STUDIA MATHEMATICA, T. LXXXIX. (1988)

Functional-analytic properties of Corson-compact spaces

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Abstract. In this paper several results on the structure of (the general class of) Corson-compact spaces (and some proper subclasses) are presented. For a Corson-compact space K we prove:

(a) The Banach space $C(K)$ is strictly convexifiable.

(b) $C(K)$ is weakly Lindelöf if and only if there exists a bounded linear one-to-one operator $T\colon C(K)^* \to l^\infty_{\omega^+}(\Gamma)$ that is weak* to pointwise continuous, where $l^\infty_{\omega^+}(\Gamma)$ is the Banach space of all bounded functions $f\colon \Gamma \to \boldsymbol{R}$ such that the set $\{\gamma \in \Gamma\colon f(\gamma) = 0\}$ is at most countable, endowed with the supremum norm.

The above theorems generalize the corresponding results of Amir–Lindenstrauss for the subclass of Eberlein-compact spaces. We also give a variety of examples (most of them assuming the continuum hypothesis) of "pathological" Corson-compact spaces, such as: (a) assuming CH, a Corson-compact ccc space Ω (and so with $C(\Omega)$ strictly convexifiable) such that the Banach space $C(G(\Omega))$ is not strictly convexifiable, where $G(\Omega)$ is the Gleason space of Ω (it follows that the geometrical condition of strict convexifiability is not a chain condition); (b) assuming CH, a nonmetrizable Corson-compact Ω with a strictly positive (normal) measure (of countable type), and so with $C(\Omega)$ not weakly Lindelöf.

Introduction. The class of Corson-compact spaces and the Σ-products (see Definition 0.3) were introduced by Corson in $[\mathrm{Co}]_1$ and studied by several authors, especially by Alster [A], Alster and Pol [A-P], Pol $[\mathrm{P}]_2$, Michael and Rudin [Mi-R], Gul'ko $[\mathrm{Gu}]_1$, Benyamini–Rudin–Wage [B-R-W].

In the present paper we continue the study of the structure of K and $C(K)$ for K a Corson-compact space.

In Section 1 we extend some classical results of Amir–Lindenstrauss for Eberlein-compact spaces; in particular, we prove that every Banach space of the form $C(K)$, where K is a Corson-compact space, is strictly convexifiable. As a consequence of our methods we answer in Theorem 1.11 a question posed in [C-N], p. 179.

Section 2 is a study of "chain conditions" in the class of Corson-compact spaces. We give several examples, assuming the continuum hypothesis, of "pathological" Corson-compact spaces (Theorem 2.3) that indicate the complicated structure and the extent of the class of Corson-compact spaces.

We also prove, at least assuming the continuum hypothesis, that the geometrical condition of strict convexifiability for Banach spaces of the form $C(K)$, where K is a compact space, is not a chain condition for K (Theorem 2.11).

Section 3 contains characterizations of a significant subclass of Corson-compact spaces (Theorem 3.5), namely those with property (M) (see Definition 3.1). We prove that this class is identical with the class of Corson-compact spaces K for which $C(K)$ is weakly Lindelöf. We also establish a characterization similar to that of Amir–Lindenstrauss for WCG Banach spaces (a Banach space E is WCG if and only if there is a one-to-one bounded linear operator $T: E^* \to c_0(\Gamma)$ that is weak* to pointwise continuous; cf. [Am-L], see also $[M]_2$, Ths. 4.1, 4.2). We also prove that if we assume the continuum hypothesis then the subclass of Corson-compact spaces with property (M) is strictly contained in the general class of Corson-compact spaces (Theorem 3.12).

In the last Section 4 we introduce a (new) subclass of Corson-compact spaces (with property (M)), namely the Corson-compact spaces of bounded order type (see Definition 4.6). Using combinatorial methods, we are able to construct an example of a non-Gul'ko-compact Corson-compact space of order type ω (Theorem 4.4), a rather surprising result.

0. Preliminaries. The ordinals are defined in such a way that an ordinal is the set of smaller ordinals. A cardinal is an ordinal not in one-to-one correspondence with any smaller ordinal. The least cardinal strictly greater that α is denoted by α^+.

The cardinality of a set A is denoted by $|A|$, the cardinality of the family $\mathscr{P}(\alpha)$ of all subsets of (a set of cardinality) α is denoted by 2^α. We set, for a set S, $[S]^k = \{A \subset S: |A| = k\}$.

The (cardinality of the) set of natural numbers is denoted by ω. The *continuum hypothesis* (CH) is the proposition $\omega^+ = 2^\omega$.

By the term *space* we mean a Hausdorff completely regular topological space.

For a space X, $w(X)$, $d(X)$ and $C(X)$ denote the (topological) weight, the density character, and the Banach space of continuous bounded real-valued functions on X with supremum norm, respectively. The *pointwise topology* on $C(X)$ is determined by the requirement: a net $(f_i)_{i\in I}$ converges to f in the pointwise topology (for $f_i, f \in C(X)$) if $\lim f_i(x) = f(x)$ for all $x \in X$.

A family $\mathscr{B}$ of nonempty open subsets of a space X is called a *pseudobase* for X if every nonempty open subset of X contains an element of the family $\mathscr{B}$.

A compact space K is said to be *extremally disconnected* if the closure of every open subset of K is open, and *angelic* if for every $A \subset K$ and $x \in \bar{A}$ there is a sequence $(x_n) \subset A$ with $x_n \to x$.

If X, Y are spaces, a continuous surjective mapping $f: X \to Y$ is called *irreducible* if for every closed set $F \subset X$ with $F \neq X$ we have $f(F) \neq Y$. A closed subset A of X is called a *retract* of X if the identity id: $A \to A$ ($\mathrm{id}(x) = x$) can be extended to a continuous function $r: X \to A$, which is called a *retraction* of X.

Let X be a space. Then:

(a) X has *caliber* ω^+ if for every family $\{U_\xi: \xi < \omega^+\}$ of nonempty open subsets of X there is $A \subset \omega^+$ with $|A| = \omega^+$ such that $\bigcap_{\xi\in A} U_\xi \neq \emptyset$.

(b) X has *property* (K_n) (where $2 \leqslant n < \omega$) if for every family $\{U_\xi: \xi < \omega^+\}$ of nonempty open subsets of X there is $A \subset \omega^+$, $|A| = \omega^+$, such that any n elements of the family $\{U_\xi: \xi \in A\}$ have nonempty intersection. We denote by (K) property (K_2). It is clear that if X has caliber ω^+ then X has property (K_n) for all n with $2 \leqslant n < \omega$.

(c) The *Suslin number* $S(X)$ of X is the smallest cardinal number α such that there is no family of cardinality α of pairwise disjoint nonempty open subsets of X. A space X satisfies the *countable chain condition* (*ccc*) if $S(X) \leqslant \omega^+$.

(d) X has *property* $(*)$ if the family of nonempty open subsets of X, $\mathscr{T}^*(X)$, can be written in the form

$$\mathscr{T}^*(X) = \bigcup_{n<\omega} \mathscr{T}_n$$

where for every $n < \omega$ there exist at most n pairwise disjoint sets in the family $\mathscr{T}_n$.

(e) X has *property* (P) if for every family $\{U_\xi: \xi < \omega^+\}$ of nonempty open subsets of X there are $A \subset \omega^+$ with $|A| = \omega^+$ and an ordinal ϱ with $2 \leqslant \varrho < \omega^+$ such that if $B \subset A$ and $\operatorname{ordertype}(B) = \varrho$ then there exist $\xi, \xi' \in B$ with $\xi \neq \xi'$ such that $U_\xi \cap U_{\xi'} \neq \emptyset$.

(f) (i) If $\mathscr{F} = \{U_1, \ldots, U_N\}$ is a nonempty finite family of nonempty subsets of X, indexed (not necessarily faithfully) by $\{1, \ldots, N\}$, then $\operatorname{cal}\mathscr{F}$ denotes the largest integer k such that there is $S \subset \{1, \ldots, N\}$ with $|S| = k$ such that $\bigcap_{i\in S} U_i \neq \emptyset$.

(ii) If $\mathscr{S}$ is a nonempty family of nonempty open subsets of X, then

$$k(\mathscr{S}) = \inf\{(\operatorname{cal}\mathscr{F})/N: 1 \leqslant N < \omega,\ \mathscr{F} = \{U_1, \ldots, U_N\} \subset \mathscr{S}\}.$$

(iii) X has *property* $(**)$ if the set $\mathscr{T}^*(X)$ of nonempty open subsets of X can be written in the form

$$\mathscr{T}^*(X) = \bigcup_{n<\omega} \mathscr{T}_n$$

with $k(\mathscr{T}_n) > 0$ for $n < \omega$.

We call any of the above properties (a) to (f) a *chain condition* on the space X. The chain conditions on X are related as follows:

$(**) \Rightarrow (*) \Rightarrow$ (K) $\Rightarrow$ (P) $\Rightarrow$ productively-ccc (i.e. for every ccc space Y, the space $X \times Y$ is ccc) $\Rightarrow$ ccc,

$(**) \Rightarrow (\mathrm{K}_n)$ for $2 \leqslant n < \omega$.

We notice that if X is compact, then X has property $(**)$ if and only if X has a strictly positive (regular Borel) measure μ (i.e. $\mu(U) > 0$ for every nonempty open subset U of X). For detailed information on chain conditions the reader should consult [C-N].

We mention several examples that separate some of the chain conditions.

0.1. Theorem ([C-N]). (a) *There is a compact space K with property $(*)$ and property (K_n) for every n and with no strictly positive measure. Moreover, if we assume CH, K does not have caliber ω^+.*

(b) *For every n with $2 \leqslant n < \omega$, there is a compact space K with property $(*)$ and (K_n) and with no strictly positive measure, and, if we assume CH, without (K_{n+1}).*

(c) *We assume CH. Then there is a compact ccc space K such that $K \times K$ is not ccc.*

(d) *We assume CH. Then there is a compact space with property* (P) *but without* (K).

The examples of Theorem 0.1 belong to Gaifman ([C-N], Th. 6.23), Argyros ([Ar]; see also [C-N], Th. 6.25), Laver–Galvin ([C-N], Th. 7.13), and Kunen ([C-N], Th. 7.4), respectively.

0.2. Definition ([C-N]). For spaces X, Y we write:

(a) $X \leqslant Y$ if there exists a base $\mathscr{B}$ for the topology of X and a mapping Φ: $\mathscr{B} \subset \mathscr{T}^*(X) \to \mathscr{T}^*(Y)$ such that, for every finite subset $\mathscr{F} \subset \mathscr{B}$ with $\bigcap\{U\colon U \in \mathscr{F}\} = \emptyset$, it follows that $\bigcap\{\Phi(U)\colon U \in \mathscr{F}\} = \emptyset$.

(b) $X \equiv Y$ if $X \leqslant Y$ and $Y \leqslant X$.

It is clear that $\equiv$ is an equivalence relation in the class of spaces.

If X, Y are spaces and $X \leqslant Y$ then if Y is ccc (resp. satisfies $(*)$, (K_n) or (P)) it follows that X is ccc (resp. satisfies $(*)$, (K_n) or (P)). If α is an infinite cardinal, then X is ccc (resp. satisfies $(*)$, (K_n) or (P)) if and only if $X \times \{0, 1\}^\alpha$ is ccc (resp. satisfies $(*)$, (K) or (P)) (see [C-N], Th. 2.17 and Cor. 2.28).

0.3. Definition. A compact space K is said to be a *Corson-compact* if K is homeomorphic to a subset of

$$\Sigma(\boldsymbol{R}^\Gamma) = \{x \in \boldsymbol{R}^\Gamma\colon \operatorname{supp}(x) \text{ is countable}\},$$

where $\operatorname{supp}(x) = \{\gamma \in \Gamma\colon x_\gamma \neq 0\}$ for $x \in \boldsymbol{R}^\Gamma$ (here Γ is any set and $\boldsymbol{R}^\Gamma$ is endowed with the product topology).

The space $\Sigma(\boldsymbol{R}^\Gamma)$ is called a *Σ-product* of the real line. Similarly, the Σ-product $\Sigma(X^\Gamma)$ is defined for every subset X of $\boldsymbol{R}$.

Corson-compact spaces and Σ-products were studied by Corson $[\mathrm{Co}]_1$ and more recently by Gul'ko ($[\mathrm{Gu}]_1$, $[\mathrm{Gu}]_2$), Alster [A], Alster–Pol [A-P], Pol $[\mathrm{P}]_2$, Michael Rudin [Mi-R], Benyamini–Rudin–Wage [B-R-W].

It is not difficult to prove that every Corson-compact is angelic.

A (real) Banach space E is called:

(a) *Weakly compactly generated — WCG*, if E contains a weakly compact total subset ([Am-L]), and

(b) *Weakly countably determined — WCD* (resp. *weakly K-analytic*) if E, endowed with the weak topology, is a continuous image of a closed subset of a space of the form $M \times K$, where K is a compact space and M a separable metric space (resp. M a Polish space) ($[\mathrm{Ta}]_1$, [N]).

It follows that every weakly K-analytic Banach space is WCD.

M. Talagrand has proved ($[\mathrm{Ta}]_1$, Th. 3.2) that every WCG Banach space is weakly K-analytic.

Let K be a compact space; K is called an *Eberlein-compact* if K is homeomorphic to a weakly compact subset of a Banach space ([L]), and a *Gul'ko-* (resp. *Talagrand-*) *compact* if $C(K)$ is WCD (resp. weakly K-analytic) ([Ar-M-N], [N]).

It is known that K is an Eberlein-compact if and only if $C(K)$ is WCG ([Am-L]).

Talagrand has constructed a Talagrand-compact space that is not an Eberlein-compact ($[\mathrm{Ta}]_1$) and a Gul'ko-compact that is not a Talagrand-compact ($[\mathrm{Ta}]_2$).

In [Ar-N] it is proved that if K is a Gul'ko-compact then $S(K) = (w(K))^+$. So every ccc Gul'ko-compact is metrizable. S. P. Gul'ko has proved that every Gul'ko-compact is a Corson-compact ($[\mathrm{Gu}]_2$; see also [N] and $[\mathrm{M}]_2$).

For the theory of WCG, weakly K-analytic and WCD Banach spaces we refer to [Am-L], [N], $[\mathrm{Ta}]_1$ and $[\mathrm{M}]_2$.

Given a set Γ, $l^\infty(\Gamma)$ is the Banach space of all bounded real-valued functions defined on Γ with the supremum norm; $c_0(\Gamma)$ denotes the (closed linear) subspace of $l^\infty(\Gamma)$ consisting of all f such that for every $\varepsilon > 0$ there exists a finite subset F_ε of Γ with $|f(\gamma)| < \varepsilon$ for all $\gamma \notin F_\varepsilon$. It is clear that $c_0(\Gamma) \subset \Sigma(\boldsymbol{R}^\Gamma)$. Also, $l^1(\Gamma)$ denotes the Banach space of all functions $f\colon \Gamma \to \boldsymbol{R}$ such that $\sum_{\gamma \in \Gamma} |f(\gamma)| < +\infty$, with the obvious norm.

0.4. Definition ([Da]). (a) A norm $\|\cdot\|$ of a Banach space E is *strictly convex* if for all $x, y \in E$ with $\|x\| = \|y\| = 1$ we have $\|(x+y)/2\| < 1$ whenever $x \neq y$.

A Banach space E is *strictly convexifiable* if there is a strictly convex norm $|||\cdot|||$ on E equivalent to the original norm of E.

(b) A norm $\|\cdot\|$ of a Banach space E is *locally uniformly convex* if for every sequence $(x_n) \subset E$ and every $x \in E$ such that $\|x_n\| = \|x\| = 1$ for all $n < \omega$, if $\lim \|(x_n + x)/2\| = 1$ then $\lim \|x_n - x\| = 0$.

If K is a compact space then we identify the dual $C(K)^*$ of $C(K)$, via the Riesz Representation Theorem, with the space $M(K)$ of all finite real-valued regular Borel measures on K (with $\|\mu\| = |\mu|(K)$, where $|\mu|$ is the total variation of μ).

If $\mu \in M(K)$ we say that:

(a) μ is of *countable type* if the Banach space $L^1(\mu)$ of μ-integrable functions is separable.

(b) μ is *normal* if $\mu(A) = 0$ for every closed nowhere dense subset A of K.

The *support* of a nonnegative measure $\mu \in M(K)$, denoted by $\operatorname{supp}(\mu)$, is the set of all $x \in K$ for which $\mu(U) > 0$ for every open set containing x. The support of a measure is a closed subset of K. It is clear that a nonnegative probability measure $\mu \in M(K)$ is strictly positive if and only if $\operatorname{supp}(\mu) = K$.

We follow, as closely as possible, the notation and terminology of [Ar-M-N] and [N].

1. Corson-compact spaces and strict convexity. In this section we study the geometry of Banach spaces of the form $C(K)$ for a Corson-compact space K.

We begin with the following

1.1. *Notation.* For $J \subset \Gamma$, we define $r_J\colon \boldsymbol{R}^\Gamma \to \boldsymbol{R}^J$ by

$$r_J(x) = \begin{cases} x_\gamma & \text{if } \gamma \in J, \\ 0 & \text{if } \gamma \in \Gamma \setminus J. \end{cases}$$

Then r_J is a continuous retraction of $\boldsymbol{R}^\Gamma$ onto $\boldsymbol{R}^J$ (or more precisely, onto $\boldsymbol{R}^J \times \{0\}^{\Gamma \setminus J}$). If $K \subset \boldsymbol{R}^\Gamma$ and $J \subset \Gamma$, then J is called *K-good* if $r_J(K) \subset K$ (then clearly the mapping $r_J\colon K \to r_J(K) \subset K$ is a continuous retraction and $r_J(K)$ is a retract of K). We sometimes denote the set $r_J(K)$ by $K|J$.

If $D \subset \Sigma(\boldsymbol{R}^\Gamma)$ then it is easy to see that $w(D) = d(D) = |\bigcup_{x \in D} \operatorname{supp}(x)|$.

The following lemma, containing a saturation-type argument, has been used in various forms by Isbell, Kuratowski, Gul'ko, Benyamini ([C-N]).

1.2. Lemma. *Let K be a compact subset of $\boldsymbol{R}^\Gamma$ containing a dense subset D of points of $\Sigma(\boldsymbol{R}^\Gamma)$ and I an infinite subset of Γ. Then there is J with $I \subset J \subset \Gamma$, $|I| = |J|$, and J K-good.*

Proof. We define inductively suitable sets $I = J_0 \subset J_1 \subset \ldots \subset J_n \subset \ldots$ for $n < \omega$, with $|J_n| = |I|$, and set $J = \bigcup_{n<\omega} J_n$. Suppose J_n has been defined. We note that $r_{J_n}(K)$ has density character at most $|I|$, so let $x_\alpha^{(n)}$, $\alpha < |I|$, be elements of D with $\{r_{J_n}(x_\alpha^{(n)})\colon \alpha < |I|\}$ dense in $r_{J_n}(K)$. Set

$$J_{n+1} = J_n \cup \bigcup \{\operatorname{supp}(x_\alpha^{(n)})\colon \alpha < |I|\}.$$

The compactness of K now easily implies the lemma.

1.3. Lemma. *Let K be a compact subset of $\boldsymbol{R}^\Gamma$ as in Lemma 1.2. Then there is a family $\{K_\xi\colon \omega \leqslant \xi \leqslant |\Gamma|\}$ of closed subsets of K, and continuous retractions $r_\xi\colon K \to K_\xi$ for $\omega \leqslant \xi \leqslant |\Gamma|$ such that:*

(1) *$r_\xi \circ r_\zeta = r_\zeta \circ r_\xi = r_\zeta$ for $\omega \leqslant \zeta < \xi \leqslant |\Gamma|$.*

(2) *$w(K_\xi) \leqslant |\xi|$ for $\omega \leqslant \xi \leqslant |\Gamma|$.*

(3) *If ξ is a limit ordinal, $\omega \leqslant \xi \leqslant |\Gamma|$, then*

$$K_\xi = \overline{\bigcup_{\eta<\xi} K_\eta}, \qquad \lim_{\eta<\xi} r_\eta(x) = r_\xi(x) \quad \text{for all } x \in K.$$

(4) *$K_{|\Gamma|} = K$, and $r_{|\Gamma|}$ is the identity on K.*

(5) *$\{\xi\colon \omega \leqslant \xi \leqslant |\Gamma|,\ \|r_\xi^*(f) - r_{\xi+1}^*(f)\| \geqslant \varepsilon\}$ is finite for all $f \in C(K)$ and $\varepsilon > 0$ (where $r_\xi^*\colon C(K) \to C(K)$ is given by $r_\xi^*(f) = f \circ r_\xi$).*

Proof. The existence of K_ξ, r_ξ, $\omega \leqslant \xi \leqslant |\Gamma|$, satisfying conditions (1) to (4) of this lemma is easily proved by transfinite induction, using Lemma 1.2.

Property (5) easily follows from the fact that every sequence in $K \cap \Sigma(\boldsymbol{R}^\Gamma)$ has a convergent subsequence (cf. [Am-L] and [Gu]$_2$, Lemma 9).

1.4. Theorem. *Let K be a compact subset of $\boldsymbol{R}^\Gamma$ containing a dense subset of points of $\Sigma(\boldsymbol{R}^\Gamma)$. Then there are a set Γ and a one-to-one bounded linear operator $T\colon C(K) \to c_0(\Gamma)$ that is also pointwise to pointwise continuous.*

Proof. The proof proceeds by induction on the weight of K. A compact space K of countable weight is metrizable and in this case certainly such an operator exists (with $\Gamma = \omega$).

Let τ be an uncountable cardinal, and suppose that the theorem holds for all cardinals $\alpha < \tau$. Let K be a compact space of weight τ, so we may assume $K \subset \boldsymbol{R}^\tau$ (of course $K \cap \Sigma(\boldsymbol{R}^\tau)$ is dense in K). Let K_ξ, r_ξ, $\omega \leqslant \xi \leqslant \tau$, be as in Lemma 1.3. By the inductive assumption there is a 1-1 bounded linear operator $T_\xi\colon C(K_\xi) \to c_0(\Gamma_\xi)$ for some set Γ_ξ, with $\|T_\xi\| = 1$, which is pointwise to pointwise continuous. We may assume that the sets Γ_ξ are pairwise disjoint (and disjoint from ω) and we set

$$\Gamma = \omega \cup \bigcup \{\Gamma_{\xi+1}\colon \omega \leqslant \xi < \tau\}.$$

We define $T\colon C(K) \to c_0(\Gamma)$ by

$$T(f)(n) = T_\omega(r_\omega^*(f))(n) \qquad \text{for } n < \omega,$$

$$T(f)(\gamma) = \tfrac{1}{2} T_{\xi+1}(r_{\xi+1}^*(f) - r_\xi^*(f))(\gamma) \qquad \text{for } \gamma \in \Gamma_{\xi+1},\ \omega \leqslant \xi < \tau.$$

It follows easily from properties (1) to (5) of Lemma 1.3 that T has the required properties.

Remark. As the referee pointed out to us, Theorem 1.4 above has also been proved by S. P. Gul'ko ([Gu]$_3$, § 9) by a similar method. We have included the proof for completeness.

1.5. Corollary. *If K is a compact subset of $\boldsymbol{R}^\Gamma$ containing a dense subset of points of $\Sigma(\boldsymbol{R}^\Gamma)$, then $C(K)$ is strictly convexifiable* (Definition 0.4(a)).

Proof. According to a classical result due to Day [Da], $c_0(\Gamma)$ is strictly convexifiable, say with a norm $|\cdot|$. Let $T\colon C(K)\to c_0(\Gamma)$ be a 1-1 linear bounded operator given by Theorem 1.4. Then $|||x||| = \|x\| + |T(x)|$ for $x\in C(K)$ defines an equivalent strictly convex norm on $C(K)$.

1.6. Remark. Following a technique of Troyanski [Tr] we may show that there is on $C(K)$, for K as in Theorem 1.4, an equivalent locally uniformly convex norm (Definition 0.4(b)).

Theorem 1.4 and Corollary 1.5 immediately imply the following result.

1.7. Theorem. *Let K be a Corson-compact space. Then there are a set Γ and a one-to-one bounded linear operator $T\colon C(K)\to c_0(\Gamma)$ that is also pointwise to pointwise continuous, and so $C(K)$ is strictly convexifiable.*

There exists another class of compact subsets K of $\boldsymbol{R}^\Gamma$ such that $K\cap\Sigma(\boldsymbol{R}^\Gamma)$ is dense in K, of considerable importance for the construction of suitable examples. This is the class of those compact spaces that result from an adequate family of sets, according to the following

1.8. Definition ([Ta]$_1$). If T is a nonempty set, a family A of subsets of T is called *adequate* if:

(a) If $A\in \mathcal{A}$, $B\subset A$ then $B\in\mathcal{A}$.

(b) $\{t\}\in\mathcal{A}$ for all $t\in T$.

(c) If $A\subset T$ and every finite subset of A belongs to $\mathcal{A}$, then $A\in\mathcal{A}$.

If $\mathcal{A}$ is an adequate family of subsets of T, we set

$$K = K_{\mathcal{A}} = \{\chi_A\colon A\in\mathcal{A}\}\subset\{0,1\}^T.$$

It is immediate that K is a closed subset of $\{0,1\}^T$ and hence compact. We often identify A with χ_A, and thus consider K as a space of certain subsets of T.

1.9. Theorem. *If τ is an infinite cardinal and $\mathcal{A}$ an adequate family of subsets of τ, then there are a set Γ and a one-to-one bounded linear operator $T\colon C(K_{\mathcal{A}})\to c_0(\Gamma)$ that is also pointwise to pointwise continuous, and so $C(K_{\mathcal{A}})$ is strictly convexifiable.*

Proof. It is clear that the family of finite subsets of τ belonging to $\mathcal{A}$ is a dense subset of K; so Theorem 1.4 implies the claim.

Remark. A more direct proof of Theorem 1.9 is as follows. For every

$A\subset\tau$, the family $\{A\cap B\colon B\in\mathcal{A}\}$ is an adequate family of subsets of A; and moreover, this family is a subset of $\mathcal{A}$. So the mapping $r_A\colon K\to r_A(K)\subset K$ defined by $r_A(\chi_B)=\chi_{A\cap B}$ for $B\in\mathcal{A}$ is a continuous retraction.

For every ξ with $\omega\leqslant\xi\leqslant\tau$ we set $A_\xi=[0,\xi)$, $K_\xi=r_\xi(K)$, where $r_\xi\colon K\to K$ is defined by $r_\xi(\chi_B)=\chi_{A_\xi\cap B}$ for $B\in\mathcal{A}$. It is easily proved that the family of retractions $\{r_\xi\colon \omega\leqslant\xi\leqslant\tau\}$ has properties (1) to (5) of Lemma 1.3.

Now by using these properties we can proceed as in the proof of Theorem 1.4.

1.10. Remark. It is known that if a compact space K has a strictly positive regular probability measure, then there exists a bounded linear one-to-one operator $T\colon C(K)\to c_0(\Gamma)$ (see [C-N], p. 179).

By using Theorem 1.9, we can answer in the negative the following question in [C-N] (p. 179): if K is a compact ccc space and $C(K)$ is strictly convexifiable, is then K necessarily the support of a strictly positive measure?

We prove that the known example of Gaifman (see Theorem 0.1(a)) of a compact ccc space K without a strictly positive measure provides a negative answer to this question.

1.11. Theorem. *There exists a compact ccc space K (moreover, with the stronger property $(*)$) without a strictly positive measure such that $C(K)$ is strictly convexifiable.*

Proof. We briefly describe the structure of this example. Let $\{T_n\colon n<\omega\}$ be a one-to-one enumeration of the set of nonempty open intervals of real numbers with rational ends. For every $n<\omega$ with $2\leqslant n<\omega$ we consider a family $\{T_{n,k}\colon 1\leqslant k<n^2\}$ of subintervals of T_n with rational ends such that $T_{n,k}\cap T_{n,k'}=\emptyset$ for $1\leqslant k<k'\leqslant n^2$, and we set

$$\mathcal{A}=\{A\subset\boldsymbol{R}\colon |\{k\colon 1\leqslant k\leqslant n^2,\ A\cap T_{n,k}=\emptyset\}|<n,\ 2\leqslant n<\omega\}.$$

It is clear that $\mathcal{A}$ is an adequate family of (nowhere dense) subsets of $\boldsymbol{R}$. The compact space $K=K_{\mathcal{A}}$ is Gaifman's example.

The proof follows immediately from Theorem 1.9.

1.12. Remark. It follows from Theorem 1.11 that the existence of a one-to-one bounded linear operator $T\colon C(K)\to c_0(\Gamma)$ does not imply, in general, that K admits a strictly positive measure. With the next result we show that for a large class of compact spaces the above implication holds.

1.13. Theorem. *Let K be an extremally disconnected compact space. Then there exists a linear bounded one-to-one operator $T\colon C(K)\to c_0(\Gamma)$ if and only if K admits a strictly positive measure.*

Proof. The "if" part is well known. So we prove the "only if" part. Assume that such an operator $T\colon C(K)\to c_0(\Gamma)$ exists. We denote by

$T^*: l^1(\Gamma) \to M(K)$ the conjugate operator of T, which is w^*-w^* continuous and $T^*(l^1(\Gamma))$ is a w^*-dense subset of $M(K)$. We denote by S the closed unit ball of $l^1(\Gamma)$ which is w^*-sequentially compact. Hence $T^*(S)$ has a similar property. On the other hand, $C(K)$ is a Grothendieck space (i.e. the w^*- and w-sequential convergences in $M(K)$ coincide).

Combining these with the w^*-sequential compactness of $T^*(S)$ we see that every sequence in $T^*(S)$ contains a weakly convergent subsequence, which implies that $T^*(S)$ is a weakly compact subset of $M(K)$. The desired result follows from a result of Rosenthal [R].

We still do not know the answer to the following

1.14. Question. Does K admit a strictly positive measure whenever $C(K)$ is strictly convexifiable and K is extremally disconnected?

Remark. As the referee pointed out to us, the following result, which is somewhat related to Theorem 1.13, has been obtained by A. V. Arkhangel'skii and V. V. Tkachuk ([Ark-T], Th. 10.8): For an extremally disconnected compact space K, $C(K)$ has a continuous linear injection into $c_0(\Gamma)$ (where both spaces are considered with the pointwise topology) if and only if K is separable.

2. Chain conditions in Corson-compact spaces. The content of this section is the reproduction in the class of Corson-compact spaces, assuming CH, of some of the known examples which separate several chain conditions, such as the examples of Argyros, Gaifman, Laver–Galvin, etc. (see Theorem 0.1).

2.1. Definition. Let Ω be a topological space, α an infinite cardinal, and $\mathcal{S} = \{V_\xi: \xi < \alpha\}$ a subset of the family $\mathcal{T}^*(\Omega)$ of nonempty open subsets of Ω.

We set $A = \{A \subset \alpha: \{V_\xi: \xi \in A\}$ has the finite intersection property$\}$ and $K = K_A$.

Then we have the following

2.2. Proposition. *K is a compact space and $K \leqslant \Omega \times \{0, 1\}^\alpha$* (cf. Definition 0.2(a)).

Proof. It is clear that A is an adequate family of subsets of α, so K is a compact subset of $\{0, 1\}^\alpha$.

Now we observe that the sets of the form

$$V_A = \{\varepsilon_1\}_{\xi_1} \times \{\varepsilon_2\}_{\xi_2} \times \ldots \times \{\varepsilon_k\}_{\xi_k} \times \prod \{0, 1\}^{\alpha \setminus A} \cap K,$$

where $A = \{\xi_1, \ldots, \xi_k\} \subset \alpha$ and $\varepsilon_i = 0$ or 1, form a base $\mathcal{B}$ of open-and-closed sets for the topology of K. We set $B = \{\xi_i: \varepsilon_i = 1\}$, $\Gamma = \{\xi_i: \varepsilon_i = 0\}$, so that $A = B \cup \Gamma$, $B \cap \Gamma = \emptyset$, and

$$V_A = \{1\}_B \times \{0\}_\Gamma \times \prod \{0, 1\}^{\alpha \setminus A} \cap K.$$

We define a mapping $\Phi: \mathcal{B} \subset \mathcal{T}^*(K) \to \mathcal{T}^*(\Omega \times \{0, 1\}^\alpha)$ by

$$\Phi(V_A) = \bigcap_{\xi \in B} V_\xi \times [\{1\}_B \times \{0\}_\Gamma \times \prod \{0, 1\}^{\alpha \setminus A}] \quad \text{if } B \neq \emptyset,$$

and in case $B = \emptyset$ we replace $\bigcap_{\xi \in B} V_\xi$ with the space Ω.

It is easily proved that Φ satisfies the properties of Definition 0.2(a). So the proof of the proposition is complete.

2.3. Theorem ("Pathological" Corson-compact spaces). *We assume CH. Then*:

(a) *There is a Corson-compact space K with property* $(*)$ *and property* (K_n) *for every $n \geqslant 2$, but without any strictly positive measure.*

(b) *For every $n \geqslant 2$ there is a Corson-compact space K with property* $(*)$ *and property* (K_n) *but without* (K_{n+1}).

(c) *There is a Corson-compact space K with property* (P) *which does not have property* (K).

(d) *There is a Corson-compact space K with ccc such that $K \times K$ does not have ccc.*

Proof. (a) We prove this claim with a variant of the example of Gaifman.

We observe that if instead of $\boldsymbol{R}$ we have a second category subset L of $\boldsymbol{R}$, then the same construction gives us a compact space K_L (resulting from an adequate family A_L of subsets of L) with properties of Gaifman's example. It is clear that if $A \in A_L$ then A is a nowhere dense subset of $\boldsymbol{R}$.

Now choose a Lusin set L in $\boldsymbol{R}$ (i.e. an uncountable subset L of $\boldsymbol{R}$ such that every uncountable subset of L is of the second category in $\boldsymbol{R}$). Since we assume CH, a Lusin set in $\boldsymbol{R}$ exists (cf. [C-N]).

If $A \in A_L$, then A is nowhere dense in $\boldsymbol{R}$, and so (since L is a Lusin set) A is countable. It follows immediately that $K = K_L$ is a Corson-compact space.

(b) In [Ar] a compact space Ω has been constructed for every $n \geqslant 2$ with property $(*)$ and without a strictly positive measure, and such that, assuming CH, it has property (K_n) but does not have (K_{n+1}) (see Theorem 0.1(b)).

Let $\mathcal{S} = \{V_\xi: \xi < \omega^+\}$ be a family of nonempty open subsets of Ω that witnesses the failure of property (K_{n+1}). Set

$$A = \{A \subset \omega^+: \{V_\xi: \xi \in A\} \text{ has the finite intersection property}\}$$

(see Definition 2.1). It is clear that every $A \in A$ is at most countable so $K = K_A$ is a Corson-compact.

From Proposition 2.2 (and preliminaries) K has properties $(*)$ and (K_n). For $\xi < \omega^+$ we set

$$V'_\xi = \{1\}_\xi \times \prod \{0, 1\}^{\omega^+ \setminus \{\xi\}} \cap K,$$

and we observe that the family $\{V'_\xi: \xi < \omega^+\} \subset \mathcal{T}^*(K)$ fails property (K_{n+1}).

(c) Let Ω be the compact space (under CH) of Kunen (which has property (P) but does not have (K); see Theorem 0.1(d)).

We consider a family $\mathcal{S} = \{V_\xi\colon \xi < \omega^+\}$ of nonempty open subsets of Ω that witnesses the failure of property (K). As in (b) we set $\mathcal{A} = \{A \subset \omega^+\colon \{V_\xi\colon \xi \in A\}$ has the finite intersection property$\}$, $K = K_{\mathcal{A}}$ and we observe that K is a Corson-compact space.

From Proposition 2.2 (and preliminaries) K has property (P).

It follows that the family $\{V'_\xi\colon \xi < \omega^+\}$ where V'_ξ is as in (b) fails property (K).

(d) If we assume CH, there is a compact ccc space Ω such that $\Omega \times \Omega$ is not ccc (see Theorem 0.1(c)); this is the space of Laver–Galvin. Pick a family $\{V_\xi, V_{\xi+1}\colon \xi < \omega^+, \xi$ limit ordinal$\}$ of nonempty open subsets of Ω such that the family $\{V_\xi \times V_{\xi+1}\colon \xi < \omega^+, \xi$ limit$\}$ is pairwise disjoint and let $\mathcal{A}$ be as in (b) and (c) and $K = K_{\mathcal{A}}$. It is easy to see that K is a Corson-compact. It follows from Proposition 2.2 (and preliminaries) that K is ccc.

We set $V'_\xi = \{1\}_\xi \times \{0, 1\}^{\omega^+ \setminus \{\xi\}} \cap K$, $\xi < \omega^+$, and it is not difficult to see that the family $\{V'_\xi \times V'_{\xi+1}\colon \xi < \omega^+, \xi$ limit$\}$ is an uncountable pairwise disjoint family of nonempty open subsets of $K \times K$.

2.4. Remarks. 1) Every compact space in Theorem 2.3 is (under CH) a Corson-compact ccc space without a strictly positive measure, since if a compact space K has a strictly positive measure then K satisfies (K_n) for all $n < \omega$ and hence (P) (see preliminaries). We also note that as follows from Theorem 1.7 (since K is a Corson-compact) or from Theorem 1.9 (since K results from an adequate family of sets) $C(K)$ is strictly convexifiable, so every such K satisfies Theorem 1.11.

2) Since every Gul'ko-compact ccc space is metrizable (see preliminaries), it follows that all compact spaces in Theorem 2.3 are (under CH) examples of Corson-compact spaces which are not Gul'ko-compact.

3) Dashiell–Lindenstrauss obtain in [D-L] some significant examples of Banach spaces of the form $C(K)$ (where K is a compact space) which are strictly convexifiable, though there is no bounded linear one-to-one operator $T\colon C(K) \to c_0(\Gamma)$ for any set Γ (see also $[\mathrm{M}]_3$ for some other examples of this type). In [D-L] the problem is also posed of the characterization of those compact spaces K for which $C(K)$ is strictly convexifiable.

It follows from our results (in Theorem 2.3) that the answer to this is difficult and not related to refined chain conditions which K satisfies, at least for ccc spaces.

2.5. Corollary. *We assume CH. Then:*

(a) *There is a topological space with a point countable base which has property* $(*)$ *and property* (K_n) *for every* $n \geqslant 2$, *but fails* $(**)$.

(b) *For every* $n \geqslant 2$ *there is a topological space with a point countable base with property* $(*)$ *and property* (K_n), *but without* (K_{n+1}).

(c) *There is a topological space with a point countable base with property* (P), *but without* (K).

(d) *There is a topological space* X *with a point countable base which is ccc while* $X \times X$ *is not ccc.*

Proof. We recall that a space X has a *point countable* base $\mathcal{B}$ if every point x of X belongs to an at most countable number of members of $\mathcal{B}$; it is clear that every such space is first countable.

Shapirovskii has proved that every Corson-compact space has a dense subset with a point countable base [S]. It follows that, since all chain conditions in Theorem 2.3 pass to dense subsets, there are, assuming CH, topological spaces with a point countable base which satisfy claims (a), (b), (c) and (d).

2.6. Remark. It was known that some of the claims of Corollary 2.5 hold for first countable spaces.

So Eric van Douwen has proved that assuming CH, there is a first countable ccc space X such that $X \times X$ is not ccc ([Ga]).

A problem given by Eric van Douwen and Negrepontis is the following: are there compact first countable spaces which satisfy some of the claims of Corollary 2.5, under CH?

We notice that we can give a direct proof of Corollary 2.5 without recourse to Shapirovskii's result. That means: if Ω is any of the spaces in Theorem 2.3 and $X = \{\chi_A \in \Omega\colon$ if $A \subset B$ and $\chi_B \in \Omega$ then $A = B\} \subset \Omega$, then it is not difficult to prove that the nonempty subspace X of Ω has a point countable base and in any case it has the required properties.

2.7. Definition (cf. [C-N]). Let X be a space. An open subset U of X is said to be *regular open* if it is equal to the interior of its closure, i.e. $U = \mathrm{int}_X \mathrm{cl}_X U$.

It is known that the set of regular open sets of a space X is a complete Boolean algebra with the following operations: $U \wedge V = U \cap V$, $U \vee V = \mathrm{int}_X \mathrm{cl}_X (U \cup V)$, $U' = \mathrm{int}_X (X \setminus U)$.

The Stone space $G(X)$ of this algebra is called the *Gleason space* of X. Since the algebra of regular open sets is complete, $G(X)$ is a (compact) extremally disconnected space (cf. [C-N]).

2.8. Theorem ([C-N]). *For every space* X, $X \equiv G(X)$ (see Definition 0.2). *In other words, a space* X *satisfies a chain condition if and only if the space* $G(X)$ *satisfies the same chain condition. Therefore the space* $G(X)$ *can be considered the compact equivalent of* X *relative to the chain condition.*

Let Ω denote the compact ccc space which was constructed in [Ar] (see Theorem 0.1(b), case $n = 2$). The space Ω not only is a space without a strictly positive measure, but moreover $C(\Omega)$ is not strictly convexifiable and it is the unique example that we know with these properties.

We shall construct, assuming CH, a Corson-compact ccc space K such that the Gleason space of K is Ω. So we can prove that, at least under CH, the (geometrical) condition of strict convexity on Banach spaces of the form $C(K)$ is not – at least for ccc compact spaces K – a (combinatorial) chain condition.

We shall need the following

2.9. THEOREM (cf. [C-N]).(a) *For every space X, $G(X) = G(\beta X)$ where βX is the Stone–Čech compactification of X.*

(b) *Let K be a compact space and Ω a compact extremally disconnected space. Then $\Omega = G(K)$ if and only if there is an irreducible mapping $f\colon \Omega \to K$.*

2.10. DEFINITION. (a) A partially ordered set $(T, \leqslant)$ is a *chain* if for $x, y \in T$ we have either $x \leqslant y$ or $y \leqslant x$.

(b) A partially ordered set $(T, \leqslant)$ is a *tree* if for every $x \in T$ the set $\{y \in T\colon y \leqslant x\}$ is well ordered with the induced order. A subset B of a tree $(T, \leqslant)$ is called a *branch* of the tree if it is a chain and is not contained in any other chain of $(T, \leqslant)$.

2.11. THEOREM. *We assume CH. Then there exists a Corson-compact ccc space K (so $C(K)$ is strictly convexifiable) such that $C(G(K))$ is not strictly convexifiable (where $G(K)$ is the Gleason space of K).*

For the proof of this theorem, we need a description of Ω (2.12) and two lemmas (2.13, 2.14).

2.12. *The space Ω.* We define a tree $(T, \leqslant)$ with $T = \bigcup_{n<\omega} T_n \subset [\omega]^2$.

Let $\{S_n^j\colon n < \omega,\ 1 \leqslant j \leqslant 3^n\}$ be a family such that $S_n^j \in [\omega]^3$ and $S_n^j \cap S_{n'}^{j'} = \emptyset$ if $(n, j) \neq (n', j')$. We set $T_n = \bigcup \{[S_n^j]^2\colon 1 \leqslant j \leqslant 3^n\}$ for $n < \omega$ (so $|T_n| = 3^{n+1}$).

Let $T_n = \{s_j\colon 1 \leqslant j \leqslant 3^{n+1}\}$ be an enumeration of T_n and $T = \bigcup_{n<\omega} T_n$.

The order $\leqslant$ in T is defined in such a way that the immediate successor of every element of T_n is in T_{n+1}, as follows: for $s \in T_n$ and $t \in T_{n+1}$, $s < t$ if and only if there exists j with $1 \leqslant j \leqslant 3^{n+1}$ such that $s = s_j$ and $t \in [S_{n+1}^j]^2$.

For $s = (k, l) \in T$ we set

$$V_s = \{0\}_k \times \{1\}_l \times \prod \{0, 1\}^{\omega \setminus s} \cup \{0\}_l \times \{1\}_k \times \prod \{0, 1\}^{\omega \setminus s} \subset \{0, 1\}^\omega.$$

Let Σ be the set of branches of the tree T; so if $\Sigma \in \mathbf{\Sigma}$ then $\Sigma = \{s_1 < \ldots < s_n < \ldots\}$ with $s_n \in T_n$ for $n < \omega$.

For every $\Sigma \in \mathbf{\Sigma}$, we set $V_\Sigma = \bigcap_{s \in \Sigma} V_s$.

We notice that:

(a) If $S \subset \omega$ with $|S| = 3$, and there exists $n < \omega$ with $[S]^2 = \{s_1, s_2, s_3\} \subset T_n$, then the family $\{V_{s_1}, V_{s_2}, V_{s_3}\}$ has empty intersection, while any two sets of this family have nonempty intersection.

(b) If $\Sigma_1, \Sigma_2 \in \mathbf{\Sigma}$ are two branches of the tree T, then $V_{\Sigma_1} \cap V_{\Sigma_2} \neq \emptyset$.

Now we define the space $(Y, \mathscr{T})$. We set $Y = \{0, 1\}^\omega$. The topology $\mathscr{T}$ is defined via a subbase which contains:

(i) the open-and-closed subsets of $\{0, 1\}^\omega$, in the usual product topology, and

(ii) the sets of the form V_Σ, for $\Sigma \in \mathbf{\Sigma}$.

It is clear that the base $\mathscr{B}$ of the topology $\mathscr{T}$ which is defined by this subbase consists of sets of the form $U \cap \left(\bigcap_{j=1}^m V_{\Sigma_j}\right)$, where U is open and closed in the usual topology of $\{0, 1\}^\omega$ and $\{\Sigma_j\colon 1 \leqslant j \leqslant m\} \subset \mathbf{\Sigma}$. We observe that the elements of $\mathscr{B}$ are closed in the usual topology of $\{0, 1\}^\omega$, therefore also in the topology $\mathscr{T}$. So $(Y, \mathscr{T})$ has a base (namely the family $\mathscr{B}$) of open-and-closed sets and hence $(Y, \mathscr{T})$ is a Hausdorff completely regular space.

We note that if $V \in \mathscr{B}$ and $V = U \cap \left(\bigcap_{j=1}^m V_{\Sigma_j}\right)$ then there is $n < \omega$ such that V is separated at level n. Namely:

(c) (i) $(\Sigma_{j_1} \cap T_n) \cap (\Sigma_{j_2} \cap T_n) = \emptyset$ for $1 \leqslant j_1 \leqslant j_2 \leqslant m$.

(ii) If F denotes the (finite) set of coordinates on which U depends, then $F \cap A(n) = \emptyset$, where $A(n) = \bigcup \{s \in T\colon s \in \bigcup_{n \leqslant p < \omega} T_p\}$.

It is easy to verify (c).

Let Ω be the Gleason space of $(Y, \mathscr{T})$. It is proved in [Ar] that Ω has property $(*)$ (and therefore it is ccc), that Ω does not have a strictly positive measure, and more precisely, if we assume CH, Ω does not have (K_3).

2.13. LEMMA. *The Banach space $C(\Omega)$ is not strictly convexifiable.*

Proof. Let βY denote the Stone–Čech compactification of Y; so (by the fundamental property of the Stone–Čech compactification) the Banach spaces $C(Y)$ and $C(\beta Y)$ are isometrically isomorphic.

By Theorem 2.9, $G(Y) = G(\beta Y)$ (so $\Omega = G(\beta Y)$) and there is a continuous irreducible surjective mapping $f\colon G(\beta Y) \to \beta Y$. Thus it is sufficient to prove that $C(\beta Y)$ is not strictly convexifiable.

Suppose, by contradiction, that $C(\beta Y)$ admits an equivalent strictly convex norm, say $|||\cdot|||$; we denote by $\|\cdot\|$ the usual supremum norm of $C(\beta Y)$.

We set $B_1 = B =$ the closed unit ball of $C(\beta Y)$, $a_1 = \sup\{|||f|||\colon f \in B_1\}$, and we choose $f_1 \in B_1$ with $|||f_1||| \geqslant \frac{3}{4} a_1$.

Let $\mu_1 \in M(\beta Y)$ be such that $|||\mu_1||| = 1$ and $\mu_1(f_1) = |||f_1|||$. Since the norms are equivalent there exists $\delta > 0$ such that $\|\mu\| \leqslant \delta$ for all $\mu \in M(\beta Y)$ with $|||\mu||| \leqslant 1$.

We choose $k_1 < \omega$ such that $(\frac{2}{3})^{k_1} < a_1/(4\delta)$, so there exists $s_1 \in T_{k_1}$ such that $|\mu_1|(\bar{V}_{s_1}) < \frac{1}{4}a_1$. This inequality follows from the fact that the caliber of the family $\{V_s\colon s \in T_{k_1}\}$ is 2^{k_1} while $|T_{k_1}| = 3^{k_1+1}$ and $\|\mu_1\| \leqslant \delta$.

We set $U_1 = \bigcap_{s\leqslant s_1} V_s$, $B_2 = \{f \in B_1\colon f\,|\,\beta Y\setminus\bar{U}_1 = f_1\,|\,\beta Y\setminus\bar{U}_1\}$, and we notice that

$$\Big|\int_{\beta Y\setminus\bar{U}_1} f_1\,d\mu_1\Big| \geqslant \int_{\beta Y} f_1\,d\mu_1 - \Big|\int_{\bar{U}_1} f_1\,d\mu_1\Big| \geqslant \tfrac{3}{4}a_1 - \tfrac{1}{4}a_1 = \tfrac{2}{4}a_1,$$

so for $f \in B_2$

$$|||f||| \geqslant \Big|\int_{\beta Y} f\,d\mu_1\Big| \geqslant \Big|\int_{\beta Y\setminus\bar{U}_1} f_1\,d\mu\Big| - \Big|\int_{\bar{U}_1} f\,d\mu\Big| \geqslant \tfrac{1}{4}a_1.$$

Let $a_2 = \sup\{|||f|||\colon f \in B_2\}$, so $a_1 \geqslant a_2$; also choose $f_2 \in B_2$ with $|||f_2||| \geqslant \frac{4}{5}a_2$ and $\mu_2 \in M(\beta Y)$ with $|||\mu_2||| = 1$ and $\mu_2(f_2) = |||f_2|||$. We choose $k_2 < \omega$ such that $(\frac{2}{3})^{k_2} \leqslant \frac{1}{5}a_2$. Since $s_1 \in T_{k_1}$ has 3^{k_1} immediate successors in $T_{k_1+k_2}$ and the caliber of the corresponding family is 2^{k_2}, an immediate successor s_2 of s_1 (in $T_{k_1+k_2}$) exists such that $|\mu_2|(\bar{V}_{s_2}) < \frac{1}{5}a_2$.

We set $U_2 = \bigcap_{s\leqslant s_2} V_s$; since $s_1 \leqslant s_2$, it follows that $U_2 \subset U_1$. It is clear from the above that $|\mu_2(f_2)| \geqslant \frac{4}{5}a_2 - \frac{1}{5}a_2 = \frac{3}{5}a_2$. Also if we set $B_3 = \{f \in B_2\colon f\,|\,\beta Y\setminus\bar{U}_2 = f_2\,|\,\beta Y\setminus\bar{U}_2\}$, then $|||f||| \geqslant \frac{2}{5}a_2$ for every $f \in B_3$.

So proceeding inductively, we construct sequences

$$B_1 \supset B_2 \supset \ldots \supset B_n \supset \ldots,$$

$$f_1, f_2, \ldots, f_n, \ldots,$$

$$s_1, s_2, \ldots, s_n, \ldots,$$

for $n < \omega$, such that $f_n \in B_n$,

$$\frac{na_n}{n+3} \leqslant |||f_n||| \leqslant a_n = \sup\{|||f|||\colon f \in B_n\},$$

and if $U_n = \bigcap_{s\leqslant s_n} V_s$, then $f \in B_{n+1}$ if and only if $f \in B_n$ and $f\,|\,\beta Y\setminus\bar{U}_n = f_n\,|\,\beta Y\setminus\bar{U}_n$. It is clear that if $f \in \bigcap_{n<\omega} B_n$ then $|||f||| = \inf a_n$.

Since B_n is convex, so is $\bigcap_{n<\omega} B_n$, and if $f, g \in \bigcap_{n<\omega} B_n$ with $f \neq g$, then $|||f||| = |||g||| = |||(f+g)/2|||$, a contradiction. So it suffices to prove that there are $f, g \in \bigcap_{n<\omega} B_n$ with $f \neq g$.

Indeed, the sequence $s_1 < \ldots < s_n < \ldots$ defines in a unique way a branch Σ of the tree T for which, by our construction, $V_\Sigma = \bigcap_{n<\omega} U_n$ and V_Σ is open and closed.

We define continuous bounded functions $f, g\colon Y \to [-1, 1]$ in the following way:

$$f(x) = \begin{cases} f_1(x) & \text{for } x \in Y\setminus U_1,\\ f_n(x) & \text{for } x \in U_{n-1}\setminus U_n,\\ \chi_{V_\Sigma}(x) & \text{for } x \in V_\Sigma,\end{cases}$$

$$g(x) = \begin{cases} f_1(x) & \text{for } x \in Y\setminus U_1,\\ f_n(x) & \text{for } x \in U_{n-1}\setminus U_n,\\ -\chi_{V_\Sigma}(x) & \text{for } x \in V_\Sigma.\end{cases}$$

The functions f, g can clearly be extended to some functions $\tilde{f}, \tilde{g} \in C(\beta Y)$, also belonging to the set $\bigcap_{n<\omega} B_n$.

The proof of the lemma is now complete.

2.14. Lemma. *We assume CH. Then Ω has a pseudobase $\{V_\xi\colon \xi < \omega^+\}$ of open-and-closed sets that witnesses the failure of property caliber ω^+ (namely, for every $A \subset \omega^+$ with $|A| = \omega^+$ we have $\bigcap_{\xi\in A} V_\xi = \emptyset$).*

Proof. We shall construct a pseudobase of open-and-closed sets which satisfies the lemma for the space Y; since $\Omega = G(Y)$, it follows that Ω has such a pseudobase.

Claim. *If $\{x_n\colon n < \omega\} \subset Y$ and V is a nonempty basic open subset of Y, then there exists $\Sigma \in \Sigma$ with $V_\Sigma \cap V \neq \emptyset$ and $\{x_n\colon n < \omega\} \cap V_\Sigma = \emptyset$.*

Indeed, V is of the form $V = U \cap \big(\bigcap_{j=1}^m V_{\Sigma_j}\big)$, where U is an open-and-closed subset in the usual topology of $\{0, 1\}^\omega$, and $\{\Sigma_j\colon 1 \leqslant j \leqslant m\} \subset \Sigma$.

Consider $n < \omega$ such that V is separated at level n. According to 2.12(c), for some fixed $j \in \{1, \ldots, m\}$ let Σ_j be the branch $\{s_1^j < \ldots < s_n^j < \ldots\}$ of T.

Let q be the index of the element $s_n^j \in T_n \cap \Sigma_j$ in the enumeration of T_n (recall that $|T_n| = 3^{n+1}$). Then according to 2.12(a) there exists $s_1 \equiv s_{q_1} \in [S_{n+1}^q]^2$ with $s_q < s_1$ and $x_1 \notin V_{s_1}$.

So by induction (and the observation 2.12(a)) we can construct a sequence $\{s_1 < \ldots < s_k < \ldots\} \subset T$ with $s_k \in T_{n+k}$ and such that $x_k \notin V_{s_k}$, $k < \omega$. It is clear that $\Sigma = \{s_1^j < \ldots < s_n^j < s_1 < \ldots < s_k < \ldots\}$ is a branch of the tree T.

Set $V_\Sigma = \bigcap_{s\in\Sigma} V_s$; it is easily proved by the above remarks that $V_\Sigma \cap V \neq \emptyset$ and $\{x_n\colon n < \omega\} \cap V_\Sigma = \emptyset$.

The proof of the claim is complete.

Let now $\{x_\xi\colon \xi < \omega^+\}$ and $\{U_\xi\colon \xi < \omega^+\}$ be one-to-one well-orderings of $\{0, 1\}^\omega$ and of the base $\mathscr{B}$ of Y, respectively. We can choose, using the claim, for every $\xi < \omega^+$ a nonempty set V_ξ in $\mathscr{B}$ such that:

(i) $V_\xi \subset \{x_\zeta\colon \xi < \zeta < \omega^+\}$.

(ii) $V_\xi \subset U_\xi$.

It follows immediately that the family $\{V_\xi\colon \xi < \omega^+\}$ is the desired pseudobase for Y.

We are now ready to complete the

Proof of Theorem 2.11. Let $\Omega = G(Y)$ and $\{V_\xi: \xi < \omega^+\}$ be a pseudobase of Ω as in Lemma 2.14.

Set $A = \{A \subset \omega^+: \bigcap_{\xi\in A} V_\xi \neq \emptyset\}$ and $L = K_A$; it is clear from the properties of $\{V_\xi: \xi < \omega^+\}$ that L is a Corson-compact space.

We define a mapping $T: \Omega \to L$ as follows:

$$T(x)(\xi) = \begin{cases} 1, & x \in V_\xi, \\ 0, & x \notin V_\xi. \end{cases}$$

It is easily seen that T is continuous, and so the space $K = T(\Omega) \subset L$ is a Corson-compact ccc space, as a continuous image of a ccc space.

The mapping $T: \Omega \to K$ is irreducible. Indeed, if U is a nonempty open subset of Ω, then since the family $\{V_\xi: \xi < \omega^+\}$ is a pseudobase for Ω, there exists $\xi < \omega^+$ such that $V_\xi \subset U$, hence $T(V_\xi) \subset T(U)$. It follows easily that $T(U)$ has nonempty interior, so T is irreducible.

Now, Ω is compact and extremally disconnected (since $\Omega = G(Y)$) and $T: \Omega \to K$ is a continuous, irreducible and onto mapping, so by Theorem 2.9(b), $\Omega = G(K)$. Thus Lemma 2.13 completes the proof of the theorem.

3. Corson-compact spaces and property (M). The principal results of this section are: a characterization of Corson-compact spaces with property (M) (see Definition 3.1(b)) given in Theorem 3.5, and an example, assuming CH, of a Corson-compact space without (M) (Theorem 3.12).

3.1. Definition. (a) A Banach space E has *property* (C) if every family of closed convex subsets of E with countable intersection property (i.e. every countable subfamily of the given family has nonempty intersection) has nonempty intersection.

(b) A compact space K has *property* (M) if every positive regular Borel measure on K has separable support.

3.2. Remarks. 1) Property (C), which is clearly a "convex analogue" of the property for a Banach space to be Lindelöf in the weak topology, is introduced by Corson in [Co]$_2$. In that paper Corson asked whether from property (C) for a Banach space E it follows that E is Lindelöf in its weak topology (clearly the converse is true). The answer is no, as Pol proved in [P]$_2$; moreover, Pol proved the following statements:

(a) If the closed unit ball of the dual of a Banach space E is an angelic space in the weak* topology, then E has property (C).

(b) If K is a compact space such that $C(K)$ has property (C) then K has (M).

2) G. Godefroy proves in [Go] that every *Rosenthal-compact* (i.e. a pointwise compact subset of the space of Baire-1 real-valued functions on a Polish space) has property (M). Since every WCD Banach space is weakly Lindelöf ([Ta]$_1$), it follows by 1) that every Gul'ko-compact has property (M). The same result follows easily from the fact that every ccc Gul'ko-compact space is metrizable (see preliminaries).

3) It is consistent with ZFC to assume that every Corson-compact has property (M) (indeed, if we assume the negation of the continuum hypothesis and Martin's Axiom then every Corson-compact ccc space is metrizable (cf. [Ar-M-N] and [C-N]). Since the support of a positive measure is ccc, the conclusion follows.

3.3. Lemma. *Let Γ be a set and K a pointwise compact convex subset of $\Sigma(\mathbf{R}^\Gamma)$. Then K has property* (M).

Proof. We assume without restriction of generality that $K \subset \Sigma([-1, 1]^\Gamma)$.

For every $\gamma \in \Gamma$, $\pi_\gamma: K \to [-1, 1]$ denotes the projection on the coordinate γ, i.e. $\pi_\gamma(x) = x(\gamma)$ for $x \in K$, so $\pi_\gamma \in C(K)$ for all $\gamma \in \Gamma$.

Set $L = \{\pi_\gamma: \gamma \in \Gamma\}$,

$$L^n = L \dots L\ (n \text{ times}) = \{f_1 \dots f_n: f_i \in L,\ i = 1, \dots, n\} \subset C(K)$$

for $n < \omega$, $W = \{1\} \cup (\bigcup_{n<\omega} L^n)$, and let $P(K)$ be the weak* compact convex set of regular Borel probability measures on K.

We notice that L separates the points of K so W separates the points of K. Therefore, since the linear hull of W in $C(K)$ is an algebra that contains the constant functions, it follows from the Stone–Weierstrass Theorem that W is a total subset of $C(K)$.

We define a mapping $\Phi: P(K) \to [-1, 1]^W$ by $\Phi(\mu) = (\mu(f))_{f\in W}$. Since μ is a probability measure and $\|f\| \leqslant 1$ for $f \in W$, Φ is well defined, and since W is a total subset of $C(K)$, Φ is 1-1. Moreover, Φ is a continuous affine mapping (i.e. Φ preserves convex combinations) if $P(K)$ has the weak* and $[-1, 1]^W$ the product topology. It follows that $P(K)$ is affinely homeomorphic to a compact convex subset of $[-1, 1]^W$.

The proof of the lemma will be complete if we prove that $\Phi(P(K)) \subset \Sigma([0, 1]^W)$, since clearly in this case the support of every measure μ would be metrizable (and so separable).

Indeed, since K is a pointwise compact convex subset of $\Sigma([-1, 1]^\Gamma)$, Φ an affine continuous 1-1 mapping and every measure $\mu \in P(K)$ is a weak* limit of convex combinations of Dirac measures, it follows easily that the space $r_\Gamma(\Phi(P(K))) = \Phi(P(K)) | \Gamma \subset \mathbf{R}^\Gamma$ is identified with K, namely, if $\mu \in P(K)$ then there exists $x \in K$ such that $\mu(\pi_\gamma) = x(\gamma)$ for all $\gamma \in \Gamma$.

Let now $\mu \in P(K)$. Suppose that there exists an uncountable subset A of W with $\mu(f) \neq 0$ for all $f \in A$. Then clearly there is $B \subset L$ uncountable with $\mu(\pi_\gamma) \neq 0$ for $\pi_\gamma \in B$, in contradiction to the preceding remarks.

3.4. Definition. Given a set Γ, $l^{\infty}_{\omega+}(\Gamma)$ denotes the set of those $f \in l^{\infty}(\Gamma)$ for which there exists an at most countable subset A of Γ such that $f(x) = 0$ for all $x \in \Gamma \setminus A$ (so $l^{\infty}_{\omega+}(\Gamma) = l^{\infty}(\Gamma) \cap \Sigma(\boldsymbol{R}^{\Gamma})$). It is clear that $l^{\infty}_{\omega+}(\Gamma)$ is a Banach space, since it is a closed linear subspace of $l^{\infty}(\Gamma)$.

Remark. If Γ is uncountable then the Banach space $l^{\infty}_{\omega+}(\Gamma)$ is not strictly convexifiable, as M. Day proved in [Da].

Now we state the characterization of Corson-compact spaces with (M). First, we notice that property (M) for a Corson-compact K is equivalent to the property that every measure on K has metrizable support (since every separable Corson-compact is metrizable, according to a remark in 1.1).

3.5. Theorem. *For a Corson-compact K the following are equivalent*:

(a) *K has property* (M).

(b) *The Banach space $C(K)$ is weakly Lindelöf.*

(c) *There exist a set Γ and a bounded linear one-to-one operator T: $M(K) \to l^{\infty}_{\omega+}(\Gamma)$ which is also weak* to pointwise continuous.*

(d) *The closed unit ball $B_{M(K)}$ of the dual space $M(K)$ of $C(K)$ is Corson-compact in the weak* topology.*

(e) *$B_{M(K)}$ is angelic in the weak* topology.*

(f) *$C(K)$ has property* (C).

Proof. Suppose that $K \subset \Sigma([-1, 1]^{\Gamma})$ and consider the sets L, W as in Lemma 3.3.

(a) $\Rightarrow$ (c). Consider the operator T: $M(K) \to l^{\infty}(W)$ given by $T(\mu) = (\mu(f))_{f \in W}$. Then T is clearly linear, weak* to pointwise continuous and because W is a total subset of $C(K)$ it is also one-to-one. Since the support of every μ is metrizable, it follows that the range of T is contained in $l^{\infty}_{\omega+}(W)$. (Cf. also the proof of Lemma 3.3.)

We verify that T is bounded: indeed, since W is contained in the closed unit ball B of $C(K)$ we have for every $\mu \in M(K)$

$$\|T(\mu)\| = \sup_{f \in W} |\mu(f)| \leqslant \sup_{f \in B} |\mu(f)| = \|\mu\|.$$

(c) $\Rightarrow$ (d). It is clear that $(B_{M(K)}, w^*)$ is (affinely) homeomorphic to the pointwise compact (and convex) subset $T(B_{M(K)})$ of $l^{\infty}_{\omega+}(\Gamma)$, and so it is a Corson-compact.

(d) $\Rightarrow$ (e). Every Corson-compact is angelic as noted in the preliminaries.

(e) $\Rightarrow$ (f) and (f) $\Rightarrow$ (d) follow by Remark 3.2.

So far we have established the equivalence of claims (a), (c), (d), (e) and (f).

(d) $\Rightarrow$ (b). Consider the natural isometric embedding Φ of $C(K)$ in $C(B_{M(K)})$ given by $\Phi(\mu)(f) = \mu(f)$. Then clearly Φ: $C(K) \to \Phi(C(K))$ is also a homeomorphism if $C(K)$ is endowed with the weak topology and $\Phi(C(K))$ with the pointwise topology; moreover, $\Phi(C(K))$ is a pointwise closed linear subspace of $C(B_{M(K)})$.

Now the conclusion follows immediately from the fact that for every Corson-compact K, $C(K)$ is pointwise Lindelöf, as proved by Alster–Pol [A-P] and Gul'ko $[\mathrm{Gu}]_1$.

(b) $\Rightarrow$ (d) follows from the implication (b) $\Rightarrow$ (f) (cf. Remark 3.2) and (f) $\Rightarrow$ (d) shown already.

The proof of the theorem is now complete.

The following corollary is essentially a "Krein-type" theorem for the locally convex space $\Sigma(\boldsymbol{R}^{\Gamma})$.

3.6. Corollary. *Let K be a compact subset of $\Sigma(\boldsymbol{R}^{\Gamma})$. Then the following are equivalent*:

(a) *K has property* (M).

(b) *The* (*pointwise*) *closure of the convex hull of K in $\Sigma(\boldsymbol{R}^{\Gamma})$ is compact.*

Proof. (b) $\Rightarrow$ (a) follows immediately from Lemma 3.3.

(a) $\Rightarrow$ (b). It is easy to see that the closure of the convex hull of K is the subset $\Phi(P(K))|\Gamma$ of $\Sigma([-1, 1]^{\Gamma})$ where Φ: $P(K) \to \Sigma([-1, 1]^{W})$ is the mapping given in the proof of Lemma 3.3. Since $\Phi(P(K))$ is pointwise compact and convex and the projection mapping r_{Γ}: $\boldsymbol{R}^{W} \to \boldsymbol{R}^{\Gamma}$ is continuous and affine, we have the proof of the claim.

It is known that if K is a Gul'ko-compact space, then the density character of $C(K)$ is equal to the weak* density character of the dual space $M(K)$ (cf. $[\mathrm{Ta}]_1$, Th. 6.2, Prop. 6.3). The same result holds for the much wider class of Corson-compact spaces with property (M).

3.7. Corollary. *Let K be a Corson-compact space with property* (M). *Then the density character of the Banach space $C(K)$ is equal to the weak* density character of the space of measures $M(K)$ on K.*

Proof. Suppose that $K \subset \Sigma(\boldsymbol{R}^{\Gamma})$ and $\Gamma = \bigcup \{\mathrm{supp}(x): x \in K\}$. Then according to standard results and remarks in 1.1 we have

$$\begin{aligned} (1) \qquad |\Gamma| = d(K) = w(K) = \dim C(K) = w(B_{M(K)}, w^*) \\ \geqslant d(B_{M(K)}, w^*) \geqslant d(M(K), w^*). \end{aligned}$$

Let T: $M(K) \to l^{\infty}_{\omega+}(\Gamma)$ be the operator given in Theorem 3.5(c). Then since T is weak* to pointwise continuous, it follows that

$$(2) \qquad |\Gamma| = d(T(M(K))) \leqslant d(M(K), w^*)$$

(see also the proof of the implication (a) $\Rightarrow$ (c) of Theorem 3.5). From (1) and (2) we have

$$|\Gamma| = \dim C(K) = d(M(K), w^*),$$

which is the desired conclusion.

3.8. Proposition. *Let K be a Corson-compact space with property* (M). *Then*:

(a) *Every Hausdorff continuous image of K is Corson-compact with* (M).

(b) *If Ω is a compact space and $C(\Omega)$ is isomorphic to a closed linear subspace of $C(K)$, then Ω is Corson-compact with* (M).

(c) *If $\{K_n: n < \omega\}$ is a sequence of Corson-compact spaces with* (M), *then the space $K = \prod_{n<\omega} K_n$ is Corson-compact with* (M).

Proof. Let Ω be a compact Hausdorff space.

(a) If φ: $K \to \Omega$ is a continuous onto mapping, then (see $[\mathrm{Gu}]_1$ and [Mi-R]) Ω is a Corson-compact. Let $\mu \in P(\Omega)$; by the Hahn–Banach Theorem there exists $\nu \in P(K)$ such that $\mu = \varphi(\nu)$ (μ is the image of ν under φ). Since ν has separable support and the support of μ is the image of the support of ν under φ, it follows immediately that μ has separable support.

(b) By Theorem 3.5, the closed unit ball of $M(K)$ is Corson-compact (with property (M)). If T: $C(\Omega) \to C(K)$ is an isomorphic embedding, then the dual operator T^*: $M(K) \to M(\Omega)$ is a weak* to weak* onto mapping, so Ω has property (M) (use claim (a)).

(c) We notice that K is Corson-compact. Let $\mu \in P(K)$, let π_n: $\prod_{n<\omega} K_n \to K_n$ be the projection onto the nth coordinate space, and let $\mu_n = \pi_n(\mu) \in P(K)$ be the image of μ under π_n for $n < \omega$.

Since the support of μ is contained in the cartesian product of the supports of μ_n, $n < \omega$, and the support of every μ_n is metrizable, the proof of the claim follows.

3.9. Remark. From claim (b) of Proposition 3.8 it follows that if K, Ω are compact spaces such that $C(K)$ and $C(\Omega)$ are isomorphic Banach spaces, and K is a Corson-compact space with property (M), then Ω is a Corson-compact with (M).

We do not know the answer to the following

Problem. If K is a Corson-compact space, Ω a compact space, and $C(K)$, $C(\Omega)$ isomorphic Banach spaces, does it follow that Ω is Corson-compact? (The answer is yes if we assume the negation of the continuum hypothesis and Martin's Axiom, because then every Corson-compact space has property (M), according to Remark 3.2, 3).)

If we assume CH, there exist Corson-compact spaces without property (M); such an example is the Kunen–Haydon–Talagrand space ([N], Th. 5.9).

We will give, also assuming CH, a simpler example of this type, by using the known example of Erdös of a compact nonseparable space with a strictly positive measure and (assuming CH) without caliber ω^+ (see [C-N], Ths. 6.28, 6.21 and C6).

We describe the space of Erdös.

3.10. *The space of Erdös.* Let $I = [0, 1] \subset \boldsymbol{R}$ and let λ be the Lebesgue measure on I. Let Ω denote the Stone space of the quotient algebra M_λ/N_λ, where M_λ is the algebra of λ-measurable sets and N_λ the ideal of λ-null sets. Since M_λ/N_λ is complete, Ω is a compact extremally disconnected space (cf. [C-N]). Ω is the space of Erdös.

On Ω there exists a unique strictly positive (regular Borel) normal measure $\tilde{\lambda}$ determined by the condition that $\tilde{\lambda}(V) = \lambda(U)$ for V any open-and-closed subset in Ω, and U a measurable subset of I such that $U + N_\lambda = V$. Ω, as Erdös proved assuming CH, does not have caliber ω^+. We shall prove a stronger result.

3.11. Lemma. *We assume CH. Then the Erdös space Ω has a pseudobase $\{V_\xi: \xi < \omega^+\}$ that witnesses the failure of property caliber ω^+.*

Proof. Let $\{x_\xi: \xi < \omega^+\}$ and $\{K_\xi: \xi < \omega^+\}$ be one-to-one well-orderings of I and of the class of compact subsets of I with strictly positive Lebesgue measure respectively.

For every $\xi < \omega^+$ we choose a compact subset U_ξ of I such that:

(a) $U_\xi \subset \{x_\zeta: \xi < \zeta < \omega^+\}$.

(b) $U_\xi \subset K_\xi$.

(c) $\lambda(U_\xi) > 0$.

It is clear, by the regularity of the Lebesgue measure and since $\lambda(\{x_\zeta: \xi < \zeta < \omega^+\}) = 1$ for every $\xi < \omega^+$, that such a choice is possible.

The family $\{V_\xi: \xi < \omega^+\}$ of open-and-closed subsets of Ω, where $V_\xi = U_\xi + N_\lambda$ (= the λ-equivalence class of U_ξ), is the desired pseudobase.

Indeed, let V be an open-and-closed subset of Ω; then $V = U + N_\lambda$ for some measurable subset U of I. Since $\tilde{\lambda}(V) = \lambda(U) > 0$, there exists $\xi_0 < \omega^+$ with $K_{\xi_0} \subset U$, therefore $U_{\xi_0} \subset U$. It follows that $V_{\xi_0} = U_{\xi_0} + N_\lambda \subset U + N_\lambda = V$.

Now if $\bigcap_{\xi \in A} V_\xi \neq \emptyset$ for some uncountable subset A of ω^+, then the family $\{U_\xi: \xi \in A\}$ has the finite intersection property; but this family consists of compact subsets of I, so $\bigcap_{\xi \in A} U_\xi \neq \emptyset$, which is impossible according to (a).

3.12. Theorem. *We assume CH. Then there is a Corson-compact ccc nonmetrizable space L with the following properties*:

(a) *L is the support of a strictly positive* (*regular Borel*) *normal measure of countable type.*

(b) *The Gleason space $G(L)$ of L is the space Ω.*

Proof. Let Ω be the space of Erdős (defined in 3.10) and let $\{V_\xi\colon \xi < \omega^+\}$ be a pseudobase of Ω as in Lemma 3.11.

Set $A = \{A \subset \omega^+\colon \bigcap_{\xi\in A} V_\xi \neq \emptyset\}$. It is easily seen, by the properties of the family $\{V_\xi\colon \xi < \omega^+\}$, that A is an adequate family of countable subsets of ω^+, so $K = K_A$ is a Corson-compact space.

We define a continuous mapping $T\colon \Omega \to K$ by

$$T(x)(\xi) = \begin{cases} 1, & x \in V_\xi, \\ 0, & x \notin V_\xi, \end{cases}$$

and set $L = T(\Omega) \subset K$.

We shall prove that L is the desired Corson-compact space.

Let U be a nonempty open-and-closed subset of Ω. Then there exists $\xi < \omega^+$ such that $V_\xi \subset U$, so $T(V_\xi) \subset T(U)$; hence from the definition of T it follows that $T(U)$ has nonempty interior; hence T is an irreducible mapping. Since Ω is extremally disconnected, it follows that the Gleason space $G(L)$ of L is the space Ω (see Theorem 2.9(b)); so L is a nonmetrizable ccc space.

Now set $\mu = T(\tilde\lambda)$, the image of the measure $\tilde\lambda$ under T. It follows that μ is a strictly positive (regular Borel) probability measure of countable type on L, since $\tilde\lambda$ on Ω has the same properties.

Moreover, μ is a normal measure. Indeed, if A is a closed nowhere dense subset of L, then since T is irreducible, $T^{-1}(A)$ is a closed nowhere dense subset of Ω; therefore $\mu(A) = \tilde\lambda(T^{-1}(A)) = 0$, because $\tilde\lambda$ on Ω is normal; so μ is a normal measure on L.

The following theorem also indicates that the space $C(K)$ for K Corson-compact could have rich structure.

3.13. Theorem. *Let K be a Corson-compact space which is defined by an adequate family of sets. Then the following are equivalent*:

(a) *$l^1(\omega^+)$ is isomorphically embedded into $C(K)$.*

(b) *K does not have property* (M).

Proof. (a) $\Rightarrow$ (b). The usual base of $l^1(\omega^+)$ is a weakly closed, discrete and uncountable subset of $C(K)$, so $C(K)$ is not weakly Lindelöf, hence Theorem 3.5 implies the claim.

(b) $\Rightarrow$ (a). Let A be the adequate family that defines the set K as a closed subset of $\{0, 1\}^\alpha$ for some cardinal α. Consider a positive regular Borel probability measure μ on K without metrizable support. Hence there exists an uncountable subset I of α such that, for all $\xi \in I$, $\mu(V_\xi) > \vartheta$ for some $\vartheta > 0$, where $V_\xi = \pi_\xi^{-1}(\{1\}) \cap K$ and $\pi_\xi\colon \{0, 1\}^\alpha \to \{0, 1\}$ is the projection on the coordinate ξ.

Claim. *For any finite nonempty disjoint subsets I_1, I_2 of I there exists a positive regular Borel probability measure μ_1 on K so that*

$$\mu_1(V_\xi) \geqslant \vartheta \quad \textit{for all } \xi \in I_1, \qquad \mu_1(V_\zeta) = 0 \quad \textit{for all } \zeta \in I_2.$$

Consider the projection $r_{I_1}\colon \{0, 1\}^\alpha \to \{0, 1\}^{I_1}$. Then the set $L = r_{I_1}(K) \times \{0\}^{\alpha \setminus I_1}$ is a retract of K; hence on L we have the image $r_{I_1}(\mu)$ of the measure μ. It is easy to see that this is the desired measure μ_1.

We now show that the family $\{\chi_{V_\xi}\colon \xi \in I\}$ is equivalent to the usual base of $l^1(I)$. Indeed, we consider real numbers $\lambda_{\xi_1}, \ldots, \lambda_{\xi_n}$ and we set $I_1 = \{\xi_i\colon \lambda_{\xi_i} > 0\}$, $I_2 = \{\xi_i\colon \lambda_{\xi_i} < 0\}$. We may assume that

$$\sum_{\xi_i \in I_1} \lambda_{\xi_i} \geqslant \tfrac{1}{2} \sum_{i=1}^n |\lambda_{\xi_i}|.$$

Now we have

$$\Big\|\sum_{i=1}^n \lambda_{\xi_i} \chi_{V_{\xi_i}}\Big\| \geqslant \mu_1\Big(\sum_{i=1}^n \lambda_{\xi_i} \chi_{V_{\xi_i}}\Big) = \mu_1\Big(\sum_{\xi_i \in I_1} \lambda_{\xi_i} \chi_{V_{\xi_i}}\Big)$$

$$\geqslant \vartheta \sum_{\xi_i \in I_1} \lambda_{\xi_i} \geqslant \tfrac{1}{2}\vartheta \sum_{i=1}^n |\lambda_{\xi_i}|.$$

This completes the proof of the theorem.

Remarks. 1) The above Theorem 3.13 does not hold if the Corson-compact K does not result from an adequate family of sets, as the Kunen–Haydon–Talagrand example ([N], Th. 5.9) shows.

2) If K is a Corson-compact then $l^1(\omega^+)$ is not isometrically embedded in $C(K)$, since in that case the compact space $[0, 1]^{\omega^+}$ would be a continuous image of K.

The following proposition is a generalization of a result of Corson.

3.14. Proposition. *Let K be a compact, convex and balanced subset of the locally convex space $\Sigma(\mathbf{R}^\Gamma)$. Then there exists a compact subset D of K with a unique limit point* (*and so D is homeomorphic to the one-point compactification of a discrete set*) *such that the cardinal of D is equal to the* (*topological*) *weight of K.*

Proof. Suppose that $\Gamma = \bigcup_{x\in K} \operatorname{supp}(x)$ and $|\Gamma| \geqslant \omega^+$ (in the case $|\Gamma| \leqslant \omega$, such a set D exists: indeed, consider a convergent sequence $x_n \to x$ in K and set $D = \{x_n\colon n < \omega\} \cup \{x\}$). It is easy to construct by transfinite induction, using Lemma 1.2, a family $\{A_\alpha\colon \omega \leqslant \alpha < |\Gamma|\}$ of subsets of Γ with the following properties (cf. [B]):

(a) $A_\alpha \cap A_\beta = \emptyset$ for $\omega \leqslant \alpha < \beta < |\Gamma|$.

(b) $\bigcup_{\omega \leqslant \alpha < |\Gamma|} A_\alpha = \Gamma$, $|A_\alpha| \leqslant |\alpha| < |\Gamma|$ for $\omega \leqslant \alpha < |\Gamma|$.

(c) If $\omega \leqslant \alpha < |\Gamma|$ then $K | \bigcup \{A_\alpha\colon \beta < \alpha\} \subset K$ and $K | A_\alpha \subset 2\cdot K$ where $2\cdot K = \{2\cdot k\colon k \in K\}$.

For every α with $\omega \leqslant \alpha < |\Gamma|$ we choose $x_\alpha \in K | A_\alpha \subset 2\cdot K$ and we set $y_\alpha = \frac{1}{2} x_\alpha$, so $y_\alpha \neq y_\beta$ if $\alpha \neq \beta$.

It is easily seen that the set $D = \{y_\alpha\colon \omega \leqslant \alpha < |\Gamma|\} \cup \{0\}$ has the desired properties.

The above proposition implies the following classical result of Corson ([L], Prop. 3.4).

3.15. Corollary (Theorem of Corson). *Let K be a weakly compact, convex and balanced subset of a Banach space E. Then K contains a weakly compact set D with a unique weak limit point such that the cardinal of D is equal to the weight of K.*

Proof. Suppose that E is a WCG Banach space (if it is not WCG, then we replace E with the closed linear hull of K in E). From the Amir–Lindenstrauss Theorem [Am-L] there exists a bounded linear one-to-one operator $T\colon E \to c_0(\Gamma)$ for some set Γ; since the set $\Omega = T(K)$ is a subset of $c_0(\Gamma)$ affinely homeomorphic to K and $c_0(\Gamma)$ is clearly contained in $\Sigma(\boldsymbol{R}^\Gamma)$, the conclusion follows from Proposition 3.14.

Remarks. 1) It is necessary to assume that K is a balanced subset of $\Sigma(\boldsymbol{R}^\Gamma)$. Indeed, there exists a nonmetrizable weakly compact convex subset of $c_0(\Gamma)$ ([L]) which is a first countable space. It is clear that such a space cannot contain an uncountable compact subset with a unique weak limit point.

2) In the following section we shall give an example of a non-Gul'ko-compact Corson-compact space (Theorem 4.4) with property (M).

We conclude this section with some comments on smoothness and some open questions.

It is known that if E is a WCG Banach space then there exists on E an equivalent smooth norm $\|\cdot\|$ (i.e. for every $x \in E$ with $\|x\| = 1$ there is only one $x^* \in E^*$ with $\|x^*\| = 1 = x^*(x)$) whose dual norm is strictly convex ([Am–L]). The same result is proved in $[\mathrm{M}]_2$ for the more general class of WCD Banach spaces.

Since every WCD Banach space is weakly Lindelöf and in view of Theorem 3.5, the following two questions seem natural.

Q1. Let K be a Corson-compact space and have property (M) (equivalently, $C(K)$ is weakly Lindelöf according to Theorem 3.5). Does $C(K)$ admit an equivalent smooth norm?

Q2. Suppose that K is a Corson-compact space and admits an equivalent (necessarily smooth) norm having strictly convex dual. Does it follow that K has property (M)?

We note that if the Corson-compact K results from an adequate family of sets, then Q2 has an affirmative answer, by using Theorem 3.13 and the fact that $l^1(\omega^+)$ does not admit an equivalent smooth norm ([Da]).

4. Corson-compact spaces of bounded order type. If K is a compact subset of $\{0,1\}^\alpha$ for some infinite cardinal α, and $\operatorname{supp}(x)$ is finite for all $x \in K$, then K is Eberlein-compact since clearly $K \subset c_0(\alpha)$.

On the other hand, there are examples of compact subsets K of $\{0,1\}^{\omega^+}$ such that

$$\text{order type}(\operatorname{supp}(x)) \leqslant \omega + 1 \quad \text{for all } x \in K$$

which are not Gul'ko-compact ([Ar-M-N], $[\mathrm{M}]_1$; see also Remark 6.59 of [N]).

We have asked in [Ar-M-N] what happens if

$$\text{order type}(\operatorname{supp}(x)) \leqslant \omega \quad \text{for all } x \in K.$$

We give an example below of a non-Gul'ko-compact that satisfies this "rareness" condition (and so is Corson-compact) and in addition has some other interesting properties.

We need some preliminaries.

Notation (cf. [N]). We denote by Σ the Baire space of infinite sequences of natural numbers ω^ω. S denotes the set of finite sequences of natural numbers. For $s \in S$ we denote by $|s|$ the length (i.e. the domain) of s. If $\sigma \in \Sigma$ and $n < \omega$, $\sigma | n$ denotes the finite sequence of the first n terms of σ. If $s \in S$ and $n < \omega$, then s, n denotes the finite sequence of length $|s|+1$ whose first $|s|$ terms are s and whose last term is n.

4.1. Lemma ([N]). *Let X be a countably determined space, i.e. a continuous image of a closed subset of a space of the form $M \times K$, where K is a compact space and M a separable metric space. Then there is a family $\{A_s\colon s \in S\}$ of subsets of X such that*

$$A_\emptyset = X, \qquad \bigcup_{k<\omega} A_{s,k} = A_s \quad \textit{for } s \in S,$$

and for every $x \in X$ there is $\sigma \in \Sigma$ such that:

(i) $x \in \bigcap_{k<\omega} A_{\sigma|k}$.

(ii) *If $x_k \in A_{\sigma|k}$ for $k < \omega$, then the sequence (x_k) has a limit point in X.*

If A is an adequate family of subsets of some set T (see Definition 1.8), then we consider $T^* = T \cup \{\infty\}$ and define on T^* a topology as follows: every element of T is isolated; a subbase for the neighborhoods of ∞ is the family

$$\{\{\infty\} \cup (T \setminus A)\colon A \in A\}.$$

We state without proof the following fact, due to Talagrand ($[\mathrm{Ta}]_1$, Th. 4.2).

4.2. Proposition. *Let A be an adequate family of subsets of T. Then K_A is Gul'ko-compact if and only if T^* is countably determined.*

A subset S of a cardinal α is called *stationary* if S intersects every closed unbounded subset of α (in the order topology on α). We have the following result of Fodor ([N], Th. 0.2) on stationary sets.

4.3. THEOREM. *Let S be a stationary subset of an uncountable regular cardinal α, and let $f: S \to \alpha$ be a function such that $f(\xi) < \xi$ for $\xi \in S$, $\xi \neq 0$. Then there is a stationary subset T of S, and $\zeta < \alpha$, such that $f(\xi) = \zeta$ for all $\xi \in T$.*

4.4. THEOREM. *There is a Corson-compact totally disconnected space K such that*:

(a) *K is not a Gul'ko-compact.*

(b) *K is a Rosenthal-compact and so has property* (M).

(c) *$K \subset \{0, 1\}^{\omega^+}$ and* $\operatorname{order\,type}(\operatorname{supp}(x)) \leqslant \omega$ *for all $x \in K$.*

Proof. We define a family $\{N_\xi: \xi < \omega^+\}$ of subsets of ω as follows: We choose $\{N_k: k < \omega\}$ such that $|N_k| = \omega$, $N_k \cap N_l = \emptyset$ for $k < l < \omega$. Let $\omega \leqslant \xi < \omega^+$, and suppose that $\{N_\zeta: \zeta < \xi\}$ have been defined, and are infinite and almost disjoint. Let $\{\zeta_n^\xi: n < \omega\}$ be a 1-1 well-ordering of ξ. We choose

$$F_n^\xi \subset N_{\zeta_n^\xi} \setminus \bigcup_{k<n} N_{\zeta_k^\xi}$$

such that $|F_n^\xi| = n$ for $n < \omega$. We set $N_\xi = \bigcup_{n<\omega} F_n^\xi$. It is clear that $\{N_\xi: \xi < \omega^+\}$ is a family of infinite almost disjoint subsets of ω.

We define $\Phi: (\omega^+)^{[2]} \to \omega$ (see preliminaries) by $\Phi(\xi, \zeta) = |N_\xi \cap N_\zeta|$. Let T be a subset of the interval $[0, 1]$ with $|T| = \omega^+$. Also let $\{x_\xi: \xi < \omega^+\}$ be a 1-1 well-ordering of T. A finite subset $J = \{\xi_1 < \ldots < \xi_n\} \subset \omega^+$ is called *Φ-admissible* if for all $\xi, \zeta \in J$ with $\xi < \zeta$ we have:

a) $|\{\eta \in J: \xi < \eta < \zeta\}| \leqslant \Phi(\xi, \zeta)$.

b) $|x_{\xi_i} - x_{\xi_j}| \leqslant 1/i$ for $1 \leqslant i \leqslant j \leqslant n$.

We set $\mathcal{A} = \{A \subset \omega^+$: every finite subset of A is Φ-admissible$\}$. It follows immediately that $\mathcal{A}$ is an adequate family of subsets of ω^+. We set $K = K_{\mathcal{A}}$. It is clear by using condition a) that if $x \in K$, then $x = \chi_A$ for $A \in \mathcal{A}$, and $\operatorname{order\,type}(A) \leqslant \omega$. (So K is a Corson-compact.) Also we note that every $A \in \mathcal{A}$, by b), has a unique limit point in $[0, 1]$, so A is a countable G_δ subset of $[0, 1]$ and hence $\chi_A: [0, 1] \to \boldsymbol{R}$ is a Baire-1 function, implying that K is a Rosenthal-compact. Therefore according to Remark 3.2, K has property (M).

Finally, we prove that K is not Gul'ko-compact. By Proposition 4.2, it is enough to prove that $T^* = T \cup \{\infty\}$ is not countably determined. If it is then according to Lemma 4.1 there is a family $\{K_s: s \in S\}$ of subsets of ω^+ such that $K_\emptyset = \omega^+$, $\bigcup_{k<\omega} K_{s,k} = K_s$ for $s \in S$, and for every $\xi < \omega^+$ there is $\sigma \in \Sigma$ such that $\xi \in \bigcap_{k<\omega} K_{\sigma|k}$ and if $\xi_k \in K_{\sigma|k}$ for $k < \omega$ then $\{\xi_k: k < \omega\}$ has a limit point in T^* (necessarily the point ∞), and so this set does not belong to $\mathcal{A}$.

We set $L = \bigcup \{K_s: K_s$ not a stationary set in $\omega^+\}$. Then $\omega^+ \setminus L$ contains a closed unbounded subset of ω^+, and thus if G is a stationary subset of ω^+ then $|G \setminus L| = \omega^+$.

For $s \in S$, $|s| = k$, with $K_s \not\subset L$, we define $f^s: K_s \to \omega^+$ by $f^s(\xi) = \max\{\zeta_1^\xi, \ldots, \zeta_k^\xi\} < \xi$. From Fodor's Theorem (Theorem 4.3), there is $L_s \subset K_s$, L_s stationary, and $\xi_s < \omega^+$, such that $f^s(\xi) = \xi_s$ for $\xi \in L_s$.

We choose $\xi < \omega^+$ with $\xi \notin L$ and $\xi > \sup\{\xi_s: s \in S, K_s \not\subset L\}$. By the above property of the family $\{K_s: s \in S\}$ there is $\sigma \in \Sigma$ such that $\xi \in \bigcap_{k<\omega} K_{\sigma|k}$ and if $\zeta_k \in K_{\sigma|k}$ for $k < \omega$ then $\{\zeta_k: k < \omega\}$ is not in $\mathcal{A}$. We choose $\zeta_1 > \xi$, $\zeta_1 \in L_{\sigma|1}$, and inductively, $\zeta_k > \zeta_{k-1}$, $\zeta_k \in L_{\sigma|k}$. We claim that $\{\zeta_k: k < \omega\}$ satisfies a). In fact, if $k < n$ then $\Phi(\zeta_k, \zeta_n) \geqslant n$, since in the enumeration of $\zeta_n = \{\zeta_k^{\zeta_n}: k < \omega\}$, ζ_k appears after the nth position; hence

$$|N_{\zeta_k} \cap N_{\zeta_n}| = \Phi(\zeta_k, \zeta_n) \geqslant n \geqslant |\{\zeta_l: k < l < n\}| = n-(k+1).$$

It is clear that there exists a subsequence A of the sequence $\{x_{\zeta_k}: k < \omega\}$ that satisfies condition b), therefore since condition a) is hereditary for subsets, A belongs to $\mathcal{A}$.

This contradiction proves that K is not Gul'ko-compact.

4.5. Remarks. 1) By Theorem 3.5, the Banach space $C(K)$ provides still another example of a weakly Lindelöf space which is not a WCG (neither a WCD) Banach space. R. Pol in $[\mathrm{P}]_1$ was the first to construct such an example.

2) Alster and Pol [A-P] were the first to construct a non-Talagrand-compact Corson-compact space. Moreover, it can be proved that this space is not a Gul'ko-compact, and that it is a Rosenthal-compact.

3) In [N] (Th. 6.55) an example is given of a Talagrand-compact whose each element has support of order type at most ω, and which is not an Eberlein-compact.

4) In $[\mathrm{M}]_1$ (Ch. 4) are produced, using the examples of Galvin–Hajnal, Kunen, and Laver–Galvin [C–N], some other examples of non-Gul'ko-compact Corson-compact spaces with property (M). Some new chain conditions are also defined.

6) In connection with Theorem 4.4, we note that all the examples of Corson-compact spaces given in the present paper are pointwise compact subsets of the space of Baire-2 real-valued functions on $[0, 1]$ (i.e. the space of pointwise limits of sequences of Baire-1 real-valued functions on $[0, 1]$). We briefly indicate how to see this: if $f \in \Sigma(\boldsymbol{R}^\Gamma)$ then it is clear that f is a pointwise limit of a sequence of functions $f_n: \Gamma \to \boldsymbol{R}$, $n < \omega$, with finite support. Since in our examples $|\Gamma| \leqslant 2^\omega$, the space $\Sigma(\boldsymbol{R}^\Gamma)$ is naturally embedded in the space of Baire-2 real-valued functions on $[0, 1]$.

The remarks at the beginning of this section and the above example lead naturally to the following:

4.6. Definition. Let K be a Corson-compact space. Then:

(a) K is said to be of *type* ξ, where ξ is a countable ordinal, if K is a compact subset of $\Sigma(\{0,1\}^\alpha)$ for some infinite cardinal α such that if $A\subset\alpha$ and $\chi_A\in K$ then $\operatorname{order\,type}(A)\leqslant\xi$.

(b) K is said to be of *bounded order type* if it is of type ξ for some countable ordinal ξ.

Remarks. 1) According to the above definition, the Corson-compact space given in Theorem 4.4 is of type ω.

2) It is easy to prove that a Corson-compact K is totally disconnected if and only if K is homeomorphically embedded in $\Sigma(\{0,1\}^\Gamma)$ for some set Γ ([Mi–R]).

We shall prove that a considerable proper subclass of Corson-compact spaces is of type ω.

4.7. Proposition. *Every scattered Corson-compact space K is of type ω.*

Proof. K. Alster proved in [A] that every such space K is homeomorphically embedded in $c_0(\Gamma)$ (and so it is an Eberlein-compact) in such a way that if $x\in K$ then x is the characteristic function of some (necessarily) finite subset of Γ; therefore K is of type ω.

4.8. Remark. The preceding proposition does not imply that a scattered Eberlein-compact is of finite order type $n<\omega$. Indeed, it is easy to see that every such space is a *uniform Eberlein-compact* (i.e. homeomorphic to a weakly compact subset of a Hilbert space). But there is an example of a scattered nonuniform Eberlein-compact ([B-S]; see also [N], 6.52), so this space is not of finite order type.

2) We do not know if an Eberlein-compact totally disconnected space is of type ω, neither if a Gul'ko-compact totally disconnected space is of bounded order type.

A Corson-compact totally disconnected space is not necessarily of bounded order type. Indeed, we have the following result similar to a result in [Ar-N] for Gul'ko-compact spaces (see also preliminaries).

4.9. Proposition. *If K is a Corson-compact of bounded order type, then $S(K)=w(K)^+$. In particular, a ccc Corson-compact of bounded order type is metrizable.*

Proof. According to Definition 4.6, we identify K with a compact subset of $\Sigma(\{0,1\}^\alpha)$ for some infinite cardinal α such that if $A\subset\alpha$ and $\chi_A\in K$ then $\operatorname{order\,type}(A)\leqslant\xi$ for some fixed countable ordinal ξ.

Suppose that $\bigcup_{x\in K}\operatorname{supp}(x)=\alpha$ (whenever $w(K)=\alpha$) and $S(K)\leqslant w(K)$. For $\zeta<\alpha$ set $V_\zeta=\pi_\zeta^{-1}(\{1\})\cap K$ every such set is open and closed in K. Now consider a countable ordinal $\xi_1<\omega^+$ with $\xi<\xi_1$. Then by Theorem 7.2 of [C-N] there exists $A\subset\alpha$ with $\operatorname{order\,type}(A)=\xi$ and $\bigcap_{\xi\in A}V_\xi\neq\emptyset$; but if $x\in\bigcap_{\xi\in A}V_\xi$ then clearly $\operatorname{order\,type}(\operatorname{supp}(x))\geqslant\xi_1$, which is absurd.

It follows immediately from the above proposition that every space described in Theorem 2.3 is (under CH) a Corson-compact totally disconnected space which is not of bounded order type.

We note that property (M) of the compact space K of Theorem 4.4 also follows from the fact that K is of bounded order type, as the following result proves.

4.10. Corollary. *Every Corson-compact of bounded order type has property* (M).

Proof. Since the support of a positive measure on a compact space is ccc, the conclusion follows from the above proposition.

We finish this section by noting that we cannot extend the result of Proposition 4.9 to the class of Corson-compact spaces with property (M).

4.11. Proposition. *Assume CH. There are nonmetrizable ccc Corson-compact spaces with property* (M).

Proof. Consider a family $\mathscr{S}=\{V_\xi:\ \xi<\omega^+\}$ of nonempty open subsets of a topological space X and let K be the compact space generated by the family $\mathscr{S}$ as in Definition 2.1.

Claim. *If K does not have property* (M) *then $\mathscr{S}$ is a witness to the failure of property* (K_n) *for no $n<\omega$.*

Proof of the claim. Indeed, let $\mu\in P(K)$ be such that $\operatorname{supp}(\mu)$ is not metrizable. Then it follows that the set $L=\{\xi<\omega^+:\ \mu(W_\xi)>0\}$ is uncountable where $W_\xi=\{1\}_\xi\times\{0,1\}^{\omega^+\setminus\{\xi\}}\cap K$.

Therefore for each $n<\omega$ there exists an uncountable subset L_n of L so that any n elements of the family $\{W_\xi:\ \xi\in L_n\}$ have nonempty intersection ([C-N]). It is easy to see that the family $\{V_\xi:\ \xi\in L_n\}$ has the same property and hence the proof of the claim is complete.

The claim implies that each of the "pathological" Corson compact spaces of Theorem 2.3(b), (c), (d) is an example of a nonmetrizable ccc space with property (M).

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Received July 15, 1986 (2194)
Revised version March 2, 1987

Added in proof (January 1988). It can be proved that the example of Corson-compact K given in Th. 2.3(a) has also property (M). Indeed, suppose there exist $\mu \in P(K)$, an uncountable subset S of L and $\delta > 0$ such that $\mu(V_x) \geqslant \delta$ for all $x \in S$, where $V_x = \pi_x^{-1}(\{1\}) \cap K$ (cf. Th. 3.13). Then $k(\{V_x\colon x \in S\}) \geqslant \delta$ (1) (see preliminaries and Lemma 6.1 of [C-N]). Let T be an open interval of $\boldsymbol{R}$ contained in $\mathrm{cl}_{\boldsymbol{R}} S$ and let $n_0 < \omega$ be such that $T_{n_0} \subseteq T$ and $1/n_0 < \delta$ (cf. the proofs of Ths. 1.11 and 2.3 (a)). For every $k \in \{1, \ldots, n_0^2\}$ pick $x_k \in T_{n_0,k} \cap S$. Then writing $V_j = V_{x_j}$ we have $\mathrm{cal}\,\{V_1, \ldots, V_{n_0^2}\}/n_0^2 < n_0/n_0^2 < \delta$, which contradicts (1).