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# ON THE NUMERICAL SOLUTION OF THE GENERALIZED ABEL INTEGRAL EQUATION

*Abstract.* A modification of the Piessens–Verbaeten type method [11] for the numerical solution of the generalized Abel integral equation is given. The modified method, which expresses the solution in terms of shifted Chebyshev polynomials, is in certain cases more effective than the original method.

**1. Introduction.** We consider the singular integral equation ([14], Section 2.3)

$$\int_0^x [t(x)-t(y)]^{-\alpha} f(y)\,dy = g(x), \quad 0 \leqslant x \leqslant 1, \tag{1.1}$$

where $\alpha \in (0, 1)$, and $t \in C^1[0, 1]$ is a strictly increasing function; we may assume without loss of generality that $t(0) = 0$ and $t(1) = 1$. Equations of this type occur in a number of mathematical and physical problems. Two special cases are of particular interest. The classical case, corresponding to Abel type integral equation, is the one in which $t(x) := x^p$, $p$ real positive. Another special case is obtained for $t(x) := (1-\cos\pi x)/2$; the corresponding equation (1.1) occurs in the theory of mixed boundary value problems [14].

The solution of (1.1) is explicitly given by (see [14])

$$f(x) = \frac{\sin\alpha\pi}{\pi}\frac{d}{dx}\int_0^x [t(x)-t(y)]^{\alpha-1} t'(y)g(y)\,dy, \quad 0 \leqslant x \leqslant 1. \tag{1.2}$$

Formula (1.2) serves as a basis for many numerical methods for the solution of (1.1). See, e.g., [11]–[13], [5] and the references given there.

In [11], Piessens and Verbaeten have described a method applicable in the case where the function $G$,

$$G(x) := g\bigl(t^{-1}(x)\bigr), \quad 0 \leqslant x \leqslant 1, \tag{1.3}$$

can be approximated accurately by a function $G_n$ of the form

$$G_n(x) = x^{\beta} \sideset{}{'}\sum_{k=0}^{n} a_k T_k^*(x), \quad 0 \leqslant x \leqslant 1, \tag{1.4}$$

where the prime means that the first term of the sum is halved, $T_k^*$ is the $k$-th shifted Chebyshev polynomial of the first kind, $T_k^*(x) = \cos k\theta$, $\cos\theta = 2x-1$, and $\beta$ ($\beta > -\alpha$) is a free parameter. The solution of (1.1) is then approximately

$$(1.5)\qquad f_n(x) := \frac{t'(x)[t(x)]^{\gamma-1}}{B(1-\alpha,\gamma)} \sum_{k=1}^{n}{}' a_k q_k(t(x)), \quad 0 \leqslant x \leqslant 1,$$

where $\gamma := \alpha+\beta$, and

$$(1.6)\qquad q_k(u) := (-1)^k {}_3F_2\left(\begin{matrix}-k,\ k,\ \beta+1\\ 1/2,\ \gamma\end{matrix}\middle| u\right), \quad k = 0, 1, \ldots,$$

is a polynomial of degree $k$ [1]. Polynomials (1.6) should be computed by a stable algorithm based on the forward use of the difference equation

$$(1.7)\qquad q_k(u)+(A_k+B_k u)q_{k-1}(u)+(C_k+D_k u)q_{k-2}(u)+E_k q_{k-3}(u) = 0,$$

where

$$A_k := \frac{1}{k-2}\left[k-3+\frac{(k-1)(2k-3)}{k+\gamma-1}\right], \quad B_k := -4\frac{k+\beta}{k+\gamma-1},$$

$$C_k := \frac{1}{k-2}\left[\frac{(k-1)(3k-\gamma-5)}{k+\gamma-1}-1\right],$$

$$D_k := -4\frac{(k-1)(k-\beta-3)}{(k-2)(k+\gamma-1)}, \quad E_k := \frac{(k-1)(k-\gamma-2)}{(k-2)(k+\gamma-1)}.$$

Starting values for (1.7) are

$$q_0(u) \equiv 1, \quad q_1(u) := \frac{2(\beta+1)}{\gamma}u-1,$$

$$q_2(u) := \frac{8(\beta+1)(\beta+2)}{\gamma(\gamma+1)}u^2-\frac{8(\beta+1)}{\gamma}u+1.$$

Although the Piessens–Verbaeten method is an efficient procedure, formula (1.5) is inconvenient for some applications. We show that the sum in (1.5) can be converted to a sum of shifted Chebyshev polynomials, so that

$$(1.8)\qquad f_n(x) = \frac{t'(x)[t(x)]^{\gamma-1}}{B(1-\alpha,\gamma)} \sum_{m=0}^{n}{}' b_m T_m^*(t(x)), \quad 0 \leqslant x \leqslant 1,$$

where the coefficients $b_m$ can be computed in terms of the coefficients $a_k$, using a stable recursive scheme (see Section 2).

[1] Formulae (1.3)–(1.6) are generalizations of the forms given in [11] for the particular case $t(x) := x$.

The sum in (1.8) is evaluated by the well-known Clenshaw's algorithm:

$$S_{n+1} := S_{n+2} := 0,$$

$$S_k := b_k + (4z-2)S_{k+1} - S_{k+2} \quad (k = n, n-1, \ldots, 0),$$

$$\sum_{m=0}^{n}{}' b_m T_m^*(z) = S_0 - (2z-1)S_1$$

(see [9], Vol. 1, Section 8.5; or [12], p. 166; or [10], Chapter 15).

Imagine that we want to tabulate the solution of (1.1) at $m$ points of the interval [0, 1]. In the Piessens–Verbaeten method we have to perform about $6m(n-2)$ multiplications and $5m(n-1)$ additions provided we have first stored the coefficients $A_k$, $B_k$, $C_k$, $D_k$ and $E_k$ ($k = 0, 1, \ldots, n$) of the equation (1.7), which costed us $2(n-2)$ divisions, $10(n-2)$ multiplications and $12(n-2)$ additions. The proposed method requires about $(n+1)(n+5)$ divisions, $(n+3)[m+4.5(n+1)]$ multiplications and $(n+1)(4n+13)+m(2n+5)$ additions (cf. formulae (2.1)–(2.5)). Actual computations show that the total cost of the latter method is lower than the total cost of the former one provided the inequality $m \geqslant 0.8n+3$ holds; if, for instance, $m = n+10$, then the ratio of the costs equals 0.8. In the case where $\gamma = 1/2$, our method simplifies considerably and the numbers of the operations required are reduced to about $0.5(n+1)(n+8)$ divisions, $m(n+3)+1.5(n+1)(n+4)$ multiplications and $m(2n+5)+(n+1)(3n+10)$ additions. See formulae (2.1), (2.2'), (2.3'), (2.4) and (2.5). The total cost of this variant is lower than the cost of the Piessens-Verbaeten method provided $m$ is greater than $0.4n$.

In (1.1), the function $g$ may be either characterized by its values on a finite set of points or given by an explicit formula. In the former case, the coefficients $a_k$ in (1.4) can be calculated by the Clenshaw's curve fitting method [2], while in the latter case one can apply the interpolation method (see [4]; or [9], Vol. 1, Section 8.5; or [10], Chapter 7) or the recurrence relation method (see [7]; or [9], Vol. 2, Section 12.5; or [10], Chapter 13) for the calculation of the Chebyshev coefficients of the function $H(x) := x^{-\beta}G(x)$ over the interval [0, 1]. The value of $\beta$ must be chosen so that $H$ is as smooth as possible on [0, 1].

Under the assumption that the function $g$ is continuously differentiable on the interval [0, 1] one can obtain, using (1.2) and an expression for $f_n(x)$ obtained from this formula by replacing $g$ by $g_n := G_n \circ t$,

$$|f(x) - f_n(x)| \leqslant \frac{\sin \alpha\pi}{\alpha\pi} t'(x)\,[t(x)]^{\alpha}\, \|G' - G_n'\|_\infty$$

for every $x \in [0, 1]$, where $\|\cdot\|_\infty$ is the supremum norm. Now, the error $\|G' - G_n'\|_\infty$ can be expressed in terms of $\|H - H_n\|_\infty$ and $\|H' - H_n'\|_\infty$, where

$$H_n(x) := x^{-\beta} G_n(x).$$

Several ways of the estimation of $\|H^{(i)}-H_n^{(i)}\|_\infty$, $i = 0, 1$, are possible. For instance, if $H_n$ is the $n$-th degree polynomial interpolating the function $H$ at the points

$$u_j := [1+\cos(j\pi/n)]/2, \quad 0 \leqslant j \leqslant n,$$

then

$$\|H^{(i)}-H_n^{(i)}\| \leqslant \lambda_n^{(i)} E_{n-i}(H^{(i)}), \quad i = 0,1, \tag{1.9}$$

where $\lambda_n^{(i)}$ are constants,

$$\lambda_n^{(0)} \leqslant \frac{2}{\pi}\log n+2, \quad \lambda_n^{(1)} \leqslant 2\log n+2,$$

and

$$E_m(F) = \inf_{p\in\Pi_m} \|F-p\|_\infty,$$

$\Pi_m$ being the set of all polynomials of degree $\leqslant m$ (see [3]).

In the case where $H_n$ is the $(n+1)$-st partial sum of the Chebyshev series of $H$, it can be shown that $H_n'$ is the $n$-th partial sum of the Chebyshev series of the second kind of $H'$. Using results of [8] or [10], Chapter 7, one can obtain inequalities of the form (1.9) with $\lambda_n^{(i)}$ such that

$$\lambda_n^{(0)} \sim \frac{4}{\pi^2}\log n, \quad \lambda_n^{(1)} \sim \frac{8}{\pi^2}n+1.$$

In both cases, estimations of the form

$$\|H^{(i)}-H_n^{(i)}\|_\infty \leqslant C_n^{(i)}\|H^{(n+1-i)}\|_\infty, \quad i = 0, 1,$$

are also available, $C_n^{(i)}$ being some constants. For instance, in the case of the truncated Chebyshev series, we have

$$C_n^{(1)} = (n+1)^2 C_n^{(0)} = \frac{n+1}{2^n n!}$$

(see [1]).

**2. Computation of the coefficients $b_m$.** We show that the coefficients $b_m$ in the right-hand member of (1.8) can be computed in the following way. Let us write

$$A_0 := \tfrac{1}{2}a_0, \quad A_k := (-1)^k k a_k, \ k = 1, 2, \ldots, n. \tag{2.1}$$

Given $m \in \{0, 1, \ldots, n\}$, let us define

$$\begin{aligned} U_r^{(m)} &:= (r^2-m^2)/(2r-1), & m \leqslant r \leqslant n+2,\\ V_r^{(m)} &:= 1/[(r+\gamma)U_{r+1}^{(m)}], & m \leqslant r \leqslant n+1, \end{aligned} \tag{2.2}$$

and

(2.3)
$$P_r^{(m)} := 2r[(2\gamma-1)U_r^{(m)}/(2r+1)-\alpha+1]V_r^{(m)}, \qquad m \leqslant r \leqslant n.$$
$$Q_r^{(m)} := (r-\gamma+1)[U_{r+1}^{(m)}-1]V_{r+1}^{(m)},$$

Next, let us define the sequence $\{S_r^{(m)}\}$, $m \leqslant r \leqslant n+2$, by

(2.4)
$$S_{n+1}^{(m)} := S_{n+2}^{(m)} := 0,$$
$$S_r^{(m)} := A_r - P_r^{(m)} S_{r+1}^{(m)} + Q_r^{(m)} S_{r+2}^{(m)}, \qquad r = n, n-1, \ldots, m.$$

Then we have

(2.5)
$$b_m = \begin{cases} 2S_0^{(0)} + \dfrac{2\alpha-2}{\gamma} S_1^{(0)}, & m = 0, \\ (-1)^m \dfrac{(\beta+1)_m}{m(\gamma)_m} S_m^{(m)}, & 1 \leqslant m \leqslant n, \end{cases}$$

where the notation

$$(a)_k := \begin{cases} 1, & k = 0, \\ a(a+1)\ldots(a+k-1), & k \geqslant 1, \end{cases}$$

is used.

To prove formulae (2.1)–(2.5) let us observe that the hypergeometric polynomial (1.6) can be represented as

(2.6)
$$q_k(u) = \sideset{}{'}\sum_{m=0}^{k} c_{km} T_m^*(u),$$

where

$$c_{km} := \begin{cases} 2, & k = m = 0, \\ (-1)^k \dfrac{(-k)_m (k)_m (\beta+1)_m}{2^{2m-1} m! (1/2)_m (\gamma)_m} {}_4F_3\left(\begin{matrix} m+1/2,\ m-k,\ m+k,\ m+\beta+1 \\ 2m+1,\ m+1/2,\ m+\gamma \end{matrix}\middle| 1\right), \\ & 0 \leqslant m \leqslant k,\ k > 0. \end{cases}$$

This follows from a general result on the Chebyshev series expansion of a generalized hypergeometric function (see [9], Vol. 1, Section 9.3; or [10], Chapter 12). Introducing the notation

(2.7)
$$\varphi_k^{(m)} := \begin{cases} 1, & k = m = 0, \\ (k-m+1)_{2m-1}\, {}_3F_2\left(\begin{matrix} m-k,\ m+k,\ m+\beta+1 \\ 2m+1,\ m+\gamma \end{matrix}\middle| 1\right), & 0 \leqslant m \leqslant k,\ k > 0, \end{cases}$$

we can write

$$c_{km} = (-1)^{k-m} \frac{2(\beta+1)_m (k+\delta_{k0})}{(2m)!(\gamma)_m} \varphi_k^{(m)}, \qquad 0 \leqslant m \leqslant k,\ k \geqslant 0.$$

Now, putting (2.6) into the polynomial

$$\sum_{k=0}^{n}{}' a_k q_k,$$

we can convert it into the finite Chebyshev series

$$\sum_{m=0}^{n}{}' b_m T_m^*,$$

where

$$b_m := (-1)^m \frac{2(\beta+1)_m}{(2m)!(\gamma)_m} \sum_{j=m}^{n} A_j \varphi_j^{(m)}, \qquad m = 0, 1, \ldots, n, \tag{2.8}$$

$A_j$ being given by (2.1).

For any fixed value of $m$, $0 \leqslant m \leqslant k$, quantities (2.7) satisfy the following second-order difference equation

$$\varphi_{r+1}^{(m)} + P_r^{(m)} \varphi_r^{(m)} - Q_{r-1}^{(m)} \varphi_{r-1}^{(m)} = 0, \qquad r \geqslant m, \tag{2.9}$$

the notation being used is that of (2.3) (see [6]). The starting values for (2.9) are

$$\varphi_m^{(m)} = \begin{cases} 1, & m = 0, \\ (2m-1)!, & m > 0, \end{cases} \qquad \varphi_{m+1}^{(m)} = (\alpha-1)(2m)!/(\gamma+m). \tag{2.10}$$

The asymptotic approximations for a fundamental set $s_1(r)$, $s_2(r)$ of (2.9) may be obtained by Birkoff–Trjitzinsky theory (see [15], Section B.2). We have

$$s_1(r) \sim r^{-2\beta-3}, \qquad s_2(r) \sim (-1)^r r^{-2\alpha}, \qquad r \to \infty,$$

which means that (2.9) cannot have any solution which increases strongly if $\beta$ is not too large. Thus we can evaluate the sum

$$\sigma_m := \sum_{j=m}^{n} A_j \varphi_j^{(m)},$$

using the following stable algorithm, called a *nesting procedure* (see [9], Vol. 1, Chapter 8; or [15], Section 10.2). Let $\{S_r^{(m)}\}$, $m \leqslant r \leqslant n+2$, be defined by (2.4). Then we have

$$\sigma_m = S_m^{(m)} \varphi_m^{(m)} + S_{m+1}^{(m)} [\varphi_{m+1}^{(m)} + P_m^{(m)} \varphi_m^{(m)}],$$

which, by (2.10) and the first equation of (2.3), simplifies to

$$\sigma_m = \begin{cases} S_0^{(0)} + (\alpha-1) S_1^{(0)}/\gamma, & m = 0, \\ (2m-1)!\, S_m^{(m)}, & m > 0. \end{cases}$$

Putting this in (2.8), we obtain equation (2.5).

We have made several heuristic experiments, using $n$ as big as 200. The results were in agreement with the above statements about the stability.

In the case where $\gamma = 1/2$, the coefficients $P_r^{(m)}$ and $Q_{r-1}^{(m)}$ of the difference equation (2.9) reduce to simpler forms and, consequently, the algorithm (2.1)–(2.5) simplifies considerably. Namely, formulae (2.2) and (2.3) may be replaced by

(2.2′) $$W_r^{(m)} := 4r/[(r+1)^2 - m^2], \quad m \leqslant r \leqslant n+1,$$

and

(2.3′) $$P_r^{(m)} := (1-\alpha) W_r^{(m)}, \quad Q_r^{(m)} := 1 - W_{r+1}^{(m)}, \quad m \leqslant r \leqslant n,$$

respectively.

**3. Examples.** The calculations reported below were carried out on the ODRA 1305 computer of the Institute of Computer Science, University of Wrocław, by using single precision arithmetic.

EXAMPLE 1. We consider the equation (see [11])

$$\int_0^x [t(x)-t(y)]^{-1/2} f(y)dy = \exp(t(x)) - 1,$$

which has the exact solution

$$f(x) = \frac{2t'(x)}{\pi} e^{t(x)} \operatorname{Erf}(\sqrt{t(x)}).$$

From the values of leading Chebyshev coefficients of the exponential function, tabulated in [9], Vol. 2, Chapter 17, one can readily obtain $a_0$, $a_1, \ldots, a_n$ such that

$$G(x) \equiv e^x - 1 \approx x \sum_{k=0}^{n}{}' a_k T_k^*(x).$$

We have tried $t(x) = x^p$, $p = .1, .5, 1, 2$, and $t(x) = (1-\cos\pi x)/2$. By using $n = 9$ and $\beta = 1$, formula (1.8) gives results with an absolute error less than $5 \cdot 10^{-11}$ for $x = k/25$, $0 \leqslant k \leqslant 25$, in all cases.

EXAMPLE 2. An integral equation of importance in plasma physics can be transformed into the Abel integral equation

$$\int_0^x (x-y)^{-1/2} f(y)dy = g(x), \quad 0 \leqslant x \leqslant 1,$$

where

$$g(x) = \frac{10}{11}\left(\frac{\pi}{x}\right)^{1/2} \exp[1.21(1-1/x)].$$

The exact solution is

$$f(x) = x^{-3/2} \exp[1.21(1-1/x)]$$

(see [11]).

Let $G_n(x) := x^{\beta} J_n(x)$, where $J_n$ is the polynomial of degree at most $n$ which interpolates the function $h(x) := x^{-\beta} g(x)$ at the points

$$x_k := [1+\cos(k\pi/n)]/2, \qquad k = 0, 1, \ldots, n.$$

It is well known that

$$J_n = \sum_{k=0}^{n}{}' a_k T_k^*,$$

where

$$a_k = \frac{2-\delta_{kn}}{n} \sum_{j=0}^{n}{}'' h(x_j) T_j^*(x_k), \qquad k = 0, 1, \ldots, n.$$

The double prime means that the first and the last terms of the sum should be halved. See, e.g., [10], Chapter 7; or [4], Chapter 4.

We have used $\beta = 0$ and $n = 30$. As $\gamma = \frac{1}{2}$, we have applied the variant of the algorithm described in the last paragraph of Section 2. The maximum absolute error of the result given by (1.8) did not exceed $5 \cdot 10^{-6}$.

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