STUDIA MATHEMATICA, T. XC. (1988)

Topological conjugacy of Morse flows over finite Abelian groups

by

WOJCIECH BUŁATEK (Toruń)

Abstract. The problem of topological conjugacy in the class of Morse flows over finite Abelian groups is investigated. A necessary and sufficient condition for two Morse flows to be topologically isomorphic is given.

Introduction. Generalized Morse sequences on n symbols and Morse flows have been defined and studied by Martin. In [6] he has described the topological structure, maximal equicontinuous factor and the form of endomorphisms of these flows. In [7] he has given a condition for the strict ergodicity and investigated the spectral properties of Morse flows. The solution of the same problems in the case of Morse flows over finite Abelian groups was given by Poch in [8].

An important class of topological systems is that of flows generated by substitutions. The classes of Morse flows and substitution flows coincide on a nonempty class of topological systems. The problem of topological conjugacy of substitution flows has been solved by Coven and Keane [1] and Markley [5].

In this paper we solve the above problem for arbitrary Morse flows over any finite Abelian group G. We show that if (Ω_x, T) and (Ω_y, T) are Morse flows generated by Morse sequences x over G and y over G', then they are topologically conjugate iff there exist blocks A, B with the same length, a generalized Morse sequence z over G and an isomorphism φ from G onto G' such that $x = A \times z$ and $y = B \times \hat{\varphi}(z)$ $(\hat{\varphi}((g_i)_{i=0}^{\infty}) = (\varphi(g_i))_{i=0}^{\infty})$. First we give a detailed description of the topological structure and of the equicontinuous structure relation of these flows. We give an algorithm to construct the set Ω_x.

In [6] Martin has shown that if

$$x = b^0 \times b^1 \times b^2 \times \ldots,$$

then the maximal equicontinuous factor of (Ω_x, T) is isomorphic to the group of $(r \cdot |b^0|, |b^1|, |b^2|, \ldots)$-adic integers with adding $\bar{1} = (1, 0, 0, \ldots)$, where $|b|$ denotes the length of the block b, and r is some divisor of $|G|$ ($|G|$ is the order of G). It turns out that the classes of the equicontinuous structure relation are uniquely determined by some subgroup H of G, described by the pairs occurring in $b^0, b^1, b^2, \ldots$ From this result we deduce that r is the order of the group G/H; so for isomorphic Morse flows the numbers r are equal and the groups of all $|b^0 \times b^1 \times \ldots \times b^t|$-roots of unity, $t \geqslant 0$, are the same.

Rojek [9] has shown that if $G = \boldsymbol{Z}_2$, if (Ω_x, T) is strictly ergodic and if x is continuous, then each class of measure-theoretic isomorphism is uncountable.

I. Definitions and preliminaries. We use $\boldsymbol{Z}$ to denote the integers. Let G be a nontrivial finite Abelian group. We call the elements of G^n ($n \in \boldsymbol{Z}$, $n \geqslant 1$) *n-blocks*, and if $B \in G^n$, then we say that $n = |B|$ is the *length* of B. Let $\Omega = G^{\boldsymbol{Z}}$ be the set of all bisequences over G and let $T\colon \Omega \to \Omega$ be the *shift transformation*, i.e. $\big(T(\omega)\big)_k = \omega_{k+1}$, $k \in \boldsymbol{Z}$, $\omega = (\omega_k)_{k=-\infty}^{+\infty} \in \Omega$.

We denote by $\omega(k, l)$ the block $(\omega_k, \omega_{k+1}, \ldots, \omega_{k+l-1})$, $k, l \in \boldsymbol{Z}$, $l \geqslant 1$. Similarly, if $B = (b_0, \ldots, b_{n-1})$ is a block, then $B(k, l) = (b_k, \ldots, b_{k+l-1})$ for $0 \leqslant k \leqslant n-1$, $1 \leqslant l \leqslant n-k$. We write $B(k)$ instead of $B(k, 1)$. If $B = (b_0, \ldots, b_{n-1})$ and $C = (c_0, \ldots, c_{m-1})$ are blocks, then the block BC is defined by

$$BC = (b_0, \ldots, b_{n-1}, c_0, \ldots, c_{m-1}).$$

Let $B+g$ denote the block $(b_0+g, \ldots, b_{n-1}+g)$ for $g \in G$. Now we can define the block $B \times C$ as follows:

$$B \times C = (B+c_0)\ldots(B+c_{m-1}). \tag{1}$$

The symbol $\omega+g$ ($\omega \in \Omega$, $g \in G$) will denote the sequence such that

$$(\omega+g)_k = \omega_k + g \quad \text{for } k \in \boldsymbol{Z}.$$

Taking now a sequence of blocks $(b^t)_{t=0}^{\infty}$ over G such that each b^t contains every symbol from G and $b^t(0) = 0$ for $t = 0, 1, \ldots$, and putting

$$x = \ldots\big((b^0 \times b^1) \times b^2\big) \times \ldots,$$

we obtain a one-sided sequence ($|b^t| = \lambda_t \geqslant |G|$). Aperiodic sequences of this form are called *generalized Morse sequences*. Since the operation (1) is associative we shall write

$$x = b^0 \times b^1 \times b^2 \times \ldots \tag{2}$$

It is well known ([6]) that if x is a generalized Morse sequence, then there exists an almost periodic point $\omega \in \Omega$ with $\omega(n) = x(n)$ for $n = 0, 1, 2, \ldots$

Now, let x be a fixed generalized Morse sequence over G and let $\omega \in \Omega$ be an almost periodic extension of x. We denote by Ω_x the orbit-closure of ω under T (it is independent of ω). In this way we obtain a topological dynamical system (Ω_x, T) which we call a *Morse flow*.

We start with a description of the set Ω_x. Let $\bar{\lambda} = (\lambda_t)_{t=0}^{\infty}$ ($\lambda_t = |b^t|$). Let $\Delta(\bar{\lambda}) = \prod_{t=0}^{\infty} \boldsymbol{Z}_{\lambda_t}$ ($\boldsymbol{Z}_n$ is the cyclic group of order n) be the group of $\bar{\lambda}$-adic integers and let $S(z) = z + \bar{1}$, where $z \in \Delta(\bar{\lambda})$ and $\bar{1} = (1, 0, 0, \ldots)$. If $z \in \Delta(\bar{\lambda})$ and

$$z = (i_0, i_1, i_2, \ldots),$$

we put

$$z_0 = i_0, \quad z_t = i_0 + i_1 n_0 + \ldots + i_t n_{t-1} \quad \text{for } t > 0,$$

where $n_t = \lambda_0 \cdot \ldots \cdot \lambda_t$.

Let $c_t = b^0 \times \ldots \times b^t$, $t = 0, 1, \ldots$ The blocks $c_t + g$, $g \in G$, will be called *t-symbols*.

Let $\bar{\Delta}$ be the subset of $\Delta(\bar{\lambda})$ obtained by removing the orbit of $\bar{0} = (0, 0, \ldots)$ under S. It is obvious that $z_{t+1} \geqslant z_t$ for any $z \in \Delta(\bar{\lambda})$, and the condition $z \in \bar{\Delta}$ is equivalent to

$$z_t \to \infty \quad \text{and} \quad n_t - z_t \to \infty \quad \text{as } t \to \infty.$$

We shall need the following two lemmas proved by Martin in [6].

Lemma 1. *For any $t \geqslant 0$ there is a $k > 0$ such that whenever $x(n, k) = x(m, k)$, then $m \equiv n \pmod{n_t}$.*

Lemma 2. *There is a flow homomorphism $f\colon (\Omega_x, T) \to (\Delta(\bar{\lambda}), S)$ such that if $z = (i_0, i_1, \ldots) \in \Delta(\bar{\lambda})$, then $f(\eta) = z$ iff $\eta(-z_t, n_t)$ is a t-symbol for every $t \geqslant 0$. For $g \in G$, $f(\eta+g) = f(\eta)$. Furthermore, if $z \in \bar{\Delta}$ and $\eta \in f^{-1}(z)$, then $f^{-1}(z) = \{\eta+g;\ g \in G\}$.*

In the remainder of this paper the symbol f will denote the flow homomorphism from Lemma 2.

Lemma 2 describes exactly the set $f^{-1}(\bar{\Delta}) \subset \Omega_x$. Now we describe the remaining elements of Ω_x. It is enough to examine the set $f^{-1}(\bar{0})$.

Let

$$x^t = b^t \times b^{t+1} \times \ldots$$

and let $N_t \subset G$, $t = 0, 1, \ldots$, be the set of all elements of the form

$$a - b + e_0 + e_1 + \ldots + e_t$$

such that the pair (a, b) appears in x^{t+1} and

$$e_0 = 0, \quad e_i = b^i(\lambda_i - 1) \quad \text{for } i = 1, \ldots, t.$$

If a pair (a, b) occurs in x^{t+1} then the pair $(a + e_t, b)$ occurs in x^t, so $N_{t+1} \subset N_t$. Let

$$N = \bigcap_{t \geqslant 0} N_t.$$

Remark 1. Observe that $|N| \geqslant 2$. Indeed, by the above there is a $t_0 \geqslant 0$ such that $N = N_t$ for $t \geqslant t_0$, so $|N| = |\{a-b;\ (a, b) \text{ appears in } x^{t_0+1}\}|$. But x is by our assumption aperiodic so x^{t_0+1} is aperiodic too. Moreover, every element of G appears in x^{t_0+1}. Hence at least $|G|+1$ different pairs must appear in x^{t_0+1}. Thus $|N| \geqslant 2$ because at most $|G|$ different pairs may have the same difference.

Now we construct sequences $\eta \in f^{-1}(\bar{0})$ such that $\eta(n) = x(n)$ for $n \geqslant 0$. Take $h \in N$ and define $\eta = \xi_0(h)$ as follows:

$$\eta(-n_t, n_t) = c_t + h_t$$

where

$$h_0 = h, \quad h_t = h-(e_1+\ldots+e_t) \quad \text{for } t > 0.$$

It is obvious that the sequences $\xi_0(h)$, $h \in N$, are well defined. Next put

$$\xi_g(h) = \xi_0(h)+g \quad \text{for } h \in N,\ g \in G.$$

We show that

$$f^{-1}(\bar{0}) = \{\xi_g(h);\ h \in N,\ g \in G\}. \tag{3}$$

The inclusion $\supset$ easily follows from the definitions of N and $\xi_g(h)$. By Lemma 2, $\eta \in f^{-1}(\bar{0})$ iff, for every $t \geqslant 0$, $\eta(0, n_t)$ is a t-symbol. Let $\eta \in f^{-1}(\bar{0})$ and $\eta(n) = x(n)$ for $n \geqslant 0$. Fix $t \geqslant 0$. We have

$$\eta(0, n_t) = c_t.$$

Since (Ω_x, T) is minimal, every block which occurs in $\eta \in \Omega_x$ occurs in x. So $\eta(-n_t, 2n_t)$ is a block in x. Take k satisfying Lemma 1 for t (we may assume $k = n_{t'}$ for some $t' > 0$). Hence the block $\eta(0, n_{t'})$ may occur in x at the places of the form ln_t $(l \in \mathbf{Z})$ only. Since $\eta(-n_t, n_t+n_{t'})$ appears in x, $\eta(-n_t, n_t)$ is a t-symbol, say $\eta(-n_t, n_t) = c_t+h_t$. Moreover, the pair $(h_t, 0)$ occurs in x^{t+1}. That implies $h_t+e_1+\ldots+e_t \in N_t$.

Applying this argument for $t = 0, 1, 2, \ldots$ we obtain a sequence $(h_t)_{t=0}^{\infty}$. It is easy to see that $h_t = h_{t+1}+e_{t+1}$, so $h_t+e_1+\ldots+e_t = h \in N$. In this way we have shown that $\eta = \xi_0(h)$, $h \in N$. In the case $\eta \in f^{-1}(\bar{0})$, $\eta(0, \infty) = x+g$, $g \in G$, we proceed as above. This means that (3) holds.

Remark 2. Observe that from this description it follows that if $\eta \in \Omega_x$ and $z = f(\eta)$, then $\eta(-z_t+kn_t, n_t)$ is a t-symbol for any $t \geqslant 0$ and $k \in \mathbf{Z}$. Moreover, if $\eta(-z_t+kn_t, 2n_t) = (c_t+g)(c_t+g')$, then

$$g-g'+e_0+e_1+\ldots+e_t \in N_t.$$

II. Equicontinuous structure relation. Let S_e denote the equicontinuous structure relation in Ω_x ([2]). Martin [6] has shown that $(\Omega_x/S_e, T/S_e)$ is isomorphic to $(\Delta(\bar{\lambda}'), S)$, where $\bar{\lambda}' = (r\lambda_0, \lambda_1, \lambda_2, \ldots)$ and r is some divisor of $|G|$. In this section we give a detailed description of S_e. We start with some definitions and facts.

We say that points $\eta, \theta \in \Omega_x$ are *proximal* iff

$$\inf_{i \in \mathbf{Z}} d(T^i\eta, T^i\theta) = 0,$$

where

$$d(\eta, \theta) = (1+\min\{|i|;\ \eta_i \neq \theta_i\})^{-1}$$

is the standard metric in Ω.

The proximal relation (not transitive) will be denoted by P. If η, θ are not proximal we say that they are *distal*. The smallest closed T-invariant equivalence relation S_d such that any two different points in $(\Omega_x/S_d, T/S_d)$ are distal is called the *distal structure relation*. It is well known [2] that

$$P \subset S_d \subset S_e. \tag{4}$$

Now, let $H \subset G$ be any subgroup of G. We denote by Δ_H the equivalence relation on Ω_x defined as follows:

$$\begin{aligned}(\eta, \theta) \in \Delta_H \text{ iff either } & f(\eta) = f(\theta) \in \bar{\Delta},\ \eta = \theta+g \text{ and } g \in H,\\ \text{or } & f(\eta) = f(\theta) = S^i(\bar{0}),\ \eta = T^i(\xi_g(h)),\\ & \theta = T^i(\xi_{g'}(h')),\quad g-g' \in H,\ h, h' \in N.\end{aligned}$$

Remark 3. Each class of the relation Δ_H contained in $f^{-1}(\bar{\Delta})$ has $|H|$ elements and each class contained in $\Omega_x \setminus f^{-1}(\bar{\Delta})$ has $|H|\cdot|N|$ elements.

Theorem 1. *The relations Δ_M and S_e coincide on Ω_x, where M is the subgroup of G generated by $N-N$. $(\Omega_x/S_e, T/S_e)$ is isomorphic to $(\Delta(|G/M|\cdot\lambda_0, \lambda_1, \ldots), S)$.*

Proof. First we show that Δ_M is the smallest T-invariant closed equivalence relation which contains P. Then we give the homomorphism f_1 from (Ω_x, T) onto $(\Delta(|G/M|\cdot\lambda_0, \lambda_1, \ldots), S)$ such that the relation given by f_1^{-1} and the relation Δ_M coincide on Ω_x. So by (4) we obtain $\Delta_M = S_e$.

To do this we describe the proximal relation P. Observe that $(\eta, \theta) \in \Delta_G$ iff $f(\eta) = f(\theta)$. Since S_e is the smallest equicontinuous relation, Lemma 2 and the inclusion (4) imply $P \subset \Delta_G$. It is obvious that if $\eta, \theta \in f^{-1}(\bar{\Delta})$ and $(\eta, \theta) \in P$, then $\eta = \theta$. Now, let $\eta, \theta \in \Omega_x$ be such that $f(\eta) = f(\theta) = S^i(\bar{0})$, i.e. $\eta = T^i(\xi_0(h)+g)$ and $\theta = T^i(\xi_0(h')+g')$. If $g = g'$, then $(\eta, \theta) \in P$ because $\eta(n) = \theta(n)$ for $n \geqslant -i$. If $g \neq g'$, then $(\eta, \theta) \in P$ iff $g-g' = h'-h$ because $\eta(n)$

$= \theta(n)+g'-g$ for $n \geqslant -i$ and $\eta(n) = \theta(n)+h'-h+g'-g$ for $n < -i$. So we obtain:

(5) $(\eta, \theta) \in P$ iff either $\eta = \theta$,

$$\text{or } f(\eta) = f(\theta) = S^i(\bar{0}),\ \eta = T^i(\xi_g(h)),$$
$$\theta = T^i(\xi_{g'}(h')) \text{ and } g-g' \in N-N.$$

Let R be the smallest T-invariant equivalence relation such that $P \subset R$. The relation P is T-invariant, reflexive and symmetric, but not transitive. So R is the transitive closure of P; but straightforward calculation using (5) shows that

(6) $(\eta, \theta) \in R$ iff either $\eta = \theta$,

$$\text{or } f(\eta) = f(\theta) = S^i(\bar{0}),\ \eta = T^i(\xi_g(h)),$$
$$\theta = T^i(\xi_{g'}(h')) \text{ and } g-g' \in M.$$

Now, we show that the closure of R (denoted by $\bar{R}$) is equal to Δ_M. From the above and from the definition of $\xi_g(h)$ it follows that if $\eta \neq \theta$ and $(\eta, \theta) \in R$, then there exist $i \in \mathbf{Z}$ and $g, h \in M$ such that

$$\theta(n) = \eta(n)+g \quad \text{for } n \geqslant i,$$
$$\theta(n) = \eta(n)+h \quad \text{for } n < i.$$

Now, let $(\eta_k, \theta_k)_{k=1}^{\infty} \subset R$ be a sequence such that $(\eta_k, \theta_k) \to (\eta, \theta) \in \bar{R}$ as $k \to \infty$. Then there exist i_k, g_k, h_k, $k = 1, 2, \ldots$, such that

$$\theta_k(n) = \eta_k(n)+g_k \quad \text{for } n \geqslant i_k,$$
$$\theta_k(n) = \eta_k(n)+h_k \quad \text{for } n < i_k.$$

Taking a subsequence of $(\eta_k, \theta_k)_{k=1}^{\infty}$ if necessary, we may assume $g_k = g$, $h_k = h$ for some $g, h \in M$ and $k = 1, 2, \ldots$ The sequence $(i_k)_{k=1}^{\infty}$ has either a bounded subsequence, a subsequence tending to $+\infty$, or a subsequence tending to $-\infty$. In the first case, $(\eta, \theta) \in R$; in the second and third, $\theta = \eta+g$ and $\theta = \eta+h$, respectively. Therefore $\bar{R} \subset \Delta_M$. Since the inverse inclusion is obvious we have $\bar{R} = \Delta_M$.

Now, let t_0 be such that $N = N_{t_0}$ and let $\eta_0 = \xi_0(h)$ for some fixed $h \in N$. From Remark 2 we have $\eta_0(kn_{t_0}, n_{t_0}) = c_{t_0}+g_k$, $k \in \mathbf{Z}$, and $g_{i+1}-g_i+e_0+e_1+\ldots+e_{t_0} \in N$. So

$$(g_{i+1}-g_i)-(g_i-g_{i-1}) \in M.$$

Thus, for $i \in \mathbf{Z}$, $[g_{i+1}-g_i]_M = [c]_M$, where $[g]_M$ is the element of G/M which contains g. Because every x^t, $t \geqslant 0$, contains every element of G, it follows that G/M is cyclic and $[c]_M$ is its generator. By Lemma 2, if a sequence $T^{j_k}(\eta_0)$ converges, then for every t, $j_{k_1} \equiv j_{k_2} \pmod{n_t}$ for all sufficiently large k_1, k_2. By the above we may in this implication write rn_t instead of n_t, where $r = |G/M|$. So the map $\bar{f}_1$: Orb $\eta_0 \to \Delta_1 = \Delta(r\lambda_0, \lambda_1, \lambda_2, \ldots)$ given by $\bar{f}_1(T^n(\eta_0)) = n\bar{1}$ has a unique extension to a continuous map f_1 from Ω_x onto Δ_1. It is obvious that f_1 is a flow homomorphism. Analogously we can construct a flow homomorphism f_0 from Δ_1 onto $\Delta_0 = \Delta(\lambda_0, \lambda_1, \ldots)$ ($\bar{f}_0(n\bar{1}) = n\bar{1}$, where the first unity is in Δ_1 and the second in Δ_0). So we obtain

$$\Omega_x \xrightarrow{f_1} \Delta_1 \xrightarrow{f_0} \Delta_0.$$

By the construction, $f_0 f_1 = f$ on the orbit of η_0; therefore by continuity this is true on all Ω_x. Let Δ_{f_1} denote the relation on Ω_x given by f_1^{-1}. Thus the elements of Δ_G are unions of r elements of Δ_{f_1}.

From the construction of f_1 it follows that if $f_1(\eta_1) = f_1(\eta_2)$, then $g_k^1 - g_k^2 \in M$, $k \in \mathbf{Z}$, where the g_k^i's are such that

$$\eta_i(-z_{t_0}+kn_{t_0}, n_{t_0}) = c_{t_0}+g_k^i, \quad i = 1, 2, \quad z = f(\eta_1) = f(\eta_2).$$

The last two conclusions give $\Delta_{f_1} = \Delta_M$ and the proof is complete.

III. Topological conjugacy of Morse flows. Let x and y be fixed generalized Morse sequences over G and G' respectively and let

$$x = b^0 \times b^1 \times \ldots, \qquad y = \beta^0 \times \beta^1 \times \ldots$$

For $t = 0, 1, \ldots$, set

$$|b^t| = \lambda_t^x, \quad |\beta^t| = \lambda_t^y, \quad c_t = b^0 \times b^1 \times \ldots \times b^t,$$
$$d_t = \beta^0 \times \beta^1 \times \ldots \times \beta^t, \quad |c_t| = n_t^x, \quad |d_t| = n_t^y,$$
$$x^t = b^t \times b^{t+1} \times \ldots, \quad y^t = \beta^t \times \beta^{t+1} \times \ldots$$

Before the classification theorem we shall prove the following:

LEMMA 3. *If Φ: $(\Omega_x, T) \to (\Omega_y, T)$ is a flow isomorphism, then there exists a group isomorphism φ: $G \to G'$ such that the diagram*

$$\begin{tikzcd}
\Omega_x \arrow[r, "g"] \arrow[d, "\Phi"] & \Omega_x \arrow[d, "\Phi"] \\
\Omega_y \arrow[r, "\varphi(g)"'] & \Omega_y
\end{tikzcd}$$

is commutative for any $g \in G$, where $h(\eta) = \eta+h$, $\eta \in \Omega_x$ $[\Omega_y]$, $h \in G$ $[G']$.

Proof. For each $g \in G$ the map $\Phi g \Phi^{-1}$ is an automorphism of Ω_y. By Theorem 10 from [6] we have

(7) $$\Phi g \Phi^{-1} = T^k \varphi(g),$$

where $\varphi(g)$ is some element of G' depending on g, and k is an integer. Let $r(g)$ and $r(\varphi(g))$ be the orders of g and $\varphi(g)$ respectively. We put $s = r(g) \cdot r(\varphi(g))$. Then by (7) we have

$$\mathrm{Id}_{\Omega_y} = \Phi sg \Phi^{-1} = T^{sk} s \cdot \varphi(g) = T^{sk}$$

because $T\varphi(g) = \varphi(g) T$ on Ω_y. Thus $k = 0$ and $\Phi g \Phi^{-1} = \varphi(g)$. Now it is easy to see that φ is a group isomorphism from G onto G'.

THEOREM 2. *The Morse flows* (Ω_x, T) *and* (Ω_y, T) *are topologically isomorphic iff there exist blocks* A *over* G, B *over* G', *a generalized Morse sequence* z *over* G *and a group isomorphism* φ: $G \to G'$ *such that* $|A| = |B|$, $x = A \times z$ *and* $y = B \times \hat{\varphi}(z)$ $(\hat{\varphi}((z_i)_{i=0}^{\infty}) = (\varphi(z_i))_{i=0}^{\infty}$, $z = (z_i)_{i=0}^{\infty})$.

Proof. Let Φ: $\Omega_x \to \Omega_y$ be a flow isomorphism. We shall find blocks A, B, a generalized Morse sequence z and an isomorphism φ as in the theorem. Since Φ preserves the equicontinuous structure relation, the groups M for both flows have the same order. By Lemma 3 the groups G and G' also have the same order. Hence by Theorem 1 and the criterion for two $\bar{\lambda}$-adic groups to be isomorphic, the groups $\Delta(\lambda_0^x, \lambda_1^x, \ldots)$ and $\Delta(\lambda_0^y, \lambda_1^y, \ldots)$ are isomorphic. Thus for every $t \geq 0$ there exists a t' such that

(8) $$n_t^x \text{ divides } n_{t'}^y.$$

Let $\omega \in \Omega_x$ be such that

(9) $$\omega(n) = x(n) \quad \text{for } n \geq 0,$$

and let $\eta = \Phi(\omega)$. Since Φ preserves the relation S_e and the class of ω has $|M| \cdot |N| > |M|$ elements (Remark 1), the class of η has $|M| \cdot |N|$ elements too. Hence there exist $m_0 \in \boldsymbol{Z}$ and $g' \in G'$ such that

(10) $$\eta = T^{m_0}(\eta'), \quad \eta'(n) = y(n) + g', \quad n \geq 0.$$

By Hedlund's Theorem there exist $l_0, k_0 \in \boldsymbol{Z}$, $l_0 > 0$ and ψ: $G^{l_0} \to G'$ such that $\Phi = \psi_\infty T^{k_0}$, i.e. for every $\theta \in \Omega_x$

$$(\Phi(\theta))_n = \psi(\theta_{n+k_0}, \ldots, \theta_{n+k_0+l_0-1}), \quad n \in \boldsymbol{Z}.$$

By Lemma 3 it is easy to see that if C is an l_0-block, then

(11) $$\psi(C+g) = \psi(C) + \varphi(g), \quad g \in G.$$

Let $t_1 > 0$ be such that

(12) $$n_{t_1}^x > |m_0 - k_0| + l_0.$$

By (8) there exists t_2 such that

(13) $$n_{t_1}^x \text{ divides } n_{t_2}^y.$$

Let $s = n_{t_2}^y / n_{t_1}^x$ and

(14) $$A_k = \omega(kn_{t_2}^y, n_{t_2}^y), \quad k \in \boldsymbol{Z},$$

(15) $$D_k = \eta(-m_0 + kn_{t_2}^y, n_{t_2}^y), \quad k \in \boldsymbol{Z}.$$

Each of the D_k's is a t_2-symbol in y, i.e.

(16) $$D_k = d_{t_2} + h_k, \quad h_k \in G', \ k \in \boldsymbol{Z}.$$

By (13) and (14) the blocks A_k have the form

(17) $$A_k = C_0^k C_1^k \ldots C_{s-1}^k,$$

where the C_i^k's are t_1-symbols in x and have the form

(18) $$C_i^k = c_{t_1} + g_i^k, \quad g_i^k \in G, \ k \in \boldsymbol{Z}, \ i = 0, 1, \ldots, s-1.$$

By (10), (15) and (16) we have $g' = h_0$ and

(19) $$y^{t_2+1} + h_0 = (h_0, h_1, h_2, \ldots)$$

and from (9), (14)

(20) $$x = A_0 A_1 \ldots$$

Now we are able to show that

(21) $$x = A_0 \times \hat{\varphi}^{-1}(y^{t_2+1}).$$

By (17)–(20), in order to do this it remains to show that

(22) $$g_i^k - g_i^0 = \varphi^{-1}(h_k - h_0), \quad 0 \leq i \leq s-1, \ k = 0, 1, \ldots$$

First consider the case $m_0 - k_0 \leq 0$. By (12), $-m_0 + k_0 + l_0 < n_{t_1}^x$ and we have

(23) $$\psi(C_0^k(-m_0+k_0, l_0)) = D_k(0),$$

(24) $$\psi(C_i^k(0, l_0)) = D_k(in_{t_1}^x + m_0 - k_0), \quad 1 \leq i \leq s-1, \ k \geq 0.$$

Hence

$$\begin{aligned} \psi((c_{t_1} + g_0^k)(-m_0+k_0, l_0)) &= D_k(0) = D_0(0) + h_k - h_0 \\ &= \psi((c_{t_1} + g_0^0)(-m_0+k_0, l_0)) + h_k - h_0, \end{aligned}$$

$$\begin{aligned} \psi((c_{t_1} + g_i^k)(0, l_0)) &= D_k(in_{t_1}^x + m_0 - k_0) = D_0(in_{t_1}^x + m_0 - k_0) + h_k - h_0 \\ &= \psi((c_{t_1} + g_i^0)(0, l_0)) + h_k - h_0, \quad 1 \leq i \leq s-1. \end{aligned}$$

Using (11) we have

$$\begin{aligned}\psi(c_{t_1}(-m_0+k_0,\, l_0))+\varphi(g_0^k) &= \psi\big((c_{t_1}+g_0^k)(-m_0+k_0,\, l_0)\big)\\ &= \psi\big((c_{t_1}+g_0^0)(-m_0+k_0,\, l_0)\big)+h_k-h_0\\ &= \psi(c_{t_1}(-m_0+k_0,\, l_0))+\varphi(g_0^0)+h_k-h_0\end{aligned}$$

and similarly

$$\psi(c_{t_1}(0,\, l_0))+\varphi(g_i^k) = \psi(c_{t_1}(0,\, l_0))+\varphi(g_i^0)+h_k-h_0, \quad 1\leqslant i\leqslant s-1,\ k\geqslant 0.$$

Then

$$g_i^k-g_i^0 = \varphi^{-1}(h_k-h_0), \quad 0\leqslant i\leqslant s-1;\ k=0, 1, \ldots$$

Thus (22) holds and consequently (21) is valid.

If $m_0-k_0>0$, then instead of (23) and (24) we have

$$(23')\qquad C_i^k(0,\, l_0) = D_k(in_{t_1}^x+m_0-k_0), \quad 0\leqslant i\leqslant s-1,\ k\geqslant 0.$$

By (12), $0\leqslant in_{t_1}^x+m_0-k_0\leqslant n_{t_2}^y$ for $0\leqslant i\leqslant s-1$, so (23′) is correct. Now we obtain (21) in the same way as in the first case. Taking

$$z=\hat{\varphi}^{-1}(y^{t_2+1}), \quad A=A_0, \quad B=d_{t_2},$$

we obtain the result.

The converse implication follows immediately from the fact that for each $\theta\in\Omega_x$ $[\Omega_y]$ there exists a unique p_θ such that $0\leqslant p_\theta\leqslant |A|-1$ and $\theta(p_\theta+k|A|,\, |A|)=A+g_k$ $[B+h_k]$ for every $k\in \boldsymbol{Z}$. This is a simple consequence of Lemma 1. If we denote by $\tilde{\theta}$ the sequence $(g_k)_{k=-\infty}^{+\infty}$ $[(h_k)_{k=-\infty}^{+\infty}]$, then we obtain the desired isomorphism $\Phi\colon \Omega_x\to\Omega_y$ as follows:

$$\Phi(\theta)=\theta' \quad \text{iff} \quad p_\theta=p_{\theta'} \text{ and } \hat{\varphi}(\tilde{\theta})=\tilde{\theta}'.$$

Thus the proof is complete.

Corollary. *If $G=G'=\boldsymbol{Z}_2$, then the unique group isomorphism is the identity on $\boldsymbol{Z}_2$; so (Ω_x, T) and (Ω_y, T) are topologically isomorphic iff there exist blocks A, B with the same length and a Morse sequence z over $\boldsymbol{Z}_2$ such that $x=A\times z$ and $y=B\times z$.*

Remark 4. In the first part of the proof of Theorem 2 we have shown that if (Ω_x, T) and (Ω_y, T) are isomorphic, then the groups $\Delta(\lambda_0^x, \lambda_1^x, \ldots)$ and $\Delta(\lambda_0^y, \lambda_1^y, \ldots)$ are isomorphic. We shall use this fact in the proof of the next proposition.

Note that if $b=b^0=b^1=\ldots$, then $x=b\times b\times\ldots$ is the sequence generated by the substitution $\Phi(g)=b+g$, $g\in G$, of constant length. In [5] Markley has shown that if Φ and Ψ are substitutions of constant lengths $|\Phi|$, $|\Psi|$ respectively, both over two symbols, and both in the normal form, then the flows generated by them are topologically isomorphic iff there exist positive integers m, n such that $|\Phi|^m=|\Psi|^n$ and $\Phi^m=\Psi^n$. In the case of Morse flows over two symbols, this means that if $x=b\times b\times\ldots$ and $y=\beta\times\beta\times\ldots$, then (Ω_x, T) is isomorphic to (Ω_y, T) iff there exist $m, n>0$ such that $|b|^m=|\beta|^n$ and

$$b^m := b\times b\times\ldots\times b \ (m \text{ factors}) = \beta^n := \beta\times\beta\times\ldots\times\beta \ (n \text{ factors}).$$

We show the analogous result for an arbitrary finite Abelian group G.

Proposition. *If*

$$x=b\times b\times\ldots, \quad y=\beta\times\beta\times\ldots$$

are generalized Morse sequences over G and G' respectively, then (Ω_x, T) and (Ω_y, T) are topologically isomorphic iff there exist integers $m, n>0$ and a group isomorphism $\varphi\colon G\to G'$ such that

$$|b|^m=|\beta|^n, \quad (\hat{\varphi}(b))^m=\beta^n,$$

where $\hat{\varphi}(b_0, \ldots, b_{|b|-1})=(\varphi(b_0), \ldots, \varphi(b_{|b|-1}))$.

Proof. By Theorem 2, if (Ω_x, T) and (Ω_y, T) are topologically isomorphic, then there exist blocks A, B with the same length, a generalized Morse sequence z over G and a group isomorphism $\psi\colon G\to G'$ such that

$$(25)\qquad x=A\times z, \quad y=B\times\hat{\psi}(z).$$

It is easy to see that if ϱ is a group isomorphism, then for any two blocks C, D we have $\hat{\varrho}(C\times D)=\hat{\varrho}(C)\times\hat{\varrho}(D)$, so

$$\hat{\psi}^{-1}(y)=\hat{\psi}^{-1}(B)\times z, \quad \hat{\psi}^{-1}(y)=\hat{\psi}^{-1}(\beta)\times\hat{\psi}^{-1}(\beta)\times\ldots$$

Set

$$(26)\qquad y'=\hat{\psi}^{-1}(y), \quad B'=\hat{\psi}^{-1}(B), \quad d=\hat{\psi}^{-1}(\beta).$$

Thus

$$(27)\qquad y'=d\times d\times d\times\ldots,$$

$$(28)\qquad y'=B'\times z,$$

where $|B'|=|B|=|A|$ and $|d|=|\beta|$.

By Remark 4 and (25), (27), (28) there exist blocks A_1, B_1 and integers $m_1, n_1>0$ such that

$$A\times A_1=b^{m_1}, \quad B'\times B_1=d^{n_1}.$$

Hence

$$A\times A_1\times b\times b\times\ldots=A\times z, \quad B'\times B_1\times d\times d\times\ldots=B'\times z,$$

and so

$$z = A_1 \times b \times b \times \ldots = B_1 \times d \times d \times \ldots$$

Thus by Remark 4 there exist a block B_2 and an $m_2 > 0$ such that

$$B_1 \times B_2 = A_1 \times b^{m2},$$

and so

$$B_1 \times B_2 \times b \times b \times \ldots = B_1 \times d \times d \times \ldots$$

Then

(29) $$B_2 \times b \times b \times \ldots = d \times d \times \ldots$$

Reasoning as above we find a block B_3 and an integer $n_2 > 0$ such that

(30) $$B_2 \times B_3 = d^{n2},$$

and so

$$B_2 \times b \times b \times \ldots = B_2 \times B_3 \times d \times d \times \ldots,$$

(31) $$b \times b \times \ldots = B_3 \times d \times d \times \ldots$$

From (29) and (31) we obtain

$$b \times b \times \ldots = B_3 \times B_2 \times b \times b \times \ldots$$

Easy induction gives

$$x = b \times b \times \ldots = B_3 \times B_2 \times B_3 \times B_2 \times \ldots$$

On the other hand, by (30) we have

$$y' = d \times d \times \ldots = B_2 \times B_3 \times B_2 \times B_3 \times \ldots$$

Write $D = B_3 \times B_2$ and $D' = B_2 \times B_3$, so that

(32) $$x = D \times D \times \ldots,$$

(33) $$y' = D' \times D' \times \ldots,$$

(34) $$|D| = |D'|.$$

Using Remark 4 and (25), (32) and (28), (33) we find blocks A', B'' and an integer $n_3 > 0$ such that

$$A \times A' = D^{n3}, \quad B' \times B'' = D'^{n3}.$$

(Existence of one n_3 for both equalities follows from (34) and from $|A| = |B'|$.) So

(35) $$|A'| = |B''|.$$

Hence $A \times z = A \times A' \times D \times D \times \ldots$ and $B' \times B'' \times D' \times D' \times \ldots = B' \times z$, so

$$z = A' \times D \times D \times \ldots = B'' \times D' \times D' \times \ldots$$

Now from (34) and (35) we obtain $A' = B''$ and $D = D'$. Thus

(36) $$x = y'.$$

Note that if we put $\varphi = \psi$, then by (26), (36) and by the obvious equality $\hat{\varphi}(x) = \hat{\varphi}(b) \times \hat{\varphi}(b) \times \ldots$ it suffices to show that there exist $m, n > 0$ such that

(37) $$|b|^n = |d|^m.$$

By (36), Remark 4 and the criterion for two $\bar{\lambda}$-adic groups to be isomorphic we deduce that $|b|$ and $|d|$ have the same prime divisors. Suppose that (37) does not hold. Thus if $p_1, \ldots, p_k$ are all distinct prime divisors of $|b|$ or $|d|$, then

$$|b| = p_1^{\gamma_1} \ldots p_k^{\gamma_k}, \quad |d| = p_1^{\delta_1} \ldots p_k^{\delta_k},$$

and the vectors $(\gamma_1, \ldots, \gamma_k)$ and $(\delta_1, \ldots, \delta_k)$ are not proportional. So the numbers γ_i/δ_i, $1 \leqslant i \leqslant k$, are not equal ($k$ must be > 1 and $\gamma_i \neq 0 \neq \delta_i$). Let $1 \leqslant i_0 \leqslant k$ be such that

$$\gamma_{i_0}/\delta_{i_0} = \min_{1 \leqslant i \leqslant k} \gamma_i/\delta_i.$$

Then $|d|^{\gamma_{i_0}}$ divides $|b|^{\delta_{i_0}}$ and these numbers are not equal. So by (27) and (36) we obtain

$$b^{\delta_{i_0}} = d^{\gamma_{i_0}} \times c \quad \text{for some block } c.$$

Hence

$$d^{\gamma_{i_0}} \times c \times b \times b \times \ldots = d \times d \times \ldots$$

Thus

$$c \times b \times b \times \ldots = d \times d \times \ldots = b \times b \times \ldots$$

Easy induction gives

$$c \times c \times \ldots = b \times b \times \ldots$$

But $|c| = |b|^{\delta_{i_0}}/|d|^{\gamma_{i_0}}$, so p_{i_0} does not divide $|c|$. This contradiction implies (37) for some $m, n > 0$. The converse implication is obvious by Theorem 2.

References

[1] E. M. Coven and M. S. Keane, *The structure of substitution minimal sets*, Trans. Amer. Math. Soc. 162 (1971), 89–102.

[2] R. Ellis and W. H. Gottschalk, *Homomorphisms of transformation groups*, ibid. 94 (1960), 258–271.

[3] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, vol. I, Grundlehren Math. Wiss. 115, Academic Press, New York, and Springer, Berlin 1963.
[4] M. S. Keane, *Generalized Morse sequences*, Z. Wahrsch. Verw. Gebiete 10 (1968), 335–353.
[5] N. G. Markley, *Substitution-like minimal sets*, Israel J. Math. 22 (1975), 332–353.
[6] J. C. Martin, *Generalized Morse sequences on n symbols*, Proc. Amer. Math. Soc. 54 (1976), 379–383.
[7] —, *The structure of generalized Morse minimal sets on n symbols*, Trans. Amer. Math. Soc. 232 (1977), 343–355.
[8] M. G. Poch, *Suites récurrentes sur les groupes abéliens*, C. R. Acad. Sci. Paris Sér. A 283 (1976), 1111–1113.
[9] T. Rojek. *On metric isomorphism of Morse dynamical systems*, Studia Math. 84 (1986), 247–267.

INSTYTUT MATEMATYKI UNIWERSYTETU MIKOŁAJA KOPERNIKA
INSTITUTE OF MATHEMATICS, NICHOLAS COPERNICUS UNIVERSITY
Chopina 12/18, 87-100 Toruń, Poland

Received October 16, 1986 (2227)
Revised version April 3, 1987

STUDIA MATHEMATICA, T. XC. (1988)

An isomorphic Banach–Stone theorem

by

EHRHARD BEHRENDS (Berlin) and MICHAEL CAMBERN (Santa Barbara, Cal.)

Abstract. For a given Banach space Y let $\lambda_0(Y)$ denote the infimum of the Banach–Mazur distances between the two-dimensional subspaces of Y and the two-dimensional L^1-space l_2^1.

It is shown that every Banach space X such that $\lambda_0(X')$ is greater than one satisfies the following isomorphic version of the classical Banach–Stone theorem: there is a $\delta > 0$ such that two locally compact Hausdorff spaces K and L are necessarily homeomorphic provided that there is an isomorphism T between $C_0(K, X)$ and $C_0(L, X)$ with $\|T\|\|T^{-1}\| \leqslant 1+\delta$.

This result properly includes all isomorphic Banach–Stone theorems now existing in the literature. It is obtained by means of the description of small-bound isomorphisms between certain L^1-direct sums of Banach spaces.

1. Introduction. For a locally compact Hausdorff space K and a Banach space X we denote by $C_0(K, X)$ the space of X-valued continuous functions on K which vanish at infinity, provided with the supremum norm. If X is the scalar field $\mathbf{K}$ (where $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$) this space is denoted by $C_0 K$.

The classical Banach–Stone theorem states that if $C_0 K$ and $C_0 L$ are isometrically isomorphic, then K and L are homeomorphic. Various authors, beginning with M. Jerison [10], have considered the problem of determining geometric properties of X which allow generalizations of this theorem to spaces of continuous vector-valued functions $C_0(K, X)$. A compilation of results of this nature may be found in [3].

A second kind of generalization deals with scalar-valued functions, but replaces isometries by isomorphisms T with $\|T\|\|T^{-1}\|$ close to one [1], [5], [6].

These two directions have been combined in [7] and [9] where it is shown that a small-bound theorem is obtainable for spaces of vector-valued functions $C_0(K, X)$ for certain X.

Here we prove a theorem which strictly contains the results in [7] and [9]. To state this theorem we need the following definition: for any Banach space Y we denote by $\lambda_0(Y)$ the number

$$\lambda_0(Y) := \inf d(Y_2, l_2^1),$$

1980 *Mathematics Subject Classification*: 46E40, 46E15, 46B20.