP_j(\lambda)$, then both $U_j$ and $V_j$ are scalar-type spectral operators [8; XVII, Theorem

2.10] with resolution of the identity given by

$$P_j^1(E) = P_j(\{w \in C;\ \mathrm{Re}(w) \in E\}), \qquad P_j^2(E) = P_j(\{w \in C;\ \mathrm{Im}(w) \in E\}),$$

for each $E \in \mathscr{B}(C)$, respectively [8; XVII, Lemma 2.9]. It is clear that $\mathrm{Supp}(P_j^1) \subseteq \boldsymbol{R}$ and $\mathrm{Supp}(P_j^2) \subseteq \boldsymbol{R}$, $j = 1, \ldots, m$, and hence $P_j^1$ and $P_j^2$ may be considered as spectral measures defined on the Borel subsets, $\mathscr{B}(\boldsymbol{R})$, of $\boldsymbol{R}$. Since the range of each measure $P_j^1$ and $P_j^2$ is a subset of the range of $P_j$, $1 \leqslant j \leqslant m$, and $\{\boldsymbol{P}(E)x;\ E \in \mathscr{B}(C^m)\}$ is relatively weakly compact, for each $x \in X$, then the product measure, say $Q$, of the $2m$ commuting spectral measures $(P_1^1, \ldots, P_m^1, P_1^2, \ldots, P_m^2)$ is defined on $\mathscr{B}(\boldsymbol{R}^{2m})$ [11; Theorem 8]. Observe that, for each $j = 1, \ldots, m$, the product of the spectral measures $P_j^1\colon \mathscr{B}(\boldsymbol{R}) \to L(X)$ and $P_j^2\colon \mathscr{B}(\boldsymbol{R}) \to L(X)$, which is a spectral measure on the Borel sets of $\boldsymbol{R}^2 = C$, can be identified with $P_j$ and that

$$S_j = U_j + iV_j = \int_{\boldsymbol{R}^2} (\lambda + i\mu)\, dP_j(\lambda, \mu).$$

Now $\Pi(S) = (U, V)$ is a partition for $S$ and hence Lemma 2 implies that

$$\mathrm{Sp}(S) = p(\gamma(\Pi(S))),$$

where $p\colon C^{2m} \to C^m$ is the polynomial (6). It can be shown that $\mathrm{Supp}(Q) = \gamma(\Pi(S))$ and $p(\mathrm{Supp}(Q)) = \mathrm{Supp}(\boldsymbol{P})$, and hence $\mathrm{Sp}(S) = \mathrm{Supp}(\boldsymbol{P})$ follows. Then (16) follows from Proposition 4 applied to $S$.

(B) *Prescalar-type spectral operators.* An operator $S \in L(X)$ is called a *prescalar-type spectral operator of class $\Gamma$*, where $\Gamma$ is a total subspace of the continuous dual space $X'$ of $X$, if there exists a spectral measure of class $\Gamma$, say $P$ (see p. 119 of [7]), such that $S$ belongs to the commutant of the range of $P$, the spectrum of the restriction of $S$ to each closed invariant subspace $P(E)X$, $E \in \mathscr{B}(C)$, is contained in the closure of $E$ in $C$, and $S = \int_{\sigma(S)} z\, dP(z)$. The "integral" is defined via a process of continuous extension from the simple functions; see [7; p. 120]. Define $\Phi_{S,\Gamma}\colon C^\infty(C) \to L(X)$ by the formula (13). That $\Phi_{S,\Gamma}$ is a spectral distribution follows from [7; Proposition 5.9(ii)], and the regularity of $\Phi_{S,\Gamma}$ is a consequence of [7; Theorem 5.12].

Accordingly, if $S = (S_j)$ is a commuting $m$-tuple in $L(X)^m$, where $S_j$ is a prescalar-type spectral operator of class $\Gamma_j$, $1 \leqslant j \leqslant m$ (the $\Gamma_j$ may vary with $j$), then it follows from Proposition 4, Remark 4 and (12) that

$$\mathrm{Sp}(S) = \sigma_{\mathrm{H}}(S) = \sigma'(S) = \sigma''(S) = \mathrm{Supp}(\Phi_S),$$

where the spectral distribution $\Phi_S\colon C^\infty(C^m) \to L(X)$ of $S$ (cf. (11)) is constructed from the $\Phi_j = \Phi_{S_j,\Gamma_j}$, $1 \leqslant j \leqslant m$.

(C) *Polar operators.* For the definition of *well bounded operators of type* (B) we refer to [7; Chapter 17]. Such operators necessarily have real spectrum. An element $T \in L(X)$ is a *polar operator* [3; Definition 3.1] if there are elements $R$ and $A$ in $L(X)$ such that $R$ and $A$ are commuting well bounded operators of type (B) and $T = R\exp(iA)$. It is known that polar operators are generalized scalar operators [3; Theorem 3.2]. Furthermore, if $R\exp(iA)$ is the *canonical decomposition* of $T$ in the sense of [3; p. 441], then it follows from Theorem 3.18(i) of [3] that the spectral distribution for $T$ as constructed in the proof of Theorem 3.2 of [3] is actually regular. Accordingly, polar operators are regular generalized scalar operators.

Some of the most important examples of polar operators, not covered by either of the classes (A) or (B) above, are the translation operators in $L^p(G)$, $1 < p < \infty$, $p \neq 2$, where $G$ is a locally compact abelian group [7; Chapter 20].

(D) *Multiplication operators.* Let $\Omega$ be a locally compact Hausdorff topological space and let $\mu$ be a regular Borel measure on $\Omega$. The space of $C$-valued continuous functions on $\Omega$ which are bounded (respectively vanish at infinity) is denoted by $C_b(\Omega)$ (respectively $C_0(\Omega)$). The norm in each space is taken to be the supremum norm. The notation $L^p(\mu)$, $1 \leqslant p \leqslant \infty$, is standard.

Let $X$ denote one of the Banach spaces $C_0(\Omega)$, $C_b(\Omega)$ or $L^p(\mu)$, $1 \leqslant p \leqslant \infty$. Define an element $T_h$ of $L(X)$ by $T_h(f) = hf$ for all $f \in X$. It is assumed that $h\colon \Omega \to C$ is a bounded Borel function if $X = L^p(\mu)$ for some $1 \leqslant p \leqslant \infty$ and that $h \in C_b(\Omega)$ if $X$ is either $C_0(\Omega)$ or $C_b(\Omega)$.

Let $T = (T_j)$ be an $m$-tuple in $L(X)^m$ given by $T_j = T_{h(j)}$, $1 \leqslant j \leqslant m$, where each function $h(j)\colon \Omega \to C$ is as specified above. If $X = L^p(\mu)$ for some $1 \leqslant p < \infty$, then each operator $T_j$, $1 \leqslant j \leqslant m$, is a scalar-type spectral operator. If $X = L^\infty(\mu)$, then each operator $T_j$, $1 \leqslant j \leqslant m$, is a prescalar-type spectral operator of class $\Gamma = L^1(\mu)$. So in each case $T$ is extra strongly commuting. The case when $X$ is $C_0(\Omega)$ or $C_b(\Omega)$ requires a different argument. If $h \in C_b(\Omega)$, then the mapping $\Phi_h\colon C^\infty(C) \to L(X)$ defined by

$$\Phi_h\colon \psi \to T_{\psi\circ h}, \qquad \psi \in C^\infty(C),$$

is easily seen to be a spectral distribution for $T_h$, supported by the closure $\overline{h(\Omega)}$ of $h(\Omega) = \{h(w);\ w \in \Omega\}$. Accordingly, $T_h$ is a generalized scalar operator. The claim is that $\Phi_h$ is regular. Indeed, if $U = T_{\mathrm{Re}(h)}$ and $V = T_{\mathrm{Im}(h)}$, then $U$ and $V$ commute, $T_h = U + iV$ and the groups $s \to \exp(isU)$, $s \in \boldsymbol{R}$, and $t \to \exp(itV)$, $t \in \boldsymbol{R}$, are uniformly bounded in $L(X)$. So Lemma 3 implies that if $A \in L(X)$ commutes with $T_h = \Phi_h(\lambda)$, then $A$ also commutes with $T_h^{\#} = U - iV = \Phi_h(\bar{\lambda})$, where $\bar{\lambda}\colon C \to C$ denotes the element of $C^\infty(C)$ given by $\bar{\lambda}(s, t) = s - it$ for each $(s, t) \in \boldsymbol{R}^2 \simeq C$. Using this observation and Proposition 3 it can be shown that $T$ is extra strongly commuting. A similar argument also applies in the case of $L^\infty(\mu)$.

EXAMPLE 2. An operator $T \in L(X)$ is a *spectral operator* if there exists a scalar-type spectral operator $S$ and a quasinilpotent operator $Q \in \{S\}'$ such that $T = S + Q$ [8; XV, Theorem 4.5]. Such a decomposition is unique; $S$ is called the *scalar part* of $T$ and $Q$ the *radical part* of $T$. Spectral operators need not be generalized scalar operators [5; Ch. 4, Theorem 3.6].

PROPOSITION 5. *Let $\boldsymbol{T}=(T_j)$ be a commuting m-tuple in $L(X)^m$ consisting of spectral operators. Then $\boldsymbol{T}$ is extra strongly commuting.*

Proof. Let $S_j$ be the scalar part and $Q_j$ the radical part of $T_j$, $1\leqslant j\leqslant m$. Then define operators

$$V_j=\int_{\boldsymbol{C}}\operatorname{Im}(\lambda)\,dP_j(\lambda),\qquad U_j=Q_j+\int_{\boldsymbol{C}}\operatorname{Re}(\lambda)\,dP_j(\lambda),$$

for each $j=1,\ldots,m$, where $P_j$ is the resolution of the identity for $T_j$ (and also for $S_j$). The claim is that $\Pi(\boldsymbol{T})=(\boldsymbol{U},\boldsymbol{V})$ is a partition for $\boldsymbol{T}$ satisfying $\{U_j,V_j\}'=\{T_j\}'$.

It is clear that $T_j=U_j+iV_j$ and that each operator $V_j$, $1\leqslant j\leqslant m$, has real spectrum [8; XVII, Corollary 2.11(ii)]. Similarly, each $U_j$, $1\leqslant j\leqslant m$, also has real spectrum [8; XV, Lemma 4.4] since $Q_j$ commutes with $\int_{\boldsymbol{C}}\operatorname{Re}(\lambda)\,dP_j(\lambda)$ as it commutes with each projection $P_j(E)$, $E\in\mathscr{B}(\boldsymbol{C})$ [8; XV, Corollary 3.7]. It then follows from the commutativity of $\boldsymbol{T}=(T_j)$ and the repeated use of Corollary XV. 3.7 of [8] that $\Pi(\boldsymbol{T})=(\boldsymbol{U},\boldsymbol{V})$ is a commuting $2m$-tuple such that $\{T_j\}'=\{U_j,V_j\}'$. ■

A consequence of this result, together with Proposition 2 and (16) is the following

COROLLARY 5.1. *Let $\boldsymbol{T}=(T_j)$ be a commuting m-tuple in $L(X)^m$ consisting of spectral operators and let $\boldsymbol{S}=(S_j)$, where $S_j$ is the scalar part of $T_j$, $1\leqslant j\leqslant m$. Then*

$$\operatorname{Sp}(\boldsymbol{T})=\sigma_{\mathrm{H}}(\boldsymbol{T})=\sigma'(\boldsymbol{T})=\sigma''(\boldsymbol{T})=\operatorname{Sp}(\boldsymbol{S})=\sigma_{\mathrm{H}}(\boldsymbol{S})=\sigma'(\boldsymbol{S})=\sigma''(\boldsymbol{S}).$$

*If the joint resolution of the identity of $\boldsymbol{S}$ exists, say $\boldsymbol{P}$: $\mathscr{B}(\boldsymbol{C}^m)\to L(X)$, then each of the above spectra is also equal to* $\operatorname{Supp}(\boldsymbol{P})$.

EXAMPLE 3. For the definition of a prespectral operator of class $\Gamma$, where $\Gamma$ is a total subspace of $X'$, we refer to Definition 5.5 of [7].

PROPOSITION 6. *Let $\boldsymbol{T}=(T_j)$ be a commuting m-tuple in $L(X)^m$ where $T_j$ is a prespectral operator of class $\Gamma_j$, $1\leqslant j\leqslant m$. Then $\boldsymbol{T}$ is extra strongly commuting.*

Proof. Define

$$H_j=\int_{\sigma(T_j)}\operatorname{Re}(\lambda)\,dP_j(\lambda),\qquad K_j=\int_{\sigma(T_j)}\operatorname{Im}(\lambda)\,dP_j(\lambda),$$

$j=1,\ldots,m$, where $P_j$ is a resolution of the identity of class $\Gamma_j$ for $T_j$. Then both $H_j$ and $K_j$ have real spectrum [7; Proposition 5.9(i)]. It is shown in the proof of Theorem 5.12 in [7] that the three operators $H_j$, $K_j$ and $Q_j=T_j-H_j-iK_j$ (which is quasinilpotent) commute, and hence $U_j=H_j+Q_j$ also has real spectrum, $j=1,\ldots,m$. It remains to show that $\Pi(\boldsymbol{T})=(\boldsymbol{U},\boldsymbol{K})$ is a commuting $2m$-tuple satisfying $\{U_j,K_j\}'=\{T_j\}'$. This can be argued as in the proof of Proposition 5 after noting that if $A\in L(X)$ commutes with $T_j$, then $A$ also commutes with $H_j$, $K_j$ and $Q_j$, $1\leqslant j\leqslant m$; see the proof of Theorem 5.12 in [7]. ■



**4. Finite-dimensional spaces.** Let $X$ be a complex Banach space and let $\boldsymbol{T}=(T_j)$ be a commuting $m$-tuple in $L(X)^m$. A point $\lambda\in\boldsymbol{C}^m$ is in the *joint point spectrum*, $\sigma_{\mathrm{pt}}(\boldsymbol{T})$, of $T$, if there exists a nonzero element $x\in X$ such that $T_jx=\lambda_jx$, $1\leqslant j\leqslant m$. It is clear that $\sigma_{\mathrm{pt}}(\boldsymbol{T})\subseteq\sigma''(\boldsymbol{T})$ is always valid. If $X$ is finite-dimensional, then this is actually an equality as shown in the following result.

PROPOSITION 7. *Let $X$ be a finite-dimensional Banach space and let $\boldsymbol{T}$ be a commuting m-tuple in $L(X)^m$. Then*

$$\operatorname{Sp}(\boldsymbol{T})=\sigma_{\mathrm{H}}(\boldsymbol{T})=\sigma'(\boldsymbol{T})=\sigma''(\boldsymbol{T})=\sigma_{\mathrm{P}}(\boldsymbol{T})=\sigma_{\mathrm{pt}}(\boldsymbol{T}).\tag{17}$$

Proof. By a theorem of Lie there is a basis of $X$ such that the matrix of each $T_j$, $1\leqslant j\leqslant m$, is upper triangular with respect to this basis. Then $\sigma_{\mathrm{pt}}(\boldsymbol{T})=\{(\lambda_1^k,\ldots,\lambda_m^k);\ 1\leqslant k\leqslant n\}$ where $n=\dim(X)$ and $\{\lambda_j^1,\ldots,\lambda_j^n\}$ are the diagonal entries of the matrix for $T_j$, $1\leqslant j\leqslant m$. So, if $\sigma(T_j)\subseteq\boldsymbol{R}$ for all $j$, then it is clear that $\gamma(\boldsymbol{T})=\sigma_{\mathrm{pt}}(\boldsymbol{T})$. For general $T$, let $\Pi(\boldsymbol{T})$ denote the partition of $\boldsymbol{T}$ as given in the proof of Proposition 5 and let $p$: $\boldsymbol{C}^{2m}\to\boldsymbol{C}^m$ be the polynomial (6). Then Lemma 2 implies the inclusions

$$\operatorname{Sp}(\boldsymbol{T})=p\big(\gamma(\Pi(\boldsymbol{T}))\big)=p\big(\sigma_{\mathrm{pt}}(\Pi(\boldsymbol{T}))\big)\subseteq\sigma_{\mathrm{pt}}(\boldsymbol{T})\subseteq\sigma''(\boldsymbol{T})$$

from which (17) follows via (4), Theorem 3 and Remark 3 since $\operatorname{Sp}(\boldsymbol{T})$ and $\sigma_{\mathrm{H}}(\boldsymbol{T})$, being finite subsets of $\boldsymbol{C}^m$, are rationally convex. ■

Most of the equalities in (17) are known or follow from known results; see [4, 19], for example. However, these results are often based on different methods and techniques. The above proof, which has the advantage of giving all the equalities in (17) at the one time, illustrates once again the usefulness of the notion $\gamma(\cdot)$.

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**References**


[1] E. Albrecht and Şt. Frunză, *Non-analytic functional calculi in several variables*, Manuscripta Math. 18 (1976), 327–336.

[2] W. Arveson, *Subalgebras of C*-algebras II*, Acta Math. 128 (1972), 271–308.

[3] H. Benzinger, E. Berkson and T. A. Gillespie, *Spectral families of projections, semigroups, and differential operators*, Trans. Amer. Math. Soc. 275 (1983), 431–475.

[4] M. Chō and M. Takaguchi, *Joint spectra of matrices*, Sci. Rep. Hirosaki Univ. 26 (1979), 15–19.

[5] I. Colojoară and C. Foiaş, *Theory of Generalized Spectral Operators*, Math. Appl. 9, Gordon & Breach, New York–London–Paris 1968.

[6] A. T. Dash, *Joint Browder spectra and tensor products*, Bull. Austral. Math. Soc. 32 (1985), 119–128.

[7] H. R. Dowson, *Spectral Theory of Linear Operators*, London Math. Soc. Monographs 12, Academic Press, London 1978.

[8] N. Dunford and J. T. Schwartz, *Linear Operators III: Spectral Operators*, Wiley–Interscience, New York 1971.

[9] Şt. Frunză, *The Taylor spectrum and spectral decompositions*, J. Funct. Anal. 19 (1975), 390–421.

[10] R. Harte, *The spectral mapping theorem in several variables*, Bull. Amer. Math. Soc. 78 (1972), 871–875.

[11] I. Kluvánek and M. Kovářiková, *Product of spectral measures*, Czechoslovak Math. J. 17 (92) (1967), 248–255.

[12] A. McIntosh and A. Pryde, *The solution of systems of operator equations using Clifford algebras*, Proc. Centre for Math. Anal. 9, Canberra 1985, 212–222.

[13] —, —, *A functional calculus for several commuting operators*, Indiana Univ. Math. J. 36 (1987), 421–439.

[14] W. Ricker, *Spectral operators of scalar type in Grothendieck spaces with the Dunford–Pettis property*, Bull. London Math. Soc. 17 (1985), 268–270.

[15] Z. Słodkowski and W. Żelazko, *On joint spectra of commuting families of operators*, Studia Math. 50 (1974), 127–148.

[16] J. L. Taylor, *A joint spectrum for several commuting operators*, J. Funct. Anal. 6 (1970), 172–191.

[17] —, *The analytic-functional calculus for several commuting operators*, Acta Math. 125 (1970), 1–38.

[18] F.-H. Vasilescu, *A characterization of the joint spectrum in Hilbert spaces*, Rev. Roumaine Math. Pures Appl. 22 (1977), 1003–1009.

[19] W. Żelazko, *An axiomatic approach to joint spectra I*, Studia Math. 64 (1979), 249–261.



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# Asymptotic stability of linear differential equations in Banach spaces


by

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**Abstract.** Let $A$ be a generator of a strongly continuous bounded semigroup $T(t)$, $t \geqslant 0$. We prove that if the intersection of the spectrum of $A$ and the imaginary axis is at most countable and $A^*$ has no purely imaginary eigenvalues, then the Cauchy problem for the differential equation $\dot{x}(t) = Ax(t)$, $t \geqslant 0$, is asymptotically stable.


We consider the differential equation

$$\dot{x}(t) = Ax(t), \quad t \geqslant 0, \tag{1}$$

in a complex Banach space $X$, where $A$ is a linear closed operator with a dense domain $D(A) \subset X$. The Cauchy problem for equation (1) is *stable* (i.e., by definition, (1) has a unique bounded solution $x(t)$ which depends continuously on the initial value $x(0) \in D(A)$ w.r.t. the sup-norm topology) if and only if the operator $A$ generates a strongly continuous semigroup $T(t)$, $t \geqslant 0$, which is bounded, i.e.

$$\sup_{t \geqslant 0} \|T(t)\| = M < \infty. \tag{2}$$

This criterion, obtained by S. Krein and P. Sobolevskii, is equivalent to the fact that: (i) the spectrum of $A$ does not meet the half-plane $\operatorname{Re}\lambda > 0$, and (ii) the resolvents $R_\lambda = (A - \lambda I)^{-1}$ satisfy the Miyadera–Feller–Phillips inequality

$$\|R_\lambda^n\| \leqslant \frac{M}{(\operatorname{Re}\lambda)^n}, \quad n = 1, 2, \ldots, \ M = \text{const}. \tag{3}$$

These classical results are presented in detail in the monograph [3]. We notice that, without loss of generality, one can put $M = 1$, which can be obtained by introducing the equivalent norm $\sup_{t \geqslant 0} \|T(t)x\|$. In this case the infinite sequence of inequalities (3) is reduced to one Hille–Yosida inequality

$$\|R_\lambda\| \leqslant \frac{1}{\operatorname{Re}\lambda}, \tag{4}$$

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