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ON THE APPROXIMATION OF SOME OPTIMAL SEQUENTIAL PLAN

Abstract. The paper is devoted to the problem of approximation of an optimal sequential plan in stochastic diffusion processes and fields when observations are made at discrete points.

1. Introduction. The paper is concerned with the problem of sequential estimating the parameter of the drift coefficient in stochastic diffusion processes and fields. We consider the models when drift is known up to a multiplicative constant. For such models Novikov [12], Le Breton and Musiela [9] obtained optimal sequential plans. Their results are based on the assumption that the process is observed continuously in time. In practice we often meet the situation when the process can be observed at discrete points t_k only. So we have the problem of approximation of the derived optimal sequential plans.

A similar problem, but in the case of a fixed-time observation, was considered by Le Breton [8]. Other aspects of estimation in the discrete-observation case were treated by Stoyanov [17], Robinson [14], Żylińskas and Senkienė [22], [23], Prakasa-Rao [13], and Dohnal [4].

The paper is organized as follows: in Section 2 the theorem about the approximation of the Novikov sequential plan is formulated, the multidimensional case is considered in Section 3, in Section 4 some optimal sequential plan in diffusion random fields is constructed and some its approximation is given, in Section 5 examples are given, and Section 6 contains proofs of previously formulated theorems.

2. Sequential estimating the parameter of the drift coefficient of a stochastic diffusion process. One-dimensional case. Let $(\Omega, \mathscr{F}, P)$ be a probability space. By $(\mathscr{F}_t)_{t\geq 0}$ we denote the family of sub-σ-algebras of the σ-algebra $\mathscr{F}$. Assume as usual that $(\mathscr{F}_t)_{t\geq 0}$ is nondecreasing and right continuous with respect to t, and $\mathscr{F}_0$ contains all subsets of Ω of P-probability zero. Let us consider a stochastic diffusion process $(X_t, \mathscr{F}_t)$ satisfying the stochastic differential equation

$$dX_t = \theta a(t, X_t)dt + dW_t, \quad X_0 = 0 \quad \text{a.s.,} \tag{2.1}$$

where $(W_t, \mathscr{F}_t)$ is a Wiener process and θ is an unknown parameter. The space of continuous realizations x of the process X is denoted by C. We also assume that

$$P\{\int_0^t a^2(s, X_s)ds < \infty\} = P\{\int_0^t a^2(s, W_s)ds < \infty\} = 1 \quad \text{for each } t \geqslant 0,$$

$$P\{\int_0^\infty a^2(s, X_s)ds = \infty\} = 1.$$

Under these assumptions one can get a maximum likelihood sequential plan $(\tau_H, \hat{\theta}_H)$ for estimating the parameter θ, where

$$\tau_H(X) = \inf\{t\colon \int_0^t a^2(s, x_s)ds \geqslant H\}, \qquad \hat{\theta}_H = H^{-1}\int_0^{\tau_H} a(s, X_S)dX_s.$$

Novikov [12] proved that this sequential plan is unbiased, efficient and achieves the minimum of its variance in the class of all unbiased sequential plans for which

$$\text{(2.2)} \qquad \mathrm{E}_\theta \int_0^\tau a^2(s, H_s)ds \leqslant H.$$

Musiela [11] proved that $(\tau_H, \hat{\theta}_H)$ is minimax in the class of sequential plans satisfying (2.2). Moreover, the estimator $\hat{\theta}_H$ is normally distributed with mean value θ and variance H^{-1}. This optimal sequential plan is constructed under the assumption that we observe a realization $x(t)$ of the process X_t continuously in time up to the random moment τ_H.

In many practical situations we observe the values of the process at some points t_k only. So the natural problem arises to approximate the optimal sequential plan $(\tau_H, \hat{\theta}_H)$.

Let us assume that we observe the process X_t at the points $t_k = k\delta$, $\delta > 0$, $k \in N = \{0, 1, 2, \ldots\}$. Introducing the Markov stopping time

$$\tau_{H,\sigma} = \inf\{t_{k,\sigma} = k\delta\colon \sum_{j=1}^{k} a^2((j-1)\delta, X_{(j-1)\delta})\delta \geqslant H\}$$

and the corresponding estimator

$$\hat{\theta}_{H,\sigma} = H^{-1} \sum_{j=1}^{\tau_{H,\delta}/\delta} a((j-1)\delta, X_{(j-1)\delta}) \Delta X_{j\delta},$$

where $\Delta X_{j\delta} = X_{j\delta} - X_{(j-1)\delta}$, we prove the following

THEOREM 2.1. *If the function $a(t, x)$ in formula (2.1) is a continuous function of both arguments, then*

$$\text{(2.3)} \qquad P\{\lim_{\delta\to 0} \tau_{H,\delta} = \tau_H\} = 1,$$

$$\text{(2.4)} \qquad \hat{\theta}_{H,\delta} \to \hat{\Theta}_H \text{ in probability as } \delta \to 0.$$

3. Multidimensional case. Le Breton and Musiela [9] considered a multidimensional generalization of the model described in the previous section. Namely, let X_t be an n-dimensional stochastic diffusion process satisfying the stochastic differential equation

$$dX_t = AX_t dt + \sigma dW_t,$$
$$X_0(\omega) = x_0 \quad \text{for almost all } \omega \in \Omega,$$

where $W = (W_1, W_2, \ldots, W_r)'$ is a Wiener process in R^r defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, A and σ are $(n \times n)$- and $(n \times r)$-matrices, respectively,

$$A = \sum_{j=1}^{p} \theta_j A_j,$$

A_j and σ are known matrices. We also assume that for $B = \sigma\sigma'$

$$BB^+ A_j \neq 0, \quad A_i' B^+ A_j + A_j' B^+ A_i = 0, \quad i \neq j, \ i, j = 1, 2, \ldots, p$$

(B^+ indicates the pseudoinverse of the matrix B). Supposing that

$$\operatorname{rank}[\sigma, A\sigma, \ldots, A^{n-1}\sigma] = n$$

we can define the sequential plans $(\tau_{i,H}, \hat{\theta}_{i,H})$ with

$$\tau_{i,H} = \inf\{t\colon \int_0^t X_s' A_i' B^+ A_i X_s ds = H\},$$

$$\hat{\theta}_{i,H} = H^{-1} \int_0^{\tau_{i,H}} X_s' A_i' B^+ dX_s, \quad i = 1, 2, \ldots, p.$$

The sequential plan $(\tau_{i,H}, \hat{\theta}_{i,H})$ is unbiased, efficient and achieves the minimum of its variance in the class of all unbiased sequential plans $(\tau_i, \hat{\theta}_i)$ for which

$$\mathrm{E}_\theta \int_0^{\tau_i} X_s' A_i' B^+ A_i X_s ds \leq H.$$

The estimator $\hat{\theta}_{i,H}$ is normally distributed with mean value θ and variance H^{-1}. This sequential plan may be approximated by some sequential plan based on observations of the process X_t at the points $t_{k,\delta} = k\delta$ only.

Let

$$\tau_{i,H,\delta} = \inf\{t_{k,\delta} = k\delta\colon \sum_{j=1}^{k} X_{(j-1)\delta}' A_i' B^+ A_i X_{(j-1)\delta}\delta \geq H\},$$

$$\hat{\theta}_{i,H,\delta} = H^{-1} \sum_{j=1}^{\tau_{i,H,\delta}/\delta} X_{(j-1)\delta}' A_i' B^+ \Delta X_{j\delta}.$$

Using the same methods as in the proof of Theorem 2.1 we get

THEOREM 3.1. *Under the assumptions mentioned above the sequential plans* $(\tau_{i,H,\delta}, \hat{\theta}_{i,H,\delta})$, $i = 1, 2, \ldots, p$, *have the following properties*:

$$P\{\lim_{\delta \to 0} \tau_{i,H,\delta} = \tau_{i,H}\} = 1,$$

$\hat{\theta}_{i,H,\delta} \to \hat{\theta}_{i,H}$ *in probability as* $\delta \to 0$.

4. Sequential estimating the parameter of the drift coefficient of a diffusion random field. Let R_+^2 be the positive quadrant of the plane. In this set one can define a relation of partial ordering as follows: for each $z_1, z_2 \in R_+^2$,

$$z_1 = (s_1, t_1), \quad z_2 = (s_2, t_2),$$

$$z_1 \leqslant z_2 \quad \text{iff} \quad s_1 \leqslant s_2 \text{ and } t_1 \leqslant t_2.$$

Let us introduce the notation

$$z_1 \otimes z_2 = (s_1, t_2).$$

For each $z_0 \in R_+^2$,

$$R_{z_0} = [0, z_0] = \{z \in R_{z_0}\colon z \leqslant z_0\}.$$

Let $(\Omega, \mathscr{F}, (\mathscr{F}_z)_{z\in R_+^2}, P)$ be a probability space, where $(\mathscr{F}_z)_{z\in R_+^2}$ is a family of sub-σ-algebras of $\mathscr{F}$ such that for each $z_1, z_2 \in R_+^2$

$$z_1 \leqslant z_2 \Rightarrow \mathscr{F}_{z_1} \subseteq \mathscr{F}_{z_2},$$

for each $z \in R_{z_0}$

$$\mathscr{F}_z = \bigcap_{z \leqslant z'} \mathscr{F}_{z'},$$

$\mathscr{F}_z^1 = \mathscr{F}_{z\otimes z_0}$ and $\mathscr{F}_z^2 = \mathscr{F}_{z_0 \otimes z}$ are conditionally independent given $\mathscr{F}_z$ (see [3]), and $\mathscr{F}_0$ contains all sets of P-probability zero, where 0 denotes the origin.

For the random field X_z, $z \in R_+^2$, the formula

$$X([z_1, z_2]) = X_{z_2} - X_{z_1 \otimes z_2} - X_{z_2 \otimes z_1} + X_{z_1}$$

defines the increment of the random field X_z on the rectangle

$$[z_1, z_2] = \{z \in R_+^2\colon z_1 \leqslant z \leqslant z_2\}, \quad z_1 \leqslant z_2.$$

Definition 4.1. A random field X_z, defined on $(\Omega, \mathscr{F}, (\mathscr{F}_z)_{z\in R_+^2}, P)$, belongs to the *class of random fields of diffusion type* if there exist nonanticipative functionals $a_z(\cdot)$ and $b_z(\cdot)$ defined on the space $C(R_{z_0})$ of continuous functions such that

$$P\{\int_{R_{z_0}} |a_z(X)| dz < \infty\} = 1, \quad P\{\int_{R_{z_0}} b_z^2(X) dz < \infty\} = 1,$$

$$X(R_z) = \int_{R_z} a_v(X) dv + \int_{R_z} b_v(X) dW_v$$

with P-probability 1 for each $z \in R_{z_0}$.

The above equation can be rewritten in the equivalent differential form

$$dX_z = a_z(X) dz + b_z(X) dW_z$$

with the initial condition

$$\phi_z = X_{(s,0)} + X_{(0,t)} - X_{(0,0)}.$$

Sufficient conditions for the existence of such random fields are given in [18] and [21]. For further applications the problem of the absolute continuity of measures corresponding to random fields of diffusion type is of great importance. Following [6] (Proposition 7.2) and [19] (Theorem 6.1) we can prove that if X_z, Y_z $(z \in R_{z_0})$ are random fields of diffusion type satisfying the equations

(4.1) $$X_z = \phi_z + \int_{R_{z_0}} a_v(X)dv + \sigma W_z,$$

(4.2) $$Y_z = \phi_z + \sigma W_z \text{ with } P\text{-probability 1 for each } z \in R_{z_0},$$

then the measure $\mu_X^{R_{z_0}}$ corresponding to the random field X is equivalent to the measure $\mu_Y^{R_{z_0}}$ corresponding to the random field Y and

$$\frac{d\mu_X^{R_{z_0}}}{d\mu_Y^{R_{z_0}}} = \exp\left[\frac{1}{\sigma^2}\int_{R_{z_0}} a_z(X)dX_z - \frac{1}{2\sigma^2}\int_{R_{z_0}} a_z^2(X)dz\right].$$

To consider the problem of sequential estimation for random fields of diffusion type we must introduce somehow the concept of Markov stopping "time". Let $X_z (z \in R^2)$ be a random field with the corresponding measure μ_X. Let Ψ be the space of realizations of this random field, and $\mathscr{G}$ be the σ-algebra of subsets of Ψ generated by cylindrical sets. By $\mathscr{K}$ we denote the family of compact subsets of R^2 for which the following condition holds:

CONDITION 4.1. *There exists a countable family of compact sets $P_i(n)$ with diameter $\delta(P_i(n)) \to 0$ as $n \to \infty$ such that for every $K \in \mathscr{K}$ there exists a finite covering $C_n \in \mathscr{K}$ of K by some sets among $P_i(n)$, $i \in I$, for which $C_{n+1} \subseteq C_n$ and*

$$\bigcap_{n=1}^{\infty} C_n = K.$$

By $\mathscr{G}_K$ we denote the restriction of the σ-algebra of subsets of Ψ generated by cylindrical sets

$$\{x \in \Psi\colon (x(z_1), x(z_2), \ldots, x(z_n)) \in B\}, \quad B \in \mathscr{B}_{R_n}, \ z_i \in K, \ i = 1, 2, \ldots, n.$$

The restriction of the measure μ_X to the σ-algebra $\mathscr{G}_K$ is denoted by μ_X^K.

DEFINITION 4.2 ([5], [15]). A *Markov stopping set* τ is a mapping $\tau\colon \Psi \to \mathscr{K}$ such that for each $K \in \mathscr{K}$

$$\{x\colon \tau(x) \subseteq K\} \in \mathscr{G}_K.$$

With each Markov stopping set τ we can connect the σ-algebra $\mathscr{G}_\tau$ as the family of sets $U \in \mathscr{G}$ such that for each $K \in \mathscr{K}$

$$U \cap \{x\colon \tau(x) \subseteq K\} \in \mathscr{G}_K.$$

The measure μ restricted to $\mathscr{G}_\tau$ is denoted by μ^τ.

The following theorem answers the question about the absolute continuity of the measures μ^τ corresponding to random fields.

THEOREM 4.1 ([15], [16]). *Assume that Condition 4.1 holds. If, for every* $K \in \mathcal{K}$, μ_X^K *is absolutely continuous with respect to* μ_0^K *and*

$$\frac{d\mu_X^K}{d\mu_0^K}(X) = g(K, x),$$

where g *is a function for which* $g(K_n, x) \to g(K, x)$ μ_0*-almost surely as* $K_n \downarrow K$, *then* μ_X^τ *is absolutely continuous with respect to* μ_0^τ *and*

$$\frac{d\mu_X^\tau}{d\mu_0^\tau}(x) = g\big(\tau(x), x\big).$$

We pass to the case of diffusion random fields in the sense of Definition 4.1. We put

$$\mathcal{K}' = \{R_z\colon z \in R_+^2\}.$$

If X_z, Y_z are random fields satisfying equalities (4.1), (4.2) and τ is a Markov stopping set with respect to $\mathcal{K}'$, then under the assumptions of Theorem 4.1 the measure μ_X^τ is absolutely continuous with respect to the measure μ_Y^τ and

$$\frac{d\mu_X^\tau}{d\mu_Y^\tau}(x) = \exp\left[\frac{1}{\sigma^2}\int_\tau a_z(X)dX_z - \frac{1}{2\sigma^2}\int_\tau a_z^2(X)dz\right].$$

Now we assume that X_z fulfils the equation

$$X_z = \phi_z + \theta \int_{R_z} a_v(X_v)dv + W_z$$

with

$$P\Big[\int_{R_z} a_v^2(X_v)dv < \infty\Big] = 1 \quad \text{for each } R_z,$$

$$P\Big[\int_{R_+^2} a_v^2(X_v)dv = \infty\Big] = 1,$$

where θ is an unknown parameter which we have to estimate. By Theorem 4.1 the measure μ_θ^τ corresponding to X and defined on the σ-algebra $\mathcal{G}_\tau$ is absolutely continuous with respect to $\mu_{\theta_0}^\tau$, where $\theta_0 = 0$ and

$$\frac{d\mu_\theta^\tau}{d\mu_0^\tau} = \exp\left[\theta\int_\tau a_z(X_z)dX_z - \frac{\theta^2}{2}\int_\tau a_z^2(X_z)dz\right] = g\big(\tau, S(\tau), \theta\big),$$

$$S(\tau) = \Big[\int_\tau a_z(X_z)dX_z, \int_\tau a_z^2(X_z)dz\Big].$$

DEFINITION 4.3. By a *sequential plan* for estimating the unknown parameter θ we mean a pair $\big(\tau, f(S(\tau))\big)$, where τ is a Markov stopping set with respect to $\mathcal{K}'$ and $f\big(S(\tau)\big)$ is an estimator of θ.

For a sequential plan so defined the Cramér–Rao–Wolfowitz inequality holds.

THEOREM 4.2. ([15]). *Let $(\tau, f(S(\tau)))$ be a sequential plan for which $f(S(\tau))$ is an unbiased estimator of the parameter θ. If the regularity conditions*

$$E_\theta \frac{\partial}{\partial \theta}[\ln g(\tau, S(\tau), \theta)] = 0, \quad \frac{\partial}{\partial \theta} E_\theta f = E_\theta f \frac{\partial}{\partial \theta}[\ln g(\tau, S(\tau), \theta)],$$

$$E_\theta \left[\frac{\partial}{\partial \theta} \ln g(\tau, S(\tau), \theta)\right]^2 > 0$$

hold, then

$$\text{(4.3)} \qquad \mathrm{Var}_\theta f(S(\tau)) \geqslant \left(E_\theta \left[\frac{\partial}{\partial \theta} \ln g(\tau, S(\tau), \theta)\right]^2\right)^{-1} = \left(E_\theta \int_\tau a_z^2(X_z)dz\right)^{-1}.$$

DEFINITION 4.4. A sequential plan $(\tau, f(S(\tau)))$ is said to be *efficient* if inequality (4.3) becomes an equality for $\theta \in R$.

Let

$$\mathscr{K}_0 = \{R_t = [0, t] \times [0, t] \colon t \geqslant 0\} \quad \text{and} \quad \tau_H(x) = \inf\{t \colon \int_{R_t} a_z^2(x)dz = H\}.$$

It is easy to see that $R_{\tau_H} = [0, \tau_H] \times [0, \tau_H]$ is a Markov stopping set with respect to $\mathscr{K}'$. We have

THEOREM 4.3. ([16]). *The sequential plan $(R_{\tau_H}, \hat{\theta}_H)$ with*

$$\hat{\theta}_H = H^{-1} \int_{R_{\tau_H}} a_z(X_z)dX_z$$

is an unbiased efficient maximum likelihood sequential plan for estimating the θ. Moreover, $(R_{\tau_H}, \hat{\theta}_H)$ has minimal variance and it is minimax in the class of all sequential plans for which

$$E_\theta \int_\tau a_z^2(X_z)dz \leqslant H.$$

The estimator $\hat{\theta}_H$ is normally distributed with mean value θ and variance H^{-1}.

This optimal sequential plan is constructed under the assumption that one can observe the whole realization of the random field X_z on the increasing squares. This is difficult to realize in practice. More often we meet the situation when we observe the random field X_z at the points $(s_i, t_j) = (i\sigma, j\delta)$. Fortunately, we can construct some approximation of the optimal sequential plan as in the previous sections. Let us define

$$\tau_{H,\delta} = \inf\{k\delta \colon \sum_{\substack{(i,j) \\ 1 \leqslant i,j \leqslant k}} a^2(((i-1)\delta, (j-1)\delta), x((i-1)\delta, (j-1)\delta))\delta^2 \geqslant H\},$$

$$\hat{\theta}_{H,\delta} = H^{-1} \sum_{\substack{(i,j) \\ 1 \leqslant i,j \leqslant \tau_{H,\delta}/\delta}} a(((i-1)\delta, (j-1)\delta), X((i-1)\delta, (j-1)\delta))X(P_{i,j}),$$

where $X(P_{i,j})$ is the increment of the random field X_z on the square $[(i-1)\delta, i\delta] \times [(j-1)\delta, j\delta]$. Then we have

THEOREM 4.4. *For the sequential plan* $(\tau_{H,\delta}, \hat{\theta}_{H,\delta})$ *the following holds*:

$$P\{\lim_{\delta\to 0}\tau_{H,\delta} = \tau_H\} = 1, \tag{4.4}$$

$$\hat{\theta}_{H,\delta}\to\hat{\theta}_H \text{ in probability as } \delta\to 0. \tag{4.5}$$

5. Examples.

1. A signal $S(\theta, t) = \theta f(t)$ with unknown amplitude θ is passing through a channel with additive white noise. So we can write

$$dX_t = \theta f(t)dt + dW_t \qquad X_0 = 0,$$

where θ is the unknown amplitude. If T is such that

$$\int_0^T [f(t)]^2 dt = H,$$

then

$$\tau_{H,\delta} = \inf\{k\delta\colon \sum_{j=1}^{k} [f((j-1)\delta)]^2\delta \geqslant H\},$$

$$\hat{\theta}_{H,\delta} = H^{-1}\sum_{j=1}^{\tau_{H,\delta}/\delta} f((j-1)\delta)\Delta X_{j\delta}.$$

Observe that $\tau_{H,\delta}$ is nonrandom.

2. Let the evolution of a dynamical system be described by the Ornstein–Uhlenbeck process. This means that X_t is the solution of the stochastic differential equation

$$dX_t = \theta X_t dt + dW_t, \qquad X_0 = x_0,$$

where W_t is a Wiener process and θ is an unknown parameter. In this case

$$\tau_{H,\delta} = \inf\{k\delta\colon \sum_{j=1}^{k} X^2_{(j-1)\delta}\delta \geqslant H\}, \qquad \hat{\theta}_{H,\delta} = H^{-1}\sum_{j=1}^{\tau_{H,\delta}/\delta} X_{(j-1)\delta}\Delta X_{j\delta}.$$

3 (see [12], [1], [2]). The instantaneous axis of rotation of the earth is displaced with respect to the minor axis of the terrestrial ellipsoid. This displacement is a sum of a periodic motion and some fluctuation. The latter can be modelled by the solution of the stochastic differential equation

$$dX_t = AX_t dt + \sigma dW_t,$$

where

$$A = \theta_1 A_1 + \theta_2 A_2, \qquad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$\sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\beta, \qquad X_t = (X_{1,t}, X_{2,t})',$$

θ_1 and θ_2 are unknown and β is known. So we can write

$$\tau_{1,H,\delta} = \tau_{2,H,\delta} = \tau_{H,\delta} = \inf\{k\delta\colon \sum_{j=1}^{k} [X_{1,(j-1)\delta}^2 + X_{2,(j-1)\delta}^2]\delta \geqslant H\},$$

$$\hat{\theta}_{1,H,\delta} = H^{-1}\Big(\sum_{j=1}^{\tau_{H,\delta}/\delta} X_{1,(j-1)\delta}\Delta X_{1,j\delta} + \sum_{j=1}^{\tau_{H,\delta}/\delta} X_{2,(j-1)\delta}\Delta X_{2,j\delta}\Big),$$

$$\hat{\theta}_{2,H,\delta} = H^{-1}\Big(\sum_{j=1}^{\tau_{H,\delta}/\delta} X_{1,(j-1)\delta}\Delta X_{2,j\delta} - \sum_{j=1}^{\tau_{H,\delta}/\delta} X_{2,(j-1)\delta}\Delta X_{1,j\delta}\Big).$$

4. Let us consider a dynamical system which may be described by the following stochastic differential equation:

$$dX_z = \theta X_z dz + dW_z, \qquad z = (s, t) \in R_+^2,$$

$$X_{(s,0)} = \phi_s, \qquad X_{(0,t)} = \eta_t.$$

This equation can be used to model the propagation of a plane sound wave with constant velocity v in some medium [7]. In order to estimate the unknown parameter θ we can use the sequential plan $(\tau_{H,\delta}, \hat{\theta}_{H,\delta})$ considered in Theorem 4.4.

6. Proofs.

Proof of Theorem 2.1. By the definition of τ_H, for each $\varepsilon > 0$ we get

$$\int_0^{\tau_H+\varepsilon} a^2(s, x_s)ds = \bar{H}_\varepsilon > H, \qquad \int_0^{\tau_H-\varepsilon} a^2(s, x_s)ds = \underline{H}_\varepsilon < H.$$

From the definition of the integral it follows that for each $\eta > 0$ such that $\bar{H}_\varepsilon - \eta > H$ and $\underline{H}_\varepsilon + \eta < H$ there exists $\delta_0 > 0$ such that for each $\delta < \delta_0$

$$\left(\left[\frac{\tau_H - \varepsilon}{\delta}\right] + 1\right)\delta \leqslant \tau_{H,\delta} \leqslant \left(\left[\frac{\tau_H + \varepsilon}{\delta}\right] + 1\right)\delta,$$

where $[z]$ indicates the integer part of z. Thus we obtain

$$\lim_{\delta\to 0}\left(\left[\frac{\tau_H - \varepsilon}{\delta}\right] + 1\right)\delta \leqslant \underline{\lim}_{\delta\to 0}\, \tau_{H,\delta} \leqslant \overline{\lim}_{\delta\to 0}\, \tau_{H,\delta} \leqslant \lim_{\delta\to 0}\left(\left[\frac{\tau_H + \varepsilon}{\delta}\right] + 1\right)\delta.$$

So for each $\varepsilon > 0$ we have

$$\tau_H + \varepsilon \leqslant \underline{\lim}_{\delta\to 0}\, \tau_{H,\delta} \leqslant \overline{\lim}_{\delta\to 0}\, \tau_{H,\delta} \leqslant \tau_H + \varepsilon.$$

Thus

$$P\{\lim_{\delta\to 0} \tau_{H,\delta} = \tau_H\} = 1. \tag{6.1}$$

By the assumption, $a(t, x)$ is a continuous function of both arguments. So by Lebesgue's theorem and (6.1) we conclude that

$$P\Big(\lim_{\delta\to 0} \sum_{j=1}^{\tau_{H,\delta}/\delta} a^2\big((j-1)\delta, X_{(j-1)\delta}\big)\delta = \int_0^{\tau_H} a^2(s, X_s)ds = H\Big) = 1.$$

To prove (2.4) we write

$$(6.2)\qquad \sum_{j=1}^{\tau_{H,\delta}/\delta} a\big((j-1)\delta, X_{(j-1)\delta}\big)\Delta X_{j\delta} = \theta \sum_{j=1}^{\tau_{H,\delta}/\delta} a\big((j-1)\delta, X_{(j-1)\delta}\big) \int_{(j-1)\delta}^{j\delta} a(s, X_s)ds + \sum_{j=1}^{\tau_{H,\delta}/\delta} a\big((j-1)\delta, X_{(j-1)\delta}\big)\Delta W_{j\delta}.$$

With the same arguments as previously we can show that

$$(6.3)\qquad P\Big(\lim_{\delta\to 0} \sum_{j=1}^{\tau_{H,\delta}/\delta} a\big((j-1)\delta, X_{(j-1)\delta}\big) \int_{(j-1)\delta}^{j\delta} a(s, X_s)ds = \int_0^{\tau_H} a^2(s, X_s)ds = H\Big) = 1.$$

We pass to the second term of the sum in (6.2). Let

$$h_\delta(s, X) = \sum_j a\big((j-1)\delta, X_{(j-1)\delta}\big) 1_{[(j-1)\delta, j\delta]}(s) 1_{[0,\tau_{H,\delta}]}(s).$$

By the previous considerations we obtain

$$P\{\int_0^\infty [h_\delta(s, X) - a(s, X_s) 1_{[0,\tau_H]}(s)]^2 ds \xrightarrow[\delta\to 0]{} 0\} = 1.$$

If $f_n \to f$ in probability in the L^2-norm, then $\int f_n dW \to \int f dW$ in probability [10]. So

$$\int_0^\infty h_\delta(s, X)dW_s \xrightarrow[\delta\to 0]{} \int_0^\infty a(s, X_s) 1_{[0,\tau_H]}(s)dW_s = \int_0^{\tau_H} a(s, X_s)dW_s \text{ in probability.}$$

However,

$$\int_0^\infty h_\delta(s, X)dW_s = \sum_{j=1}^{\tau_{H,\delta}/\delta} a\big((j-1)\delta, X_{(j-1)\delta}\big)\Delta W_{j\delta}.$$

Then

$$(6.4)\qquad \sum_{j=1}^{\tau_{H,\delta}/\delta} a\big((j-1)\delta, X_{(j-1)\delta}\big)\Delta W_{j\delta} \to \int_0^{\tau_H} a(s, X_s)dW_s$$

in probability as $\delta \to 0$.

From (6.3) and (6.4) it follows that

$$\sum_{j=1}^{\tau_{H,\delta}/\delta} a((j-1)\delta, X_{(j-1)\delta})\Delta X_{j\delta} \to \int_0^{\tau_H} a(s, X_s)dX_s \text{ in probability as } \delta \to 0.$$

The proof of Theorem 3.1 is the same as that of Theorem 2.1 and is therefore omitted.

Proof of Theorem 4.4. The proof of (4.4) is analogous to that of (2.3). To prove (4.5) let us consider the expression

$$\sum_{\substack{(i,j) \\ 1 \leqslant i,j \leqslant \tau_{H,\delta}/\delta}} a(((i-1)\delta, (j-1)\delta), X_{((i-1)\delta,(j-1)\delta)})X(P_{ij}).$$

We have

$$\sum_{\substack{(i,j) \\ 1 \leqslant i,j \leqslant \tau_{H,\delta}/\delta}} a(((i-1)\delta, (j-1)\delta), X_{((i-1)\delta,(j-1)\delta)})X(P_{ij}) \tag{6.5}$$
$$= \theta \sum_{\substack{(i,j) \\ 1 \leqslant i,j \leqslant \tau_{H,\delta}/\delta}} a(((i-1)\delta, (j-1)\delta), X_{((i-1)\delta,(j-1)\delta)}) \int_{P_{ij}} a(z, X_z)dz$$
$$+ \sum_{\substack{(i,j) \\ 1 \leqslant i,j \leqslant \tau_{H,\delta}/\delta}} a(((i-1)\delta, (j-1)\delta), X_{((i-1)\delta,(j-1)\delta)}) W(P_{i,j}).$$

By arguments similar to those in the proof of Theorem 2.1 we obtain

$$P\{\lim_{\delta\to 0} \sum_{\substack{(i,j) \\ 1 \leqslant i,j \leqslant \tau_{H,\delta}/\delta}} a(((i-1)\delta, (j-1)\delta), X_{((i-1)\delta,(j-1)\delta)}) \int_{P_{ij}} a(z, X_z)dz \tag{6.6}$$
$$= \int_{R_{\tau_H}} a^2(z, X_z)dz = H\} = 1.$$

Let us consider the second term of the sum in (6.5). Let

$$h_\delta(z, X) = \sum_{(i,j)} a(((i-1)\delta, (j-1)\delta), X_{((i-1)\delta,(j-1)\delta)}) 1_{R_{\tau_{H(\delta)}}(z)} 1_{P_{i,j}}(z).$$

If Γ denotes the line $s = t$, then $1_{R_{\tau_{H,\delta}}}(z)$ is $F_{z\Gamma}$-measurable (for the definition of $F_{z\Gamma}$-measurability see [19], [20]). Observe that

$$\sum_{\substack{(i,j) \\ 1 \leqslant i,j \leqslant \tau_{H,\delta}/\delta}} a(((i-1)\delta, (j-1)\delta), X_{((i-1)\delta,(j-1)\delta)}) W(P_{ij})$$
$$= \int_{R_{\tau_{H,\delta}}} \sum_{(i,j)} a(((i-1)\delta, (j-1)\delta), X_{((i-1)\delta,(j-1)\delta)}) 1_{P_{ij}}(z) W(dz).$$

Using the definition and properties of the Γ-integral with respect to a Wiener random field, introduced in [19], we can write

$$\int_{R_{\tau_{H,\delta}}} \sum_{(i,j)} a(((i-1)\delta, (j-1)\delta), X_{((i-1)\delta,(j-1)\delta)}) 1_{P_{ij}}(z) W(dz)$$
$$\int_{R_{\tau_{H,\Delta}}} \sum_{(i,j)} a(((i-1)\delta, (j-1)\delta), X_{((i-1)\delta,(j-1)\delta)}) 1_{P_{ij}}(z) W(dz).$$

As proved by Różański [16] the above integral is equal to

$$\int_{R_+^2}^{\Gamma} \sum_{(i,j)} a(((i-1)\delta, (j-1)\delta), X_{((i-1)\delta,(j-1)\delta)}) 1_{P_{ij}}(z) 1_{R_{\tau_H,\delta}}(z) W(dz)$$

$$= \int_{R_+^2}^{\Gamma} h_\delta(z, X) W(dz).$$

So we have proved that

$$\sum_{\substack{(i,j) \\ 1 \leqslant i,j \leqslant \tau_{H,\delta}/\delta}} a(((i-1)\delta, (j-1)\delta), X_{((i-1)\delta,(j-1)\delta)}) W(P_{ij}) = \int_{R_+^2} h_\delta(z, X) W(dz).$$

With arguments similar to those in the proof of Theorem 2.1 we conclude that $h_\delta(z, X)$ converges in probability in $L^2(R_+^2)$ to $a(z, X_z) 1_{R_{\tau_H}}(z)$ as $\delta \to 0$. This fact together with the results of Wong and Zakai [19], [20] implies that the integral

$$\int_{R_+^2}^{\Gamma} h_\delta(z, X) W(dz)$$

converges in probability to

$$\int_{R_+^2}^{\Gamma} a(z, X_z) 1_{R_{\tau_H}}(z) W(dz) = \int_{R_{\tau_H}} a(z, X_z) W(dz)$$

by [16]. The last statement together with (6.6) gives (4.5).

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