ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
20,2 (1990), pp. 239–244

T. ŚLIWA (Wrocław)

UNIQUENESS OF THE SOLUTION OF THE CAUCHY PROBLEM TO SOME EQUATIONS OF NONSTATIONARY FILTRATION

Consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}(\varphi(x, t, u)), \tag{1}$$

where the function $\varphi(x, t, u)$ is defined for $x \in R$, $t \geqslant 0$, $u \geqslant 0$. Assume also that

$$\varphi(x, t, u) > 0, \quad \varphi_u(x, t, u) > 0 \quad \text{if } u > 0,$$

and

$$\varphi(x, t, 0) \equiv \varphi_u(x, t, 0) \equiv 0.$$

In the special case $\varphi = \varphi(u)$, (1) reduces to the one-dimensional equation of nonstationary filtration (cf. [1]). For $u > 0$, (1) is a parabolic equation of second order, for $u = 0$ it is degenerated. For some initial or boundary data in Cauchy or boundary value problems for (1), the equation does not degenerate and via the substitution $v = \varphi(x, t, u)$ reduces to a quasilinear equation of parabolic type. The Cauchy problem and the boundary value problems are solvable (of course, with some natural additional assumptions on φ). However, in the general case these problems might not have classical solutions and their solvability is meant in some generalized sense (cf. [2] and [3]).

The paper [2] deals with the Cauchy problem and the boundary value problems of the first and second kinds. The proofs are given for $\varphi = \varphi(x, u)$ assuming the Lipschitz continuity of $\varphi(x, u(x, 0))$. The Cauchy problem for (1) with $\varphi = \varphi(x, u)$ and the initial conditions from $L^\infty(R)$ are considered in [3].

The multidimensional analogue of equation (1) is of the form

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}(\varphi(x, t, u)), \tag{2}$$

where the function $\varphi(x, t, u)$ is defined for $x = (x_1, \ldots, x_N) \in R^N$, $t \geqslant 0$, $u \geqslant 0$

and satisfies

$$\varphi(x, t, u) > 0, \quad \varphi_u(x, t, u) > 0 \quad \text{if } u > 0$$

and

$$\varphi(x, t, 0) \equiv \varphi_u(x, t, 0) \equiv 0.$$

Here and in the sequel $\partial\varphi/\partial x_i$ and $\partial\varphi/\partial t$ are the derivatives of $\varphi(x, t, u(x,t))$ with respect to x_i and t, respectively, and φ_{x_i}, φ_t, φ_u the derivatives of $\varphi(x, t, u)$ with respect to x_i, t, u.

The problem of generalization of the result from the one-dimensional to the multidimensional case seems to be interesting. The existence of the solution of the Cauchy problem can be proved similarly as in one dimension, and the solution is the limit of the solutions of some nondegenerate quasilinear parabolic equations. The method of proving the uniqueness of this solution used in [2] does not generalize to the multidimensional case.

In this paper we show that the Cauchy problem for equation (2) has at most one solution if $N = 1$ or $N = 2$. The proof is independent of the existence and regularity results.

Consider the Cauchy problem for equation (2) in the strip

$$G = \{(x, t): x \in R^N, \ 0 \leqslant t \leqslant T\}, \quad 0 < T < \infty,$$

with the initial condition

$$u(x, 0) = u_0 \in L^\infty(R^N). \tag{3}$$

The generalized solution of the problem (2), (3) is defined by the following

DEFINITION. A function $u \geqslant 0$ defined on G is called a *generalized solution* of the problem (2), (3) if

(i) $u \in C^0(G) \cap L^\infty(G)$, $\partial\varphi/\partial x_i \in L^\infty(G)$;

(ii) for each function $f \in C_0^1(G)$ which vanishes at $t = T$ we have

$$\iint_G \left[\frac{\partial f}{\partial t} u - \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}\frac{\partial\varphi}{\partial x_i}\right] dx dt + \int_{R^N} f(x, 0) u_0(x) dx = 0. \tag{4}$$

THEOREM. *Assume that the functions φ, φ_u for bounded u are continuous and bounded in G and $N < 3$. Then the problem* (2), (3) *has at most one generalized solution.*

Remark. The assumption $N < 3$ is used only in the final part of our proof. The difficulties of generalization of our proof to the case $N \geqslant 3$ become apparent when we consider an arbitrary N.

Proof of the Theorem. At first we prove that equality (4) holds for any function $f \in C_0^0(G)$ for which

$$f(x, T) = 0, \quad \partial f/\partial x_i, \ \partial f/\partial t \in L^\infty(G).$$

Let

$$\varrho \in C_0^\infty(R^N \times R), \qquad \varrho \geqslant 0, \qquad \iint_{R^N \times R} \varrho(x, t)\, dxdt = 1.$$

A function f can be extended to a continuous function $\tilde{f}$ in R^{N+1} by using the formula

$$\tilde{f}(x, t) = \begin{cases} 0 & \text{for } t > T,\ x \in R^N, \\ f(x, t) & \text{for } (x, t) \in G, \\ f(x, 0) & \text{for } t < 0,\ x \in R^N. \end{cases}$$

Let also

$$J_h\tilde{f}(x, t) = \frac{1}{h^{N+1}} \iint_{R^{N+1}} \varrho\left(\frac{x-y,\ t-\tau}{h}\right) \tilde{f}(y, \tau)\, dyd\tau \qquad \text{for } h > 0.$$

If $f \in C_0^0(G)$, $\partial f/\partial x_i$, $\partial f/\partial t \in L^\infty(G)$, then the sequence $\{J_h\tilde{f}|_G\} \subset C_0^\infty(G)$ is uniformly convergent, when $h \to 0$, to the function f and the sequences $\{(\partial/\partial x_i)J_h\tilde{f}\}$, $\{(\partial/\partial t)J_h\tilde{f}\}$ converge in $L^2(G)$ to $\partial f/\partial x_i$, $\partial f/\partial t$, respectively. Hence (as $\partial\varphi/\partial x_i \in L^\infty(G)$) equality (4) holds if we replace f by

$$f_h^n = [1-(t/T)^n]\, J_h\tilde{f}|_G.$$

Since $f(x, T) = 0$, it is easy to verify that (4) holds for the limit function

$$f(x,t) = \lim_{\substack{n\to\infty \\ h\to 0}} f_h^n(x, t).$$

Let us assume that the problem (2), (3) has two different solutions u_1 and u_2. From equality (4) written for u_2 and u_1, respectively, we obtain

$$\text{(5)} \qquad \iint_G \left\{\frac{\partial f}{\partial t}(u_1 - u_2) - \sum_{i=1}^N \frac{\partial f}{\partial x_i}\left[\frac{\partial\varphi(x, t, u_1)}{\partial x_i} - \frac{\partial\varphi(x, t, u_2)}{\partial x_i}\right]\right\} dxdt = 0.$$

Let $\{\alpha_n(x)\}$ be a sequence of functions which have the following properties: $\alpha_n(x) = 1$ for $|x| \leqslant n-1$, $\alpha_n(x) = 0$ for $|x| \geqslant n$, $0 \leqslant \alpha_n(x) \leqslant 1$ for $n-1 \leqslant |x| \leqslant n$, the functions $\{\partial\alpha_n/\partial x_i\}$ ($i = 1, 2, \ldots, N$; $n = 1, 2, \ldots$) are uniformly bounded.

With the help of $\alpha_n(x)$ we define now a new sequence of functions $\{f_n(x, t)\}$ by putting

$$f_n(x, t) = \alpha_n(x) \int_T^t [\varphi(x, \tau, u_1(x, \tau)) - \varphi(x, \tau, u_2(x, \tau))]\, d\tau.$$

It is easy to verify that

$$f_n \in C_0^0(G), \quad f_n(x, T) \equiv 0 \quad \text{and} \quad \partial f/\partial x_i,\ \partial f/\partial t \in L^\infty(G).$$

Therefore, equalities (4) and (5) hold for each function f_n.

Equality (5), with f replaced by f_n defined as above, is of the form

$$\text{(6)} \qquad I_{1,n} + I_{2,n} + I_{3,n} = 0,$$

where

$$I_{1,n} = \iint_G \alpha_n(x)[\varphi(x,t,u_1)-\varphi(x,t,u_2)](u_1-u_2)\,dxdt,$$

$$I_{2,n} = -\iint_G \alpha_n(x)\sum_{i=1}^{N}\left\{\int_T^t\left[\frac{\partial\varphi(x,\tau,u_1)}{\partial x_i}-\frac{\partial\varphi(x,\tau,u_2)}{\partial x_i}\right]d\tau\left[\frac{\partial\varphi(x,t,u_1)}{\partial x_i}-\frac{\partial\varphi(x,t,u_2)}{\partial x_i}\right]\right\}dxdt,$$

$$I_{3,n} = -\iint_{Q_n}\left\{\int_T^t[\varphi(x,\tau,u_1)-\varphi(x,\tau,u_2)]\,d\tau\right\}\left\{\sum_{i=1}^{N}\frac{\partial\alpha_n}{\partial x_i}\left[\frac{\partial\varphi(x,t,u_1)}{\partial x_i}-\frac{\partial\varphi(x,t,u_2)}{\partial x_i}\right]\right\}dxdt,$$

and $Q_n = \{(x,t)\in G\colon n-1\leqslant |x|\leqslant n\}$.

Since

$$I_{2,n} = -\frac{1}{2}\iint_G \alpha_n(x)\sum_{i=1}^{N}\frac{\partial}{\partial t}\left\{\int_T^t\left[\frac{\partial\varphi(x,\tau,u_1)}{\partial x_i}-\frac{\partial\varphi(x,\tau,u_2)}{\partial x_i}\right]d\tau\right\}^2 dxdt$$
$$= \frac{1}{2}\int_{R^N}\alpha_n(x)\sum_{i=1}^{N}\left\{\int_T^0\left[\frac{\partial\varphi(x,\tau,u_1)}{\partial x_i}-\frac{\partial\varphi(x,\tau,u_2)}{\partial x_i}\right]d\tau\right\}^2 dx,$$

we may write equality (6) in the form

$$(7)\qquad I_{1,n}+\frac{1}{2}\int_{R^N}\alpha_n(x)\sum_{i=1}^{N}\left\{\int_T^0\left[\frac{\partial\varphi(x,\tau,u_1)}{\partial x_i}-\frac{\partial\varphi(x,\tau,u_2)}{\partial x_i}\right]d\tau\right\}^2 dx = -I_{3,n}.$$

From the boundedness of the functions

$$\left|\frac{\partial\alpha_n(x)}{\partial x_i}\right| \quad\text{and}\quad \left|\frac{\partial\varphi(x,t,u_k)}{\partial x_i}\right|$$

and the Schwarz inequality we get

$$|I_{3,n}| \leqslant C_1\iint_{Q_n}|\varphi(x,t,u_1)-\varphi(x,t,u_2)|\,dxdt$$
$$\leqslant C_1\Big(\iint_{Q_n}1\,dxdt\Big)^{1/2}\Big(\iint_{Q_n}[\varphi(x,t,u_1)-\varphi(x,t,u_2)]^2dxdt\Big)^{1/2}$$
$$\leqslant Cn^{(N-1)/2}\Big\{\iint_{Q_n}[\varphi(x,t,u_1)-\varphi(x,t,u_2)]\,(u_1-u_2)\,dxdt\Big\}^{1/2}.$$

From the above inequality and from (7) we get

$$(8)\qquad \iint_G \alpha_n(x)[\varphi(x,t,u_1)-\varphi(x,t,u_2)](u_1-u_2)\,dxdt$$
$$\leqslant Cn^{(N-1)/2}\Big\{\iint_{Q_n}[\varphi(x,t,u_1)-\varphi(x,t,u_2)](u_1-u_2)\,dxdt\Big\}^{1/2}.$$

The function

$$\psi(x, t) = [\varphi(x, t, u_1) - \varphi(x, t, u_2)](u_1 - u_2)$$

is nonnegative and continuous in G, and it vanishes if and only if $u_1(x, t) = u_2(x, t)$. Hence, it suffices to show that

$$\iint_G \psi(x, t)dxdt = 0.$$

Let us define the sequences

$$k_n = \int_0^T dt \int_{|x| \leqslant n-1} \psi(x, t)dx, \qquad \lambda_n = k_n/n^{N-1}, \qquad n = 1, 2, \ldots$$

From (8) we obtain

(9)
$$k_n \leqslant C[n^{N-1}(k_{n+1} - k_n)]^{1/2}.$$

The function ψ is bounded, so

$$k_{n+1} - k_n = \iint_{Q_n} \psi(x, t)dxdt \leqslant cn^{N-1}$$

for a certain constant c.

Now, from the above and (9) we get $k_n \leqslant \text{const} \cdot n^{N-1}$, and the boundedness of $\{\lambda_n\}$ is proved.

Dividing both sides of (9) by n^{N-1}, after some calculations we get

(10)
$$\lambda_n^2 + C\lambda_n \leqslant C\left(\frac{n+1}{n}\right)^{N-1} \lambda_{n+1}.$$

Taking the upper limits we have

$$\limsup \lambda_n^2 + C \limsup \lambda_n \leqslant C \limsup \lambda_n;$$

thus $\limsup \lambda_n = 0$. Since λ_n are positive, they tend to zero.

Let

$$\bar{\lambda}_n = \max_{m \geqslant n} \lambda_m.$$

Using inequality (10) we may write

$$\bar{\lambda}_n^2 + C\bar{\lambda}_n = \lambda_m^2 + C\lambda_m \leqslant C\left(\frac{m+1}{m}\right)^{N-1} \lambda_{m+1} \leqslant C\left(\frac{n+1}{n}\right)^{N-1} \bar{\lambda}_n$$

for some $m \geqslant n$. This gives

$$\lambda_n \leqslant \bar{\lambda}_n \leqslant C\left[\left(\frac{n+1}{n}\right)^{N-1} - 1\right]$$

and

$$k_n = \lambda_n n^{N-1} \leqslant C[(n+1)^{N-1} - n^{N-1}].$$

In the case of $N = 1$ or $N = 2$ this inequality implies the boundedness and, keeping in mind the monotonicity, the convergence of $\{k_n\}$.

Let us write inequality (9) in the form

$$\frac{k_n^2}{n^{N-1}} \leqslant C(k_{n+1} - k_n).$$

Summing up the inequalities for all n we get

$$\sum_{n=1}^{\infty} \frac{k_n^2}{n^{N-1}} \leqslant C \lim k_n < \infty .$$

This inequality holds only if $k_n = 0$, $n = 1, 2, \ldots$ Hence

$$\iint_G \psi(x, t) dx dt = \lim k_n = 0,$$

which completes the proof of the Theorem.

References

[1] L. S. Leĭbezon, *Dvizhenie prirodnoĭ zhidkosteĭ i gazov v poristoĭ srede*, Gostekhizdat, Moscow 1947.

[2] O. A. Oleĭnik, A. S. Kalashnikov and Chzhou Yui-Lin', *Zadacha Cauchy i kraevye zadachi dlya uravneniĭ tipa nestatsionarnoĭ filtratsii*, Izv. Akad. Nauk SSSR, Ser. Mat., 22 (1958), pp. 667–704.

[3] T. Śliwa, *On the Cauchy problem for the equation of one-dimensional non-stationary filtration with non-continuous initial data*, Colloq. Math. 44 (1981), pp. 333–345.

MATHEMATICAL INSTITUTE
UNIVERSITY OF WROCŁAW
50-384 WROCŁAW

Received on 1987.01.06