we get

$$S_k^* \pm f_k(t) \geqslant (S_k^* \pm S_{k,n})\left[\left(\frac{t}{r}\right)^{p_n} - \left(\frac{t}{r}\right)^{p_{n+1}}\right] \quad \text{for } 0 < t < r.$$

Hence we obtain easily

$$|f_k(t)| \geqslant |S_{k,n}|\left[\left(\frac{t}{r}\right)^{p_n} - \left(\frac{t}{r}\right)^{p_{n+1}}\right] - S_k^*\left(1 - \left[\left(\frac{t}{r}\right)^{p_n} - \left(\frac{t}{r}\right)^{p_{n+1}}\right]\right)$$

for $0 < t < r$ and all k, $n = 1, 2, \ldots$

Let c be a number such that $0 < c < 1$ and $(1+c)(b^\alpha - b^{1+\alpha}) > 1$. For each k, choose n_k such that $|S_{k,n_k}| > cS_k^*$. Since $\lim_{k\to\infty} S_{k,n}/S_k^* = 0$ we obtain $\lim_{k\to\infty} n_k = \infty$. If $n_k > n_0$ then

$$\begin{aligned} M &\geqslant |f_k(t_{n_k})| \geqslant |S_{k,n_k}|\left[\left(\frac{t_{n_k}}{r}\right)^{p_{n_k}} - \left(\frac{t_{n_k}}{r}\right)^{p_{n_k+1}}\right] \\ &\quad - S_k^*\left(1 - \left[\left(\frac{t_{n_k}}{r}\right)^{p_{n_k}} - \left(\frac{t_{n_k}}{r}\right)^{p_{n_k+1}}\right]\right) \geqslant S_k^*\left\{(1+c)\left[\left(\frac{t_{n_k}}{r}\right)^{p_{n_k}} - \left(\frac{t_{n_k}}{r}\right)^{p_{n_k+1}}\right] - 1\right\} \\ &\geqslant S_k^*\{(1+c)[b^\alpha - b^{p_{n_k+1}/p_{n_k}}] - 1\} \geqslant S_k^*[(1+c)(b^\alpha - b^{1+\alpha}) - 1], \end{aligned}$$

because $rb^{\alpha/p_n} \leqslant t_n \leqslant rb^{1/p_n}$ for each n.

It follows that the sequence $\{S_k^*\}$ is bounded, contrary to what was assumed at the beginning of the proof. Thus $|Z| = 0$. ∎

Remark 1. The proof of the theorem gives a stronger result. Namely, the assertion of the theorem is valid under the assumption that the set A of points of pointwise convergence of the sequence $\{f_k\}$ has the property: $|A \cap [1-\delta, 1]| > 0$ for each $\delta > 0$.

Remark 2. Let $\{p_n\}$ be a sequence of positive numbers as in the theorem. The space F of continuous functions on the interval $[0, 1]$ which are sums of power series of the form $\sum_{n=0}^{\infty} c_n t^{p_n}$ on the interval $[0, 1)$ has the following properties:

(i) If f is a continuous function and f is the pointwise limit of a sequence $\{f_k\}$ on $[0, 1]$ with $f_k \in F$ for $k = 1, 2, \ldots$ then $f \in F$.

(ii) F is a linear space different from $C[0, 1]$.

(iii) If $t_1, t_2, \ldots, t_k \in [0, 1]$ and $s_1, s_2, \ldots, s_k$ are real numbers then there exists a function $f \in F$ such that $f(t_i) = s_i$ for $i = 1, 2, \ldots, k$.

The above example solves Problem P80, Colloquium Mathematicum 2 (1951), p. 152.

Received January 26, 1987 (2272)

STUDIA MATHEMATICA, T. LXXXIX. (1988)

Factorizations of natural embeddings of l_p^n into L_r, I

by

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Abstract. In this paper we study quantitative aspects of the local $\mathscr{L}_p$-structure and the uniform approximation property of L_p, $1 \leqslant p \leqslant \infty$. Let $K > 1$. Given a subspace X of L_p with $\dim X = n < \infty$, the parameters $m_p(X, K)$ and $k_p(X, K)$ denote, respectively, the smallest dimension m of a superspace Y, $X \subset Y \subset L_p$, such that $d(Y, l_p^m) \leqslant K$ and the smallest rank of an operator u on L_p such that $\|u\| \leqslant K$ and $ux = x$ for $x \in X$.

We consider mainly the case $p = 1$. For some natural Euclidean subspaces $X \subset L_1$ we show that $m_1(X, K)$ and $k_1(X, K)$ are at least exponential in n, which in general cannot be improved. In fact, our lower estimates lead to new L_1-characterizations of Sidon sets (cf. Section 2). Analogous estimates are obtained in Section 3 in the case where $X \subset L_1$ is spanned by n i.i.d. r-stable random variables, $1 < r < 2$.

The case $p = \infty$ is treated in Section 4. We prove that $k_\infty(X, K) \leqslant m_\infty(X, K) \leqslant \exp(A(K)n)$ and, if $n > 1$, we show cases where $k_\infty(X, K) \geqslant \exp(\delta K^{-2} n)$, for some $\delta > 0$ and each $K > 1$.

Our method depends on analysis of factorizations of the embedding map $X \hookrightarrow L_p$. In Section 5 we show that a similar scheme can be applied also in the case of quotient maps onto some subspaces of L_1.

0. Introduction. In this paper and its sequel [FJS] we investigate quantitative aspects of the local $\mathscr{L}_p$-structure and the uniform approximation property of L_p. The two most basic questions can be phrased as follows:

($\mathscr{L}$) Given a subspace X of L_p (or $C(S)$, when $p = \infty$), $\dim X = n$, and a constant $K > 1$, estimate the smallest $m = m_p(X, K)$ such that there is a subspace Y of L_p with $X \subset Y$ and $d(Y, l_p^m) \leqslant K$. In particular, estimate $m_p(n, K) = \sup\{m_p(X, K)\colon \dim X = n\}$. (Here $d(Y, Z)$ is the Banach–Mazur distance coefficient

$$\inf\{\|T\|\,\|T^{-1}\|;\ T\colon Y \to Z \text{ is an onto isomorphism}\}.)$$

($\mathscr{U}$) Given a subspace X of L_p (or $C(S)$, when $p = \infty$), $\dim X = n$, and a constant $K > 1$, estimate the smallest $k = k_p(X, K)$ such that there is an

[1] Supported in part by NSF DMS-8316627.
[2] Supported in part by NSF DMS-8200584 and the U.S.-Israel BSF.
[3] Supported in part by NSF DMS-8500764.

operator $u\colon L_p \to L_p$ with $ux = x$ for $x \in X$, $\|u\| \leqslant K$, and $\operatorname{rank} u \leqslant k$. In particular, estimate

$$k_p(n, K) = \sup\{k_p(X, K)\colon \dim X = n\}.$$

Problems $(\mathscr{L})$ and $(\mathscr{U})$ are related, at least for small values of K, because by the Dor–Schechtman theorem [D], [Sc1], for each p there is a $K_p > 1$ so that every K-isomorph of an l_p^m space in L_p with $K < K_p$ is $f_p(K)$-complemented in L_p, and $f_p(K) \to 1$ as $K \to 1$.

Although they did not consider the problem of estimating the parameters $m_p(n, K)$ and $k_p(n, K)$ precisely, the concepts were introduced by Pełczyński and Rosenthal [PR], who showed that for all $K > 1$, all n, and all $1 \leqslant p \leqslant \infty$, $m_p(n, K)$ and $k_p(n, K)$ are finite. In fact, [PR] contains an argument, due to Kwapień, that $m_p(n, 1+\varepsilon)$ and $k_p(n, 1+\varepsilon)$ are of order no larger than $(n/\varepsilon)^{Cn}$ for some constant C.

In Section 1, we show that for natural n-dimensional Euclidean subspaces X of L_1, the uniformity functions $m_1(X, K)$ and $k_1(X, K)$ satisfy a lower exponential estimate $\exp \delta n$, where $\delta = \delta(K) > 0$ is a constant which is independent of n. "Natural" is broad enough to include the cases where X is spanned by independent gaussian variables, independent Rademacher functions, or any set of characters whose Sidon constant is bounded independently of n. The analytic condition which is relevant is that there exists a constant C so that

$$\|x\|_q \leqslant C\sqrt{q}\,\|x\|_1 \tag{0.1}$$

holds for all $2 \leqslant q < \infty$ and all $x \in X$. (Here and elsewhere we assume that the L_p space is $L_p(\mu)$ for some probability measure μ.) Weaker lower estimates (which, however, examples show are best possible) are obtained for $m_1(X, K)$ and $k_1(X, K)$ if X satisfies (0.1) for *some* $q > 2$ (e.g., X is the span of n characters from a Λ_q-set [Ru1]).

Note that the results of Section 1 (also those of Section 3) say something about the necessity for nonconstructive approaches to packing problems. For example, random techniques are used in [FLM] to prove that l_2^n $1+\varepsilon$-embeds into l_1^k with $k \leqslant C(\varepsilon)n$. By the results of Section 1, this phenomenon cannot be exhibited by finding an l_1^k superspace in L_1 to a natural n-dimensional Euclidean subspace of L_1.

In Section 2 we present a result of J. Bourgain which combines with the results of Section 1 to give an L_1 characterization of Sidon sets: A set Λ of characters on a compact abelian group G is Sidon if and only if for some (or any) $K > 1$, for each n, and for each n-element subset A of Λ, every operator T on $L_1(G)$ with $\|T\| \leqslant K$ and $T\gamma = \gamma$ for $\gamma \in A$ must have rank at least $\exp \delta n$ ($\delta = \delta(K) > 0$).

In Section 3 we estimate $m_1(X, K)$ and $k_1(X, K)$ from below when X is the span of n i.i.d. random variables with p-stable distribution for some $1 < p < 2$.

In Section 4 we show that $m_\infty(n, K) \leqslant \exp A(K)n$; the corresponding lower estimate is known. This gives the same kind of upper estimate for $k_\infty(n, K)$ and we observe in Proposition 4.3 that $k_\infty(X, K) \geqslant \exp \delta(K)n$ when X is an n-dimensional Euclidean subspace of $C(S)$.

Finally, in Section 5 we generalize the results of Sections 1 and 3, which are best thought of as factorization results for embedding mappings into L_1, to the setting of mappings into L_1 which act like quotient mappings relative to suitable subspaces of L_1. This gives a fourth proof of the most basic result in Section 1 (there are two proofs in Section 1, another in Section 3, and a fifth proof will be presented in [FJS]), so our presentation is not the most economical. On the other hand, each proof yields different constants, the proofs which give stronger results are at a deeper conceptual level, and the application to harmonic analysis in Section 2 requires only the most basic argument. Moreover, the context of Section 5 and the arguments in Section 3 do not seem natural until one has thoroughly analysed the situation considered in Section 1.

In regard to the questions $(\mathscr{L})$ and $(\mathscr{U})$, we have very good information for $p = \infty$ and good information for $p = 1$. Probably the main remaining problem in the latter case is whether $m_1(n, K)$ and $k_1(n, K)$ admit upper estimates of the form $\exp A(K)n$. In [F] a proof is given of such upper estimates for $m_1(X, K)$ and $k_1(X, K)$ when X is spanned by n i.i.d. gaussian variables. While this is perhaps the easiest nontrivial example, the argument is a bit complicated and does not appear to be applicable to other subspaces of L_1.

For $p \notin \{1, 2, \infty\}$, problems $(\mathscr{L})$ and $(\mathscr{U})$ are wide open. While the techniques of Section 1 can be used to estimate $k_p(n, K)$ for $1 < p < 2$ and small $K > 1$ (this is why we prove Proposition 1.2 for general p), the conclusions are so weak that, in the sequel, we do not treat the case $1 < p \neq 2 < \infty$. It may be that for $1 < p \neq 2 < \infty$ and some (or any) $K > 1$, there exist constants $s = s(p, K)$ and $A = A(p, K)$ so that $m_p(n, K) \leqslant An^s$ and $k_p(n, K) \leqslant An^s$. A Euclidean section argument shows that

$$m_p(n, k) \geqslant \delta_p(K)\,n^{q/2} \quad \text{for } 1 < p \neq 2 < \infty,$$

where $q = \max(p, p(p-1)^{-1})$, but we do not know any lower estimate for $k_p(n, K)$ when $1 < p \neq 2 < \infty$ except when K is close to 1.

We should note that the results of Sections 1, 3, and 5 are strengthened in [FJS]. Roughly speaking, whenever we give a result that any factorization of a certain operator into L_1 through an L_1 space Z must yield an operator $u\colon Z \to L_1$ which is "big" (e.g., u has large rank or uZ contains a copy of l_1^k with k large), in [FJS] we check that in fact the operator u must preserve a copy of l_1^k.

We use standard Banach space theory notation and terminology as may be found in [LT1], [LT2]. Less familiar terms are defined where they are introduced.

Work on this paper and on [FJS] was begun several years ago. Some of the results were obtained during visits to the Mittag-Leffler Institute and the Banach Space Theory Worskhop at the University of Iowa. The authors benefited from discussions with many people; in particular, J. Bourgain, B. Maurey, A. Pełczyński, G. Pisier, and J. Zinn made comments and suggestions which were useful for us.

1. Natural Euclidean subspaces of L_1. Let $0<p<r<2$, let X be a subspace of $L_p = L_p(\mu)$ for some probability measure μ, and suppose that u is an operator from some Banach space Z into L_p. We shall obtain lower estimates for the "size" of u in terms of two quantities:

(1) The γ_2-norm of some extension $\tilde{u}$: $Z \to X$ of the restriction of u to $u^{-1}X$; and

(2) The constant of equivalence between the L_p- and L_{r^*}-norms on X.

(Recall that for $1 \leqslant p \leqslant \infty$ and u an operator between Banach spaces, $\gamma_p(u\colon X \to Y) = \inf\{\|v\|\,\|w\|\colon w\colon X \to Z,\ v\colon Z \to Y,\ Z \text{ an } L_p \text{ space},\ u = vw\}$.)

When there is a good lower estimate for $\gamma_2(\tilde{u})$, one can conclude that the range of u must be large. To illustrate this, we outline now the briefest (but not the most elementary; see Remark 1 after Corollary 1.1) application of Proposition 1.2 and Theorem 1.3 to a special case: For some natural number n, let X be a subspace of L_1 spanned by n independent gaussian random variables or n independent Rademacher functions; it is well known that $\|x\|_{r^*} \leqslant \sqrt{r^*}\,\|x\|_1$ for $x \in X$. Suppose that u is an extension of the inclusion mapping i: $X \to L_1$ to an operator of norm $\leqslant K$ from a space containing X which is K-isomorphic to some L_1 space. Proposition 1.2 implies that $T_r(u)$, the type r-stable norm of u as defined later in this section and written $T_{r,1}(u)$ in Pietsch's book [Pie, p. 291], dominates, up to a constant depending on the bound K, $(r^*)^{-1/2}$ times the γ_2-norm of some extension of the identity on X to an operator from an L_1 space to X. Since this latter quantity is of order $n^{1/2}$, $T_r(u)$ is estimated from below by a constant α depending on K times $(n/r^*)^{1/2}$. Setting $r^* = \frac{1}{2}n\alpha^2$, we get

$$T_r(u)^{r^*} \geqslant \exp\delta n \quad (\delta = \delta(K) > 0).$$

It then follows from Pisier's method [Pis 3] (see [Sc 2]) that the range of u contains a copy of l_r^m with m of order $\exp\delta n$. The choice of r guarantees that l_r^m is 2-isomorphic to l_1^m, so we conclude:

Corollary 1.1. *Suppose that $X \subset Y \subset L_1$ where X is the span of n independent gaussian random variables or n independent Rademacher functions and the inclusion i: $X \to Y$ satisfies $\gamma_1(i\colon X \to Y) \leqslant K$. Then there exists a constant $\delta = \delta(K) > 0$ independent of n so that for some $m \geqslant \exp\delta n$, l_1^m is 2-isomorphic to a subspace of Y.*

Remark 1. If we weaken the conclusion of Corollary 1.1 to "$\dim Y \geqslant \exp\delta n$" then we do not need the concept of $T_r(u)$ or the results of [Pis 3], [Sc 2]. This more basic result is worked out in detail in Corollary 1.5. The direct argument provides better constants in the lower estimate for $\dim Y$ than can be deduced from Corollary 1.1.

Remark 2. Since $\gamma_1(i\colon X \to Y) \leqslant d(Y, l_1^k)$ ($k = \dim Y$), Corollary 1.1 and Corollary 1.5 say in particular that every L_1-superspace of X in L_1 must have dimension which is exponential in n, i.e., $m_1(X, K) \geqslant \exp\delta(K)n$, in the notation of the introduction.

Remark 3. Another special case of Corollary 1.1 or Corollary 1.5 is that if v: $L_1 \to L_1$, $\|v\| \leqslant K$, and $vx = x$ for $x \in X$, then $\operatorname{rank} v \geqslant \exp\delta(K)n$. That is, in the language of the introduction, the uniformity function for the bounded approximation property of L_1 satisfies the lower estimate $k_1(X, K) \geqslant \exp\delta(K)n$.

Remark 4. The argument outlined above yields a quantity $\delta(K)$ in Corollary 1.1 which satisfies $\delta(K) \geqslant \tau K^{-2}$ for some absolute constant $\tau > 0$. Alternatively, this estimate is explicit in the derivation of Corollary 1.1 from Proposition 3.2 and the strengthened form of Corollary 1.1 given in Section 5.

Before stating Proposition 1.2, we introduce notation which will occur frequently. Given a probability measure μ and a subspace X of $L_p(\mu) \cap L_q(\mu)$, $0 < p, q \leqslant \infty$, we let $i_{p,q}^X$ denote the formal inclusion mapping from $(X, \|\cdot\|_p)$ into L_q and set

$$D_{p,q}(X) = \|i_{p,q}^X\|.$$

(The mnemonic is "divide", because

$$D_{p,q}(X) = \sup\{\|x\|_q/\|x\|_p\colon 0 \neq x \in X\}.)$$

Given an operator u: $Z \to L_p$, we set

$$C_{p,q}(u) = \inf\{\|h\|_s \|h^{-1}u\colon Z \to L_q\|\}$$

where the inf is over all appropriate *changes* (mnemonic) of measures; i.e., over all $0 < h \in L_s(\mu)$ where $1/q + 1/s = 1/p$. Thus

$$C_{p,p}(u) = \|u\colon Z \to L_p\|$$

and for $0 < p \leqslant q_0 < q_1 \leqslant \infty$, $1/q = (1-\alpha)/q_0 + \alpha/q_1$,

$$C_{p,q}(u) \leqslant C_{p,q_0}(u)^{1-\alpha} C_{p,q_1}(u)^{\alpha}.$$

If X is a subspace of $L_p(\mu)$,

$$C_{p,q}(X) \equiv C_{p,q}(i_{p,q}^X) = \inf\{D_{p,q}(h^{-s/p}X)\colon\ h > 0,\ \int h^s\,d\mu = 1\},$$

where the parameter $D_{p,q}(h^{-s/p}X)$ is computed relative to the spaces $L_p(h^s\,d\mu)$ and $L_q(h^s\,d\mu)$. Or, if one prefers to speak in terms of the one measure μ,

$$C_{p,q}(X) = \inf\{D_{p,q}(W)\}$$

where the inf is over all subspaces W of $L_p(\mu)$ which are the image of X under an isometry from $L_p(\mu)$ onto $L_p(\mu)$.

Another interpretation of $C_{p,q}(u)$ is given by a theorem of Maurey [M1, Theorem 2]: Given a subset A of $L_p(\mu)$ and $0 < p,\ q \leqslant \infty$, let

$$C'_{p,q}(A) = \sup\{[\int(\sum t_i|f_i|^q)^{p/q}\,d\mu]^{1/p}\}$$

where the sup is over all convex combinations of all finite subsets (f_i) of A. Maurey's result states that for any subset A of L_p,

$$C'_{p,q}(A) = \inf_h\{\|h^{-1}f\|_q\colon\ f\in A\},$$

where h ranges over all positive functions satisfying $\int h^s\,d\mu = 1$ and $1/q+1/s = 1/p$. (In fact, Maurey checks that the inf is obtained if one allows nonnegative h and adopts the convention that $0/0 = 0$.) Therefore, if $u\colon Z\to L_p$ is any operator, then

$$C_{p,q}(u) = C'_{p,q}(u\,\mathrm{Ball}\,Z).$$

For $0 < p < q < 2$, let $(g_i)_{i=1}^\infty$ be independent symmetric q-stable random variables normalized in L_p. Following [Pie], denote by $T_{q,p}(u)$ the smallest constant C such that

$$\Big(\int\Big\|\sum_{i=1}^n g_i u x_i\Big\|^p\Big)^{1/p} \leqslant C\Big(\sum_{i=1}^n \|x_i\|^p\Big)^{1/p}$$

for all vectors $(x_i)_{i=1}^n$ in the domain of u. It is quite easily seen that

$$T_{q,p}(u) = C'_{p,q}(u\,\mathrm{Ball}\,Z)$$

when u is an operator from Z into L_p. For $p = 1$ we write $T_q(u) = T_{q,1}(u)$ and call $T_q(u)$ the *type q-stable norm* of u. Note that $T_q(u)$ differs from Pisier's $ST_q(u)$ [Pis 3] by a constant factor which depends on q.

PROPOSITION 1.2. *Suppose that $0 < p < 2 < r^* \leqslant \infty$ with $p \leqslant r$ ($p = 1$ is the main case), $1/r+1/r^* = 1$, and define t by $1/r+1/t = 1/p$ (so $t = r^*$ when $p = 1$). Assume that X is a closed subspace of L_p and that $D_{p,r^*}(X) < \infty$. Let u be an operator from a Banach space Z into L_p for which $C_{p,r}(u) < \infty$ and set* *$Z_0 = u^{-1}X$. Then there exists an operator $\tilde{u}\colon Z\to X$ such that*

$$\tilde{u}x = ux \quad \textit{for } x\in Z_0,$$

$$\gamma_2(\tilde{u}\colon Z\to X) \leqslant \gamma_2(\tilde{u}\colon Z\to L_s) \leqslant 4^{1/t}D_{s,r^*}(X)\,C_{p,r}(u)$$

where $1/t+1/2 = 1/s$.

Proof. Since $s \geqslant p$ and

$$\gamma_2(\tilde{u}\colon Z\to X) = \gamma_2(\tilde{u}\colon Z\to L_p)$$

the left inequality is evident.

Since $C_{p,r}(u) < \infty$, we can find $0 \leqslant h_1 \in L_t$ so that $\|h_1\|_t = 1$ and

$$\|h_1^{-1}u\colon Z\to L_r\| = C_{p,r}(u).$$

Setting $h = \max\{1, h_1\}$, we have

$$\|h\|_t \leqslant 2^{1/t}, \quad \|h^{-1}u\colon Z\to L_r\| \leqslant C_{p,r}(u).$$

Let $P\colon L_2\to L_2$ be the orthogonal projection onto $h^{-1}X$. To see that this makes sense, first note that since the L_{r^*}- and L_p-norms are equivalent on the closed subspace X of L_p and $p < 2 < r^*$, X is closed in L_2. Moreover, mutliplication by h^{-1} is an L_2 isomorphism on X. Indeed, for x in X, we have

$$\|h^{-1}x\|_2 \leqslant \|h^{-1}\|_\infty\|x\|_2 \leqslant \|x\|_2,$$

$$\|x\|_2 \leqslant D_{p,2}(X)\|x\|_p \leqslant D_{p,2}(X)\|h\|_t\|h^{-1}x\|_r \leqslant 2^{1/t}D_{p,2}(X)\|h^{-1}x\|_2.$$

Define for $z\in Z$

$$\tilde{u}z = hP(h^{-1}uz).$$

Evidently, $\tilde{u}z = uz$ for $z\in Z_0$, but the norm estimate requires some comments.

FACT 1. *If P is the orthogonal projection onto the subspace H of L_2 and $1 < r < 2$, then*

$$\|P\colon L_r\to L_2\| = \|P\colon L_2\to L_{r^*}\| = D_{2,r^*}(H).$$

FACT 2. *If $0 < s < 2 < q < \infty$, $1/t+1/2 = 1/s$, and X is a subspace of L_2, then for all $0 \leqslant h\in L_t$,*

$$D_{2,q}(h^{-1}X) \leqslant \|h\|_t\|h^{-1}\|_\infty D_{s,q}(X).$$

Fact 1 is obvious. Fact 2 follows from the following computation for x in X:

$$\|h^{-1}x\|_q \leqslant \|h^{-1}\|_\infty\|x\|_q \leqslant D_{s,q}(X)\|h^{-1}\|_\infty\|x\|_s \leqslant D_{s,q}(X)\|h^{-1}\|_\infty\|h\|_t\|h^{-1}x\|_2.$$

Returning to the proof of Proposition 1.2, we have

$$\|P\colon L_r \to L_2\| = D_{2,r^*}(h^{-1}X) \quad \text{(by Fact 1)}$$
$$\leqslant \|h\|_t \|h^{-1}\|_\infty D_{s,r^*}(X) \quad \text{(by Fact 2)}$$
$$\leqslant 2^{1/t} D_{s,r^*}(X).$$

Thus

$$\gamma_2(\tilde{u}\colon Z \to L_s) \leqslant \|h\colon L_2 \to L_s\| \|P\colon L_r \to L_2\| \|h^{-1}u\colon Z \to L_r\|$$
$$\leqslant 2^{1/t}\|h\|_t D_{s,r^*}(X) C_{p,r}(u) \leqslant 2^{2/t} D_{s,r^*}(X) C_{p,r}(u). \blacksquare$$

Remark. By making a suitable change of density, the quantity $D_{s,r^*}(X)$ in Proposition 1.2 can be replaced with $C_{s,r^*}(X)$.

A slight weakening of Proposition 1.2 and the above remark are more convenient for us to apply.

THEOREM 1.3. *Let $0 < p < 2 < r^* < \infty$ with $p < r$ and let X be a closed subspace of L_p for which $C_{p,r^*}(X) < \infty$. Suppose that Z_0 is a subspace of a Banach space Z and $u_0\colon Z_0 \to X$ is an operator such that every extension $\tilde{u}\colon Z \to X$ of u_0 satisfies*

$$\gamma_2(\tilde{u}) \geqslant \beta.$$

Then every extension $u\colon Z \to L_p$ of u_0 satisfies

$$C_{p,r}(u) \geqslant 4^{-1/p} C_{p,r^*}(X)^{-1}\beta.$$

Suppose now that we are in the position described in Theorem 1.3. Then

$$C_{p,r}(u) \leqslant C_{p,2}(u)^\alpha C_{p,p}(u)^{1-\alpha} = C_{p,2}(u)^\alpha \|u\|^{1-\alpha}$$

where $\alpha/2+(1-\alpha)/p = 1/r$. Now by [M1]

$$C_{p,2}(u) \leqslant K_p \gamma_2(u)$$

where K_p is from Khintchine's inequality (so $K_p \leqslant \sqrt{2}$ for $1 \leqslant p < 2$, [H], [Sz]). Thus from Proposition 1.2 and the remark following we get

$$K_p^{-1}\|u\|\big(4^{-1/p} C_{p,r^*}(X)^{-1}\beta/\|u\|\big)^{1/\alpha} \leqslant \gamma_2(u), \tag{1.1}$$

which, when specialized to the case $p = 1$, yields

$$2^{-1/2}\|u\|\big(4^{-1} C_{1,r^*}(X)^{-1}\beta/\|u\|\big)^{r^*/2} \leqslant \gamma_2(u). \tag{1.2}$$

Since u factors through the Banach space $Z/\ker u$ which has dimension $\operatorname{rank} u$ and the identity operator on an m-dimensional Banach space has γ_2-norm at most $\sqrt{m}$ [GG], we have $\gamma_2(u) \leqslant (\operatorname{rank} u)^{1/2}\|u\|$. So inequality (1.2) can be weakened to

$$\tfrac{1}{2}\cdot 4^{-r^*} C_{1,r^*}(X)^{-r^*}(\beta/\|u\|)^{r^*} \leqslant \operatorname{rank} u. \tag{1.3}$$

Suppose that the injection $j \equiv i_{1,1}^X$ from some n-dimensional subspace X of L_1 into L_1 factors through an L_1 space Z:

$$j = uw, \quad w\colon X \to Z, \quad u\colon Z \to L_1, \quad \|w\| = 1.$$

Set $Z_0 = wX$, $u_0 = u|_{Z_0}$. Since by [Grü], [GG]

$$\gamma_1(l_2^n) \geqslant (n/2)^{1/2},$$

we see that any extension $\tilde{u}\colon Z \to X$ of u satisfies

$$\|\tilde{u}\| \geqslant d(X, l_2^n)^{-1}(n/2)^{1/2},$$

which gives a lower estimate for β in (1.3). We summarize these remarks as:

COROLLARY 1.4. *Let X be an n-dimensional subspace of L_1 and $j\colon X \to L_1$ the inclusion map. If $j = uw$ with $w\colon X \to Z$, $u\colon Z \to L_1$, and Z an L_1 space, then for all $r^* > 2$,*

$$\operatorname{rank} u \geqslant \big(8C_{1,r^*}(X)\, d(X, l_2^n)\|w\|\|u\|\big)^{-r^*} n^{r^*/2}. \tag{1.4}$$

Note that if Y is an infinite-dimensional subspace of L_1 with $C_{1,r^*}(Y) < \infty$ for some fixed $2 < r^* < \infty$ (for example, Y is the span of a Λ_{r^*}-set of characters [Ru1]), then Corollary 1.4 says that the parameters $m_1(X, K)$ and $k_1(X, K)$ discussed in the introduction can be estimated from below for n-dimensional subspaces X of Y by

$$\exp \delta (n/K^2)^{r^*/2}$$

for some constant $\delta = \delta(r^*, C_{1,r^*}(X)) > 0$. Much more can be said if there is very good control of $C_{1,r^*}(X)$ as $r^* \to \infty$. The smallest possible rate of growth for infinite-dimensional X is $\sqrt{r^*}$ (a precise version of this known fact will be proved in [FJS]); this occurs, for example, if X is spanned by independent gaussian random variables or independent Rademacher functions or any Sidon set of characters. In this case we obtain:

COROLLARY 1.5. *Suppose that X is a subspace of L_1 for which $C_{1,r^*}(X) \leqslant C\sqrt{r^*}$ for all $2 \leqslant r^* < \infty$. If j is the inclusion mapping from an n-dimensional subspace X_0 of X into L_1 and $j = uw$ with $w\colon X_0 \to Z$, $u\colon Z \to L_1$, Z an L_1 space, then*

$$\operatorname{rank} u \geqslant 2^{\Delta n}, \quad \textit{where} \quad \Delta = \big(2^4 Cd(X_0, l_2^n)\|w\|\|u\|\big)^{-2}.$$

Proof. Choose r^* so that

$$n^{1/2} = 2\big(8C\sqrt{r^*}\, d(X_0, l_2^n)\|w\|\|u\|\big).$$

Then $r^* = \Delta n$, so the conclusion follows from inequality (1.4). $\blacksquare$

Remark 1. In the conclusion of Corollary 1.5, we could of course replace $d(X_0, l_2^n)$ with $C\sqrt{2}$. However, when C is large it is better to use [FLM] and apply Corollary 1.5 to a subspace of X_0 of dimension proportional to n which is 2-isomorphic to a Hilbert space.

Remark 2. In [FJS] we improve the conclusion of Corollary 1.5 by showing that if $\|w\| = 1$, then u is a $1/\delta$-isomorphism on a subspace of Z which is 2-isomorphic to l_1^m for some $m \geqslant \exp \delta n$, where $\delta > 0$ depends only on C and on $\|u\|$.

2. An L_1-characterization of Sidon sets. G. Pisier pointed out to us that Corollary 1.5 and the Malliavin–Malliavin [MM] characterization of Sidon sets for the Cantor group yield the following:

COROLLARY 2.1. *Let $G = \{-1, 1\}^N$ be the Cantor group and let Λ be a set of characters on G. The following are equivalent:*

(1) *Λ is a Sidon set.*

(2) *There exists C so that for all $2 \leqslant p < \infty$, $D_{1,p}(\operatorname{span} \Lambda) \leqslant Cp^{1/2}$.*

(3) *For all $K > 1$ there exists $\delta = \delta(K) > 0$ so that for all finite subsets A of Λ, if $\operatorname{span} A \subset X \subset L_1(G)$ and X is K-isomorphic to l_1^m, then $m \geqslant \exp(\delta |A|)$.*

(4) *Same as (3), except that "for all $K > 1$" is replaced by "there exists $K > 1$".*

Proof. (1)$\Rightarrow$(2) is classical (see [Ru1]), (2)$\Rightarrow$(3) comes from Corollary 1.5, while (4)$\Rightarrow$(1) follows from [MM] (if Λ is not Sidon then for all $\delta > 0$ there is even a subset A of Λ so that the sigma algebra generated by A has cardinality less than $\exp(\delta |A|)$). ■

It is open whether the implication (4)$\Rightarrow$(1) (or (3)$\Rightarrow$(1)) is valid for all compact abelian groups. However, J. Bourgain has shown that in the general case there is a similar characterization which involves the uniformity function in the bounded approximation property for L_1.

THEOREM 2.2. *Let G be a compact abelian group with dual group Γ and let $\Lambda \subset \Gamma$. The following are equivalent:*

(1) *Λ is a Sidon set.*

(2) *There exists C so that for all $2 \leqslant p < \infty$, $D_{1,p}(\operatorname{span} \Lambda) \leqslant Cp^{1/2}$.*

(3) *For all $K > 1$ there exists $\delta = \delta(K) > 0$ so that for all finite subsets A of Λ and all operators T on $L_1(G)$ for which $\|T\| \leqslant K$ and $T\gamma = \gamma$ $(\gamma \in A)$, it follows that*

$$\operatorname{rank} T \geqslant \exp(\delta |A|).$$

(4) *Same as (3), except that "for all $K > 1$" is replaced by "there exists $K > 1$".*

(5) *Same as (3), except that "translation invariant" is inserted before "operators".*

(6) *Same as (4), except that "translation invariant" is inserted before "operators".*

Proof. In view of Corollary 1.5, the only implication requiring proof is (6)$\Rightarrow$(1), which is due to J. Bourgain. He kindly allowed us to present here our exposition of this argument.

The main fact needed for the proof is Bourgain's L_∞-entropy estimate [B], which is both an improvement of and a fairly simple consequence of an entropy estimate due to Pisier [Pis1]. We restate Bourgain's lemma in a form suitable for our use. (Given a set B of characters, $S(B)$ is the *Sidon constant* of B, i.e. the constant of equivalence between B in $(\operatorname{span} B, \|\cdot\|_\infty)$ and the unit vector basis for l_1^k, $k = |B|$.)

PROPOSITION 2.3 [B]. *There exist constants $\alpha > 0$ and C so that if Λ is a set of characters on some compact abelian group G and Λ is not a Sidon set, then for each $M < \infty$ there is a finite subset B of Λ with $S(B) > M$ so that for each $\varepsilon > 0$, there exists a partition of G into*

$$N \leqslant \exp[CS(B)^{-\alpha} |B| \log(1/\varepsilon)]$$

measurable sets $\{G_i\}_{i=1}^N$ so that for each $\gamma \in B$ and all $g, h \in G_i$ $(1 \leqslant i \leqslant N)$,

$$|\gamma(g) - \gamma(h)| < \varepsilon^2.$$

To prove (6)$\Rightarrow$(1) in Theorem 2.2, assume that Λ is not Sidon and that $K > 1$. Given any $M > 2$ satisfying

$$(1 + M^{-1})^2 + 2/M < K, \tag{2.1}$$

take $B \subset \Lambda$ to satisfy the conclusions of Proposition 2.3. Now choose ε of order $S(B)^{-1}$; say,

$$(1 - \varepsilon^2)/\varepsilon = S(B), \tag{2.2}$$

so that $\frac{1}{2} S(B)^{-1} \leqslant \varepsilon \leqslant S(B)^{-1}$, and let N and $\{G_i\}_{i=1}^N$ satisfy the conditions of Proposition 2.3 for this choice of ε.

Our goal is to find a signed measure μ on G with total mass $\|\mu\|_1 \leqslant K$ so that $\hat{\mu}(\gamma) = 1$ for γ in B and the support of $\hat{\mu}$ satisfies

$$|\{\gamma \in \Gamma:\ \hat{\mu}(\gamma) \neq 0\}| \leqslant 2N^3.$$

Indeed, then the operator T of convolution with μ has rank at most $3N^3 \leqslant \exp[3CS(B)^{-\alpha} |B| \log(2S(B))]$, T is the identity on B, and $\|T\colon L_1(G) \to L_1(G)\| \leqslant K$.

Let P be the conditional expectation projection onto the sigma algebra generated by $\{G_i\}_{i=1}^N$. The entropy condition yields that for each $\gamma \in B$, (2.3) $$\|P\gamma-\gamma\|_1 \leqslant \|P\gamma-\gamma\|_\infty \leqslant \varepsilon^2.$$

Replace P by its average R over G, defined by

$$R = \int T_{g^{-1}} P T_g \lambda(dg)$$

where T_g is translation by g and λ the normalized Haar measure on G. The positive translation invariant operator R satisfies the norm condition

$$\|R\colon L_p \to L_p\| \leqslant \|P\colon L_p \to L_p\| = 1$$

in particular for $p = \infty$ and hence [Ru2, p. 74] is given by convolution with some measure ν with total mass $\|\nu\|_1 \leqslant 1$.

Since property (2.3) is inherited by R, we have

(2.4) $$1-\varepsilon^2 \leqslant \hat{\nu}(\gamma) \leqslant 1 \quad (\gamma \in B).$$

Moreover,

(2.5) $$\sum_{\gamma\in\Gamma} |\hat{\nu}(\gamma)| \leqslant \|R\|_{\mathrm{TC}} \leqslant \|P\|_{\mathrm{TC}} = N$$

where $\|\cdot\|_{\mathrm{TC}}$ is the trace class norm of an operator on $L_2(G)$.

We want to perturb ν by a signed measure of small mass so that the resulting signed measure has Fourier coefficients which are 1 on B and still satisfy (2.5). Since (2.5) is preserved by convolution with a signed measure of mass 1, we want to find a signed measure τ so that

(2.6) $$(\nu+\tau*\nu)\hat{}\,(\gamma) = 1 \quad (\gamma\in B),$$

that is,

$$\hat{\tau}(\gamma) = \big(1-\hat{\nu}(\gamma)\big)/\hat{\nu}(\gamma) \quad (\gamma\in B).$$

Since

$$\big|\big(1-\hat{\nu}(\gamma)\big)/\hat{\nu}(\gamma)\big| \leqslant \varepsilon^2/(1-\varepsilon^2) \quad (\gamma\in B),$$

there is a linear functional of norm at most

$$\varepsilon^2 S(B)/(1-\varepsilon^2) \quad (=\varepsilon, \text{ by } (2.2))$$

on $L_\infty(G)$ which for γ in B takes on the value $\big(1-\hat{\nu}(\gamma)\big)/\hat{\nu}(\gamma)$. By averaging against the T_g's we obtain a translation invariant functional on $L_\infty(G)$ with the same properties and thereby get a signed measure τ which satisfies (2.6) and has mass $\|\tau\|_1 \leqslant \varepsilon$. Setting $\nu_1 = \nu+\tau*\nu$, we see that $\hat{\nu}_1(\gamma) = 1$ for $\gamma\in B$, $\|\nu_1\|_1 \leqslant 1+\varepsilon$, and

$$\sum_{\gamma\in\Gamma} |\hat{\nu}_1(\gamma)| \leqslant N+N\|\tau\|_1 \leqslant (1+\varepsilon)N.$$

Next, we smooth ν_1 a bit by setting $\nu_2 = \nu_1 * \nu_1$, so that $\hat{\nu}_2(\gamma) = 1$ for $\gamma\in B$, $\|\nu_2\|_1 \leqslant (1+\varepsilon)^2$, and

(2.7) $$\sum_{\gamma\in\Gamma} |\hat{\nu}_2(\gamma)|^{1/2} \leqslant (1+\varepsilon)N.$$

Letting $A = \{\gamma\in\Gamma\colon |\hat{\nu}_2(\gamma)| \leqslant N^{-4}\}$ we get from (2.7)

$$\sum_{\gamma\in A} |\hat{\nu}_2(\gamma)| \leqslant N^{-2}\sum_{\gamma\in\Gamma} |\hat{\nu}_2(\gamma)|^{1/2} \leqslant (1+\varepsilon)/N < 2/N.$$

Therefore the multiplier defined by $f \to \sum_{\gamma\in A}\langle f, \gamma\rangle\gamma$ is an operator of norm (even nuclear norm) at most $2/N$ on $L_p(G)$ for all $1\leqslant p\leqslant\infty$ and hence is given by convolution with some signed measure τ_1 with total mass $\|\tau_1\|_1 \leqslant 2/N$. The signed measure $\mu = \nu_2-\tau_1$ satisfies

$$\|\mu\|_1 \leqslant (1+\varepsilon)^2+2/N \leqslant \big(1+S(B)^{-1}\big)^2+2|B|^{-1} \leqslant (1+M^{-1})^2+2/M < K$$

and $\hat{\mu}(\gamma) = 1$ for $\gamma\in B$. Finally, (2.7) gives us control over the cardinality of the support of $\hat{\mu}$:

$$\begin{aligned}|\{\gamma\in\Gamma\colon \hat{\mu}(\gamma)\neq 0\}| &= |\{\gamma\in\Gamma\colon |\hat{\nu}_2(\gamma)| > N^{-4}\}|\\ &\leqslant N^2\sum_{\gamma\in\Gamma}|\hat{\nu}_2(\gamma)|^{1/2} \leqslant (1+\varepsilon)N^3 \leqslant 2N^3. \ \blacksquare\end{aligned}$$

3. Subspaces of L_1 spanned by p-stable variables. In this section we study the natural embedding of l_p^n, $1<p<2$, into L_1 defined by mapping the unit vector basis of l_p^n onto L_1-normalized i.i.d. p-stable variables. The analysis is similar to that for natural embeddings of l_2^n into L_1; however, there are additional complications because the orthogonal projection onto the span of n i.i.d. p-stables (even divided by a change of density) does not have sufficiently small γ_2-norm as an operator from L_r, $1<r<p$, to allow us to mimic the arguments presented earlier for natural embeddings of l_2^n into L_1. Although the orthogonal projection can be replaced with another natural projection, the results we obtain for the span of p-stables are weaker than the corresponding results for, say, the span of gaussian variables. For example, if the inclusion mapping from X, the span of n i.i.d. p-stables in L_1, into some superspace $Y\subset L_1$ satisfies $\gamma_1(i\colon X\to Y)\leqslant K$, then we obtain in Theorem 3.3 that l_1^m 2-embeds into Y for some $m\geqslant \exp\delta n^{2/p^*}$ ($\delta = \delta(K) > 0$). We conjecture that this lower bound can be improved to $m\geqslant\exp\delta n$ for some $\delta = \delta(p, K) > 0$.

In the proof of the main technical result of this section, Proposition 3.2 (which for the ideal norm $\alpha = \gamma_2$ is a variation of Proposition 1.2), we use the following lemma:

LEMMA 3.1. *Let X be a subspace of $L_1 = L_1(\mu)$ for some probability measure μ such that $D = D_{1,s}(X) < \infty$ for some $s > 1$, and let $u\colon Z\to L_1$ be an* *operator satisfying $C_{1,r}(u) < \infty$ where $1 \leqslant r \leqslant s$. Then for every $\eta \geqslant 1$ there is a multiplication operator F on L_1 such that*

$$\|Fu\colon Z \to L_r\| \leqslant \eta C_{1,r}(u),$$
$$\|x - Fx\|_r \leqslant \eta^{(s^*-r^*)/s^*} D \|x\|_1 \qquad (x \in X).$$

Proof. Take $0 \leqslant h$ in L_{r^*} with $\|h\|_{r^*} = 1$ so that

$$\|h^{-1}u\colon Z \to L_r\| = C_{1,r}(u).$$

Let F be the indicator function of the set $[h \leqslant \eta]$. Then $0 \leqslant F \leqslant \eta h^{-1}$ and hence

$$\|Fu\colon Z \to L_r\| \leqslant \eta C_{1,r}(u).$$

On the other hand, if $1/r = 1/t + 1/s$, then

$$\|(1-F)x\|_r \leqslant \|1-F\|_t \|x\|_s \leqslant \|1-F\|_t D \|x\|_1.$$

Since $1-F$ is the indicator function of the set $[\eta^{-1}h > 1]$, we obtain easily

$$\|1-F\|_t^t = \|1-F\|_{r^*}^{r^*} \leqslant (\eta^{-1}\|h\|_{r^*})^{r^*} = \eta^{-r^*}.$$

Since $r^*/t = (r^*-s^*)/s^*$, this completes the proof. ■

Proposition 3.2. *In the setting of Lemma 3.1, if $uZ \supset X$, then to every operator $P\colon L_r(\mu) \to X$ such that $Px = x$ for $x \in X$ there corresponds an operator $\tilde{u}\colon Z \to X$ which satisfies*

(3.1) $$\tilde{u}z = uz \quad \text{whenever} \quad uz \in X;$$

(3.2) *If α is any operator ideal norm, then*

$$\alpha(\tilde{u}) \leqslant g(s^*/r^*)(D\|P\|)^{s^*/(r^*-s^*)} C_{1,r}(u)\,\alpha(P)$$

where $g(t) = (1-t)^{-1} t^{t/(t-1)}$.

Consequently, if $r^* \geqslant \max(2s^*, s^*[1+2\log(D\|P\|)])$, then

(3.3) $$\alpha(\tilde{u}) \leqslant 7C_{1,r}(u)\,\alpha(P\colon L_r \to X).$$

Proof. Notice that (3.3) follows from (3.2) because g is increasing for $0 < t < 1$ with $g(\frac{1}{2}) = 4$, and when $r^* \geqslant s^*(1+2\log(D\|P\|))$, $(D\|P\|)^{s^*/(r^*-s^*)} \leqslant e^{1/2} < 7/4$.

Let F be the multiplication operator obtained in Lemma 3.1. Put $w = PFu$. Then $w\colon Z \to X$ and

$$\alpha(w) \leqslant \alpha(P\colon L_r \to X)\|Fu\colon Z \to L_r\| \leqslant \eta C_{1,r}(u)\,\alpha(P).$$

Moreover, if $z \in Z_0 \equiv u^{-1}X$, then $uz \in X$ and hence

$$\|(1-F)uz\|_r \leqslant \eta^{(s^*-r^*)/s^*} D \|uz\|_1.$$

Consequently, since $Puz = uz$,

$$\|uz - wz\|_1 = \|P(1-F)uz\|_1 \leqslant \|P\colon L_r \to L_1\| \|(1-F)uz\|_r$$
$$\leqslant \|P\| \eta^{(s^*-r^*)/s^*} D \|uz\|_1.$$

Set $\delta = \eta^{(s^*-r^*)/s^*} D\|P\|$. If $\delta < 1$ and $z \in Z_0$, then

$$\|wz\|_1 \geqslant \|uz\|_1 - \|uz - wz\|_1 \geqslant (1-\delta)\|uz\|_1.$$

Since $uZ_0 = X$, we get also $wZ_0 = X$ and the correspondence $wz \to uz$ defines an operator $v\colon wZ_0 \to X$ with $\|v\| \leqslant (1-\delta)^{-1}$. Let $\tilde{u} = vw$. This clearly ensures that (3.1) holds. Also

$$\alpha(\tilde{u}) \leqslant \|v\|\,\alpha(w) \leqslant (1-\delta)^{-1}(\delta^{-1}D\|P\|)^{s^*/(r^*-s^*)} C_{1,r}(u)\,\alpha(P).$$

Now, letting $\delta = s^*/r^*$ (the best choice) we obtain the estimate (3.2). ■

Before presenting the application of Proposition 3.2 to p-stable subspaces of L_1, we note that Corollary 1.1 can easily by deduced from Proposition 3.2. Indeed, suppose that $X \subset Y \subset L_1$ and X is spanned by, say, n independent gaussian variables. Factor the inclusion map $i\colon X \to Y$ as $j\colon X \to Z$ followed by $u\colon Z \to X$ with Z an L_1 space, $\|u\| = 1$, and $\|j\| \approx \gamma_1(i)$. Since $\gamma_1(l_2^n) \geqslant (\frac{1}{2}n)^{1/2}$ ([Grü], [GG]), any $\tilde{u}\colon Z \to X$ which agrees with u on $jX \subset u^{-1}X$ admits the lower estimate

$$\|\tilde{u}\| \geqslant \|j\|^{-1}(\tfrac{1}{2}n)^{1/2}.$$

For the P in Proposition 3.2, we take the orthogonal projection, so that $\|P\colon L_r \to X\| \leqslant D_{r,r^*}(X) \leqslant (r^*)^{1/2}$ for every $1 < r < 2$. Setting $s = 2$, $r^* = (14\|j\|)^{-2} n$, and $\alpha(\cdot) = \|\cdot\|$, we get from (3.3) (at least when $n \geqslant 8$)

$$T_r(Y) \geqslant C_{1,r}(u) \geqslant \sqrt{2}.$$

Hence $T_r(Y)^{r^*} \geqslant \exp(\delta_1\|j\|^{-2}n)$ for some absolute constant $\delta_1 > 0$, whence l_r^m $\sqrt{2}$-embeds into Y by [Pis3], [Sc2] for some $m \geqslant \exp(\delta\gamma_1(i)^{-2}n)$ and some absolute constant $\delta > 0$, which in view of the choice of r gives Corollary 1.1.

We now come to the main result of this section.

Theorem 3.3. *Suppose that $X \subset Y \subset L_1(\mu)$ where for some $1 < p < 2$, $X = X_{(p,n)}$ is the span of n independent symmetric p-stable variables $x_1, \ldots, x_n$ normalized so that $\|x_i\|_1 = 1$. Let $i\colon X \to Y$ be the inclusion map. If*

$$n^{2/p^*} > 400\gamma_1(i\colon X \to Y)^2 p^* \log n,$$

then l_1^m 2-embeds into Y for some

$$m \geqslant \exp[\delta\gamma_1(i\colon X \to Y)^{-2} n^{2/p^*}]$$

where $\delta > 0$ is an absolute constant independent of n, p, and $\gamma_1(i\colon X \to Y)$.

Proof. If we set $s = \frac{1}{2}(p+1)$, then there is D independent of p and n such that $D_{1,s}(X) \leqslant D$ (see [Pie], p. 289). Next observe that there is a projection P: $L_1 \to X$ so that for all $1 < r < 2$,

$$\gamma_2(P\colon L_r \to X) \leqslant (r^*/2)^{1/2} n^{1/p-1/2}.$$

Indeed, let $r_i = \operatorname{sgn} x_i$, $1 \leqslant i \leqslant n$, so that the r_i's have the same joint distribution as the first n Rademacher functions. Define

$$Px = \sum_{i=1}^{n} \int x(t) r_i(t) d\mu(t) x_i.$$

An application of Khintchine's inequality to the natural factorization of P through l_2^n shows that $\gamma_2(P)$ satisfies the required estimate. Since clearly $\|P\colon L_r \to X\| \leqslant n$ for all $1 < r < 2$, if $n \geqslant 2$ and

$$r^* \geqslant s^*(1+2\log Dn) \tag{3.4}$$

we obtain from (3.3) in Proposition 3.2

$$\gamma_2(\tilde{u}) \leqslant 5C_{1,r}(u)(r^*)^{1/2} n^{1/p-1/2}$$

if u and $\tilde{u}$ satisfy the conditions in Proposition 3.2.

Suppose now that the inclusion i: $X \to Y$ factors as j: $X \to Z$ followed by u: $Z \to Y$ with Z an L_1 space, $\|u\| = 1$, and $\|j\| \approx \gamma_1(i\colon X \to Y)$. If $\tilde{u}$: $Z \to X$ agrees with u on $jX \subset u^{-1}X$, then we get

$$\begin{aligned} \sqrt{n} &= \pi_2(X) \quad \text{(by [GG])} \\ &\leqslant \|j\| \pi_2(\tilde{u}) \\ &\leqslant \|j\| \sqrt{2}\gamma_2(\tilde{u}) \quad \text{(by a weak form of Grothendieck's inequality [G], [LP])}, \end{aligned}$$

so that

$$C_{1,r}(u) \geqslant (5\|j\|)^{-1} 2^{-1/2} n^{1/p^*} (r^*)^{-1/2}.$$

For the choice $r^* = (10\|j\|)^{-2} n^{2/p^*}$, (3.4) is satisfied when

$$n^{2/p^*} > 400\gamma_1(i\colon X \to Y)^2 p^* \log n,$$

and the estimate for $C_{1,r}(u)$ becomes $C_{1,r}(u) \geqslant \sqrt{2}$. The rest of the proof is identical to the last step of the proof of Corollary 1.1 in Section 1 given just before the statement of Corollary 1.1. ∎

Remark 1. Notice that if $X \subset W$, then for every projection P from W onto X and every subspace X_0 of X, there is a projection $\tilde{P}$ from W onto X_0 with $\gamma_2(\tilde{P}) \leqslant \gamma_2(P)$. Thus the proof of Theorem 3.3 shows that, in the notation of Theorem 3.3, if $X_0 \subset X_{(p,n)}$, $\dim X_0 = \tau n$, $X_0 \subset Y \subset L_1$ with inclusion mapping i: $X_0 \to Y$, and

$$n^{2/p^*} > (400/\tau)^{p^*} \gamma_1(i\colon X_0 \to Y)^2 \log n,$$

then l_1^m 2-embeds into Y for some $m \geqslant \exp \delta\tau n^{2/p^*}$.

Remark 2. In Section 1 we state Proposition 1.2 for subspaces of L_p even though we only have good applications for L_1 just to draw attention to the fact that there is nothing in our approach which is basically L_1. There is an analogous version of Theorem 3.3 for p-stable subspaces of L_s for $s < p$, but in the absence of applications we decided that the extra parameter makes the statement unduly complicated.

4. Local $\mathscr{L}_\infty$-structure. In Corollary 4.2 we show that $m_\infty(n, 1+\varepsilon) \leqslant \exp c(\varepsilon) n$. This means that every n-dimensional subspace X of a $C(S)$ space is contained in a subspace Y of $C(S)$ with $\dim Y = m \leqslant \exp c(\varepsilon) n$ which is $1+\varepsilon$-isomorphic to l_∞^m and *a fortiori* $1+\varepsilon$-complemented in $C(S)$. On the other hand, the easy Proposition 4.3 implies that if X is any l_2^n subspace of $C(S)$ and u is an operator on $C(S)$ which is the identity on X and satisfies $\|u\| \leqslant K$, then $\operatorname{rank} u \geqslant \exp \delta(K) n$. (In fact, in [FJS] we show that u must preserve a copy of l_∞^m with $m \geqslant \exp \delta(K) n$.) Thus "exponential of n" is the order of magnitude for the uniformity function $k_\infty(n, K)$ in the uniform approximation property for $C(S)$ spaces.

The main new fact we prove is that $m_\infty(X, K)$ does not depend very much on the embedding of X into $C(S)$. More precisely, we have:

THEOREM 4.1. *Assume that X is an n-dimensional subspace of l_∞^N and X is K-isomorphic to a subspace of l_∞^k. Then there is a subspace Y of l_∞^N with $X \subset Y$ and $d(Y, l_\infty^m) \leqslant K$ for some $m \leqslant kn$ (real scalars) or $m \leqslant 2kn$ (complex scalars).*

Proof. The proof is more naturally phrased in the dual situation: Let Q: $l_1^N \to X^*$ be the quotient map. Note that it is sufficient to find a subset A of $\{1, \ldots, N\}$ of cardinality at most kn ($2kn$ in the complex case) so that

$$Q \operatorname{Ball} l_1^A \supset K^{-1} \operatorname{Ball} X^*.$$

Indeed, having accomplished this, we can write for each $1 \leqslant i \leqslant N$, $i \notin A$,

$$Q\delta_i = \sum_{j \in A} \lambda(i, j) Q\delta_j$$

with $\max_{1 \leqslant i \leqslant n} \sum_{j \in A} |\lambda(i, j)| \leqslant K$. Now define u: $l_1^N \to l_1^A$ by

$$u\delta_i = \begin{cases} \delta_i & \text{if } i \in A, \\ \sum_{j \in A} \lambda(i, j)\delta_j & \text{if } i \notin A. \end{cases}$$

The operator $u^*\colon l_\infty^A \to l_\infty^N$ is a K-isomorphism because u is a projection and $\|u\| \leqslant K$, while $u^* l_\infty^A \supset X$ because $Qu = Q$.

The existence of A follows from Carathéodory's theorem: Since X K-embeds into l_∞^k, there exists a subset F of $\operatorname{Ball} X^*$ of cardinality k so that $\operatorname{conv\,bal} F \supset K^{-1} \operatorname{Ball} X^*$. For each f in F there is $A_f \subset \{1, \ldots, N\}$ so that

$$f \in \operatorname{conv\,bal}\{Q\delta_i\colon i \in A_f\}$$

and, by Carathéodory's theorem, $|A_f| \leqslant n$ ($2n$ in the complex case). Just set

$$A = \bigcup_{f \in F} A_f. \ \blacksquare$$

Corollary 4.2. *There is a constant M so that if X is an n-dimensional subspace of $C(S)$ and $0 < \varepsilon < 1$, then there exists a subspace Y of $C(S)$ with $X \subset Y$, $\dim Y = m \leqslant \exp(Mn/\varepsilon)$, and $d(Y, l_\infty^m) \leqslant 1+\varepsilon$.*

Proof. By replacing X with its image under an $1+\varepsilon/2$-isometry from $C(S)$ onto $C(S)$, we may assume that X is contained in a subspace of $C(S)$ which is isometric to l_∞^N for some N. Simple volume considerations (see, for example, [FLM]) show that $\operatorname{Ball} X^*$ contains an $\varepsilon/2$-net of cardinality $k \leqslant (10/\varepsilon)^n$ and thus X $(1-\varepsilon/2)^{-1}$-embeds into l_∞^k. Now apply Theorem 4.1. $\blacksquare$

Remark. In Corollary 4.2, $C(S)$ can of course be replaced by any L_1-predual, or, if one weakens the conclusion on Y to $d(Y, l_\infty^m) \leqslant \lambda(1+\varepsilon)$, by any $\mathscr{L}_{\infty,\lambda}$-space.

Proposition 4.3. *Suppose that X is an n-dimensional subspace of l_∞^N and u is an operator from l_∞^N into a Banach space for which $\|ux\| \geqslant \|x\|$ for all x in X. Then*

$$\operatorname{rank} u \geqslant \exp\left(\delta d(X, l_2^n)^{-2} \|u\|^{-2} n\right)$$

where $\delta > 0$ is an absolute constant.

Proof. For all $1 \leqslant p \leqslant \infty$, u satisfies

$$\pi_p(u) \geqslant d(X, l_2^n)^{-1} \pi_p(l_2^n) \geqslant d(X, l_2^n)^{-1} \sqrt{n/p} \quad \text{(by [Pe])}.$$

On the other hand, for the choice $p = \log_2(\operatorname{rank} u)$, by interpolating between $\pi_\infty(u) = \|u\|$ and $\pi_2(u) \leqslant (\operatorname{rank} u)^{1/2}\|u\|$ we get $\pi_p(u) \leqslant 2\|u\|$. $\blacksquare$

Remark. Maurey [Pis2] proved that if $\dim X = n$ and the type p constant of X is not more than K for some $p > 1$ and X C-embeds into l_∞^m, then $m \geqslant \exp\delta(p, K, C)n$. Probably the assumption on X can be replaced by the weaker condition that the cotype p^* constant $C_{p^*}(X)$ of X is at most K; this is Pisier's "cotype dichotomy" problem for l_∞^N. In the context of Proposition 4.3, if X satisfies $C_q(X) \leqslant K$ for some $q < \infty$ and some constant K, then by considering large Euclidean sections in X we see that the proof of Proposition 4.3 yields in this case that $\operatorname{rank} u \geqslant \exp \delta n^{2/q}$ for some $\delta = \delta(q, K, \|u\|) > 0$. While it seems likely that $n^{2/q}$ can be replaced by n in this inequality, we do not see how to check this strengthening of Pisier's dichotomy conjecture in some cases covered by Maurey's argument (e.g., $X = l_q^n$, $2 < q < \infty$).

5. Factoring quotient mappings into L_1. In this section we generalize the situation studied in Sections 1 and 3 as follows: Suppose that X is a subspace of L_1 and Q is an operator from some Banach space Y into L_1 which is a "good" quotient mapping relative to X, i.e., $Q \operatorname{Ball} Y \supset \operatorname{Ball} X$. Estimate from below the "size" of factorizations of Q through L_1 in terms of some parameter of X. When Y^* has small cotype q constant $C_q(Y^*)$ for some $q < \infty$, we obtain exact analogues of Corollary 1.1 and Theorem 3.3 as Theorems 5.1 and 5.3. When Y^* does not have small cotype q constant, e.g. when Y is an L_1 space, there clearly are no such generalizations; nevertheless, Theorems 5.2 and 5.4 imply that if Q well-factors through l_1^m then m must be large.

We conclude this section with a simple result, Proposition 5.5, which however is rather surprising in view of Corollary 1.1 and Theorem 5.1; namely, every norm one operator from l_2^n into L_1 is a convex combination of operators whose $\gamma_{l_1^n}$-parameters are less than 3. (Recall that $\gamma_Z(u\colon X \to Y) = \inf\{\|v\|\,\|w\|\colon u = wv,\ v\colon X \to Z,\ w\colon Z \to Y\}$.) In the language of [T-J2], this result says that the ideal norm generated by $\gamma_{l_1^n}$ satisfies the inequality

$$\hat{\gamma}_1^{(n)}(u) \leqslant 3\|u\|$$

for every operator from l_2^n into L_1.

Theorem 5.1. *Suppose that $X \subset Y \subset L_1$ and $C_{1,r^*}(X) \leqslant C\sqrt{r^*}$ for all $2 \leqslant r^* < \infty$. If W is a Banach space with $C_q(W^*) < \infty$ and $Q\colon W \to Y$ is an operator for which*

$$Q \operatorname{Ball} W \supset \operatorname{Ball} X,$$

then Y contains a subspace which is 2-isomorphic to l_1^m with

$$m \geqslant \exp\left(\delta [C\sqrt{q}\, C_q(W^*)\gamma_1(Q\colon W \to Y)]^{-2} n\right)$$

where $n = \dim X$ and $\delta > 0$ is an absolute constant.

Proof. By [FLM], X has a subspace of dimension proportional to n which is 2-isomorphic to a Hilbert space. Since absolute constants can be absorbed into δ, we assume that $d(X, l_2^n) \leqslant 2$.

Write $Q = uj$ with $j\colon W \to Z$, $u\colon Z \to Y$, Z an L_1 space, $\|u\| = 1$, and

$\|j\| \approx \gamma_1(Q\colon W \to Y)$. The main fact to be added to the arguments already used in Sections 1 and 3 is Maurey's result [M2] that there is a constant B so that for all $2 \leqslant q < \infty$,

$$\pi_{2q}(v) \leqslant BC_q(W^*)\|v\|$$

for every operator v from an L_∞ space into W^*. (That the constant in this inequality is of the form $BC_q(W^*)$ instead of just some function of q and $C_q(W^*)$ follows from the proof given in [MP].) We again use Pełczyński's computation [Pe] that for all $1 < q < \infty$

$$\pi_{2q}(l_2^n) \geqslant \tfrac{1}{2}\sqrt{n/q}.$$

Now if $\tilde{u}\colon Z \to X$ extends the restriction of u to $u^{-1}X$, we have

$$\tilde{u}j \operatorname{Ball} W \supset (Q \operatorname{Ball} W) \cap X \supset \operatorname{Ball} X,$$

i.e., $(\tilde{u}j)^*$ is norm-increasing. Therefore

$$\pi_{2q}([\tilde{u}j]^*) \geqslant \pi_{2q}(X^*) \geqslant \pi_{2q}(l_2^n)/d(X, l_2^n) \geqslant \tfrac{1}{4}\sqrt{n/q}.$$

Since also

$$\pi_{2q}([\tilde{u}j]^*) \leqslant \pi_{2q}(j^*)\|\tilde{u}\| \leqslant BC_q(W^*)\|j\|\|\tilde{u}\| \leqslant BC_q(W^*)\|j\|\gamma_2(\tilde{u}),$$

from Theorem 1.3 we get for all $2 < r^* < \infty$,

$$C_{1,r}(u) \geqslant b\big(C_q(W^*)\sqrt{q}\,C_{1,r^*}(X)\|j\|\big)^{-1} n^{1/2}$$
$$\geqslant b\big(C\sqrt{q}\,C_q(W^*)\|j\|\big)^{-1}(n/r^*)^{1/2}$$

where $b > 0$ is another absolute constant. Setting $r^* = \Delta n$ with $\Delta = \frac{1}{4}(C\sqrt{q}\,C_q(W^*)\|j\|/b)^{-2}$ we have

$$T_r(Y)^{r^*} \geqslant C_{1,r}(u)^{r^*} \geqslant 2^{\Delta n},$$

which by [Pis3], [Sc2] yields that l_r^m $\sqrt{2}$-embeds into Y for some

$$m \geqslant \exp\big[\delta\big(C\sqrt{q}\,C_q(W^*)\|j\|\big)^{-2} n\big]$$

where $\delta > 0$ is another absolute constant. In view of the closeness of r to 1, this completes the proof. ■

THEOREM 5.2. *Suppose that X is an n-dimensional subspace of L_1 and $C_{1,r^*}(X) \leqslant C\sqrt{r^*}$ for all $2 \leqslant r^* < \infty$. If $u\colon l_1^m \to L_1$ is an operator for which $u \operatorname{Ball} l_1^m \supset \operatorname{Ball} X$, then*

$$m \geqslant \exp\big(\delta(C\|u\|)^{-1} n^{1/2}\big),$$

where $\delta > 0$ is an absolute constant.

Proof. As in the proof of Theorem 5.1, we can assume that $d(X, l_2^n) \leqslant 2$. Lewis [L] proved that for any $Y \subset L_1$, $C_{1,2}(Y) \leqslant (\dim Y)^{1/2}$, so that, by interpolation, $C_{1,r}(Y) \leqslant (\dim Y)^{1/r^*}$ for $1 \leqslant r \leqslant 2$. Now if $\tilde{u}\colon l_1^m \to X$ agrees with u on $u^{-1}X$, then $\tilde{u}^*\colon X^* \to l_\infty^m$ is norm-increasing. Since l_∞^m is 2-isomorphic to l_t^m with $t = \log_2 m$, by [BDGJN], p. 182, we get

$$\gamma_2(\tilde{u}) \geqslant \tfrac{1}{2} d(X, l_2^n)^{-1}(n/t)^{1/2} m^{-1/t} \geqslant \delta_1 (n/\log_2 m)^{1/2}$$

where $\delta_1 > 0$ is absolute. Letting $r^* = \log_2 m$, from Theorem 1.3 we get

$$2\|u\| = m^{1/r^*}\|u\| \geqslant C_{1,r}(u) \geqslant 2\delta C_{1,r^*}(X)^{-1}(n/\log_2 m)^{1/2}$$
$$\geqslant 2\delta(C\log_2 m)^{-1} n^{1/2},$$

where $\delta > 0$ is absolute. ■

We now prove a strengthening of Theorem 3.3.

THEOREM 5.3. *Suppose that $X \subset Y \subset L_1$ and X is a k-dimensional subspace of the span of n independent symmetric p-stable variables for some $1 < p < 2$. If Q is an operator from a Banach space W into Y for which*

$$Q \operatorname{Ball} W \supset \operatorname{Ball} X,$$

then Y contains a subspace which is 2-isomorphic to l_1^m with

$$m \geqslant \exp\big(\delta[\sqrt{q}\,C_q(W^*)\gamma_1(Q\colon W \to Y)]^{-2}(k/n) n^{2/p^*}\big)$$

for some absolute constant $\delta > 0$, as long as

$$(k/n) n^{2/p^*} \geqslant A[\sqrt{q}\,C_q(W^*)\gamma_1(Q\colon W \to Y)]^2 p^* \log k \tag{5.1}$$

where A is an absolute constant.

Proof. Again we may assume that $d(X, l_2^n) \leqslant 2$. From the proof of Theorem 3.3, there is a projection P from L_1 onto X so that for all $1 \leqslant r < 2$

$$\gamma_2(P\colon L_r \to X) \leqslant (r^*/2)^{1/2} n^{1/p - 1/2}.$$

Since in Proposition 3.2 it is OK for the projection P to depend on r, we may also assume that

$$\gamma_2(P\colon L_r \to X) \leqslant \dim X = k.$$

As in the proof of Theorem 3.3, let $s = (p+1)/2$, so that $D_{1,s}(X) \leqslant D$ for some absolute constant D, and suppose that $Q\colon W \to Y$ factors as $j\colon W \to Z$ followed by $u\colon Z \to Y$ with Z an L_1 space, $\|u\| = 1$, and $\|j\| \approx \gamma_1(Q\colon W \to Y)$.

According to Proposition 3.2, if $r^* \geqslant 2s^*$ satisfies

$$r^* \geqslant s^*(1 + 2\log(D\|P\colon L_r \to X\|)) \tag{5.2}$$

then there is an operator $\tilde{u}\colon Z \to X$ agreeing with u on $u^{-1}X$ so that

$$\gamma_2(\tilde{u}) \leqslant 5C_{1,r}(u)(r^*)^{1/2} n^{1/p-1/2}.$$

The computation in Theorem 5.1 yields

$$\tfrac{1}{4}\sqrt{k/q} \leqslant \tfrac{1}{2}\pi_{2q}(l_2^k) \leqslant \pi_{2q}([\tilde{u}j]^*) \leqslant BC_q(W^*)\|j\|\|\tilde{u}\|$$

where B is an absolute constant, so when (5.2) is satisfied, Proposition 3.2 gives

$$\sqrt{k} \leqslant B_1\sqrt{q}\,C_q(W^*)\|j\|\,C_{1,r}(u)(r^*)^{1/2} n^{1/p-1/2}$$

where B_1 is another absolute constant.

We make $C_{1,r}(u) \geqslant 2$ by setting

$$(r^*)^{1/2} = \tfrac{1}{2}\big(B_1\sqrt{q}\,C_q(W^*)\|j\|\big)^{-1} k^{1/2} n^{1/2-1/p}.$$

The rest of the proof is the same as the last steps of the proof of Theorem 5.1; however, to guarantee that (5.2) is satisfied, we need (5.1). ∎

Remark. The restricition (5.1) in Theorem 5.3 is not very important because when it is violated, the lower estimate for m in Theorem 5.3 would be less than k^{p^*}.

Next we do the p-stable analogue of Theorem 5.2.

THEOREM 5.4. *Suppose that X is a k-dimensional subspace of L_1 which is contained in the span of n independent symmetric p-stable variables for some $1 < p < 2$. If $u\colon l_1^m \to L_1$ is an operator for which*

$$u\,\mathrm{Ball}\, l_1^m \supset \mathrm{Ball}\, X,$$

then

$$m \geqslant \exp[(\delta/p^*)(k/n)\, n^{1/p^*}\|u\|^{-2}]$$

where $\delta > 0$ is an absolute constant.

Proof. We begin just as in Theorem 5.3. Assume that $d(X, l_2^k) \leqslant 2$ and $P\colon L_1 \to X$ satisfies for an appropriate $1 < r < p$,

$$\gamma_2(P\colon L_r \to X) \leqslant \min\big((r^*/2)^{1/2} n^{1/p-1/2}, k\big).$$

Set $s = (p+1)/2$, so that $D_{1,s}(X) \leqslant D$ for an absolute constant D. By Proposition 3.2, there is an operator $\tilde{u}\colon l_1^m \to X$ so that

$$\|\tilde{u}\| \leqslant \gamma_2(\tilde{u}) \leqslant 5\,C_{1,r}(u)(r^*)^{1/2} n^{1/p-1/2}$$

as long as condition (5.2) is satisfied.

We now shift to the proof of Theorem 5.2. Since $\tilde{u}^*\colon X^* \to l_\infty^m$ is norm-

increasing, from [BDGJN] we get

$$\|\tilde{u}\| \geqslant \delta_1 (k/\log m)^{1/2}$$

for some absolute constant $\delta_1 > 0$. Since $C_{1,2}(u) \leqslant (\operatorname{rank} u)^{1/2}\|u\|$, interpolation yields that for some absolute constant A,

$$C_{1,r}(u) \leqslant A\|u\| \quad \text{when } r^* \geqslant \log m.$$

Condition (5.2) holds when $r^* \geqslant Ap^*\log k$ (A another absolute constant), so the choice $r^* = Ap^*\log m$ allows us to conclude from the two estimates on $\|\tilde{u}\|$ that

$$\log m \geqslant (\delta/p^*)\, k^{1/2} n^{1/2-1/p}\|u\|^{-2},$$

where $\delta > 0$ is an absolute constant. ∎

Remark. If $p^*\log k \geqslant A(k/n)^{1/2} n^{1/p^*}$ (A is an absolute constant), then the conclusion of Theorem 5.4 can be improved to

$$m \geqslant \exp[\delta(k/n)^{1/2} n^{1/p^*}\|u\|^{-2}]$$

where $\delta > 0$ is absolute. To see this, in the proof of Theorem 5.4 make the choice $r^* = \max(\log m, Ap^*\log k)$. The further argument yields that for some absolute $\delta > 0$,

$$\log m \geqslant \delta\min\{(k/n)^{1/2} n^{1/p^*}, (k/n)\, n^{2/p^*}(p^*\log k)^{-1}\}\|u\|^{-2}.$$

PROPOSITION 5.5. *If $C > \sqrt{2\pi}$, then every operator $u\colon l_2^n \to L_1$ is a convex combination of operators $v\colon l_2^n \to L_1$ satisfying*

$$\gamma_{l_1^n}(v) \leqslant C\|u\|.$$

Proof. In Tomczak's notation [T-J2], we want to prove that

$$\gamma_1^{(n)}(u) \leqslant \sqrt{2\pi}\,\|u\|.$$

Since $\gamma_1^{(n)}(u) = \gamma_\infty^{(n)}(u^*)$ and the dual norm to $\gamma_\infty^{(n)}$ is $\pi_1^{(n)}$ (i.e., the absolutely summing norm defined in terms of n-tuples of vectors), it suffices to check the following:

If $w\colon l_2^n \to L_\infty$, then

$$|\mathrm{tr}(wu^*)| \leqslant \sqrt{2\pi}\,\|u\|\,\pi_1^{(n)}(w).$$

By Tomczak's result [T-J1], for such a w we have

$$\pi_2(w) \leqslant 2\pi_2^{(n)}(w) \leqslant 2\pi_1^{(n)}(w)$$

so that, by Grothendieck's theorem [G], [LP],

$$|\mathrm{tr}(wu^*)| = |\mathrm{tr}(u^*w)| \leqslant \pi_2(u^*)\pi_2(w) \leqslant \sqrt{\pi/2}\,\|u^*\|\,2\pi_1^{(n)}(w). \ ∎$$

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Received January 29, 1987 (2274)